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Note on the convergence of iterates

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1. This note is strictly connected with note [3] by M. Kuczma in which the following problem has been considered: Let $f(x)$ be a function of a real or a complex variable defined in a neighbourhood $U = \{x: |x-a| < r\}$ of a point a such that $f(a) = a$. For a given $x_0 \in U$ we define the sequence of iterates x_n by the relation

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots,$$

and we consider the sequence

$$y_n = \frac{|x_{n+1}-x_n|}{|x_n - x_{n-1}|}, \quad n = 1, 2, \ldots$$

(putting $y_n = 0$ whenever $x_n = x_{n-1}$). Let us fix an $x_0 \in U, x_0 \neq a$.

THEOREM. *If the function $f(x)$ fulfils the inequality*

$$(1) \qquad |f(x)-a| < |x-a| \quad \textit{for } x \in U \setminus \{a\},$$

and has at the point a the derivative $s = f'(a) \neq 1$, then there exists a limit of the sequence y_n and $\lim y_n = |s|$ (¹).

This theorem is due to Hamilton [1] who stated it also for the value $s = 1$. But in this case the theorem becomes false, viz. Kuczma [3] showed that $\lim y_n$ need not exist; however, if it exists then $\lim y_n = 1$.

The aim of the present note is to give a sufficient condition for the existence of $\lim y_n$ (Theorem 1) and to derive an asymptotic relation for the sequence y_n (Theorem 2).

2. Let k be an integer, $k > 1$. We now formulate

THEOREM 1. *Let $p(x)$ be a function defined in U and fulfilling the conditions:*

$$(2) \qquad 0 < |1+(x-a)^{k-1}p(x)| < 1 \quad \textit{for } x \in U \setminus \{a\}$$

and

$$(3) \qquad \textit{there exists } g = \lim_{x\to a} |p(x)| \textit{ and } g > 0.$$

(¹) If not otherwise stated, the symbols lim and limsup will refer to $n \to +\infty$.

If the function $f(x)$ is given by the formula

$$f(x) = x+(x-a)^k p(x) \quad \text{for } x \in U, \tag{4}$$

then for every $x_0 \in U \setminus \{a\}$ there exists a limit of the sequence y_n and $\lim y_n = 1$.

Proof. By (2) and (4) inequality (1) holds for the function $f(x)$. Thus, given an $x_0 \in U \setminus \{a\}$, we can actually form the sequence of iterates x_n. Moreover, $\lim |x_n - a| = 0$ (cf. [3], § 3), i.e. $\lim x_n = a$. By (4) and (2) $x_n \neq a$ for $n = 0, 1, \ldots$ By (3) there is an integer n_0 such that for $n \geqslant n_0$ we have $p(x_n) \neq 0$. We write y_n explicitely for $n > n_0$, making use of (4):

$$y_n = \frac{|(x_n - a)^k p(x_n)|}{|(x_{n-1} - a)^k p(x_{n-1})|} = |1+(x_{n-1}-a)^{k-1} p(x_{n-1})|^k \frac{|p(x_n)|}{|p(x_{n-1})|}.$$

Hence, an account of the preceding discussion and (3), the theorem follows immediately.

3. Let x be a real variable. We shall write $u_n \sim v_n$ iff $v_n \neq 0$ and $\lim u_n/v_n = 1$. We shall also use the notion of the class $S_a^0(U)$ of functions, which is due to Kuczma ([2], p. 20) and may be formulated as follows:

$f \in S_a^0(U)$ iff f is continuous in U and satisfies (1) together with the inequality $\big(f(x)-a\big)(x-a) > 0$ for $x \in U \setminus \{a\}$.

THEOREM 2. *For every sequence $\{u_n\}$, $n = 0, 1, \ldots$, of positive numbers such that the sequence*

$$p_n = \prod_{i=0}^{n} u_i$$

satisfies the condition

$$0 < \limsup p_n < \infty \tag{5}$$

and for every integer $k > 1$ there exists a function $p(x)$ such that the function

$$f(x) = x - x^k p(x) \tag{6}$$

has the following properties:

1° $f \in S_0^0(\langle -1, 1 \rangle)$,

2° *the sequence y_n generated by the function $f(x)$ for $x_0 = 1$ is asymptotically equal to u_n: $y_n \sim u_n$.*

Proof. Fix an integer $k > 1$. Relation (5) implies the boundedness of the product p_n: $0 < p_n < M$ for every $n \geqslant 0$. We can restrict our attention to the case $M = 1$. For otherwise, if we had $M > 1$, then we could take into consideration the sequence $v_n = u_n M^{-2^{-n}}$, $n \geqslant 0$. Then the sequence $q_n = \prod_{i=0}^{n} v_i$ fulfils (5) since $q_n = p_n M^{-s_n}$, where $s_n = 2 - 2^{-n}$; hence $\limsup q_n = M^{-2} \limsup p_n$. Moreover, $0 < q_n \leqslant p_n M^{-1} < 1$ for $n \geqslant 0$, and $v_n \sim u_n$.

Thus, let the inequality

$$0 < p_n < 1 \tag{7}$$

be fulfilled for every $n \geqslant 0$. We put

$$x_0 = 1, \quad x_{n+1} = x_n - x_n^k p_n \quad \text{for } n = 0, 1, \ldots \tag{8}$$

and we are going to prove that the inequalities

$$0 < x_{n+1} < x_n \leqslant 1 \tag{9}$$

are valid for every $n \geqslant 0$. Indeed, $x_1 = 1 - p_0$, i.e. $0 < x_1 < x_0 = 1$, and if (9) holds for an $n \geqslant 0$, then $x_{n+2} > x_{n+1} - x_{n+1}^k > 0$ on account of (7) and the induction hypothesis. The inequality $x_{n+2} < x_{n+1}$ is obvious. Thus (9) is true for $n = 0, 1, \ldots$ and the sequence x_n converges, $\lim x_n = a$. It follows that $a = 0$. In fact, consider this subsequence of the sequence p_n, say p_{n_ν}, which converges to $\limsup p_n$. By (5) we have $\lim\limits_{\nu \to +\infty} p_{n_\nu} = p > 0$. Passing to the limit in equality (8) written for $n = n_\nu$ we get $a = a - a^k p$, which yields $a = 0$. Thus $\lim x_n = 0$, and (cf. (8))

$$\lim x_{n+1}/x_n = 1. \tag{10}$$

Now let us take an arbitrary function $p(x)$ which is defined in the interval $\langle 0, 1\rangle$, continuous in $(0, 1\rangle$ and fulfils the conditions

$$0 < p(x) < 1 \quad \text{in } (0, 1\rangle, \quad p(x_n) = p_n \quad \text{for } n = 0, 1, \ldots \tag{11}$$

We extend this function onto the whole interval $\langle -1, 1\rangle$ putting for $x \in \langle -1, 0)$: $p(x) = -p(-x)$ if k is even, and $p(x) = p(-x)$ if k is odd.

Finally we define the function $f(x)$ for $x \in \langle -1, 1\rangle$ by formula (6) (with $p(x)$ defined above). The function $f(x)$ is continuous in $\langle -1, 1\rangle$, and inequality (11) implies $0 < f(x) < x$ in $(0, 1\rangle$. We can easily verify that this function satisfies the relation $f(x) = -f(-x)$, whence $x < f(x) < 0$ for $x \in \langle -1, 0)$. These remarks imply assertion 1°.

Sequence (8) is then the sequence of iterates of the function $f(x)$, starting from the point $x_0 = 1$. We calculate the corresponding sequence y_n and, making use of formula (11), we get

$$y_n = x_n^k p(x_n)/[x_{n-1}^k p(x_{n-1})] = u_n (x_n/x_{n-1})^k.$$

From this and (10) we obtain the relation $y_n \sim u_n$, i.e. assertion 2° is true. This completes the proof of the theorem.

Remark. Theorem 2 allows us to find other examples of a divergent sequence y_n in the case $s = 1$. For instance if $u_n = [(n+2)/2]^{\varepsilon_n}$, where $[x]$ denotes the integral part of x, and $\varepsilon_n = (-1)^{n+1}$, $n = 0, 1, \ldots,$

then $p_{2m} = (m+1)^{-1}$, $p_{2m+1} = 1$ for $m = 0, 1, \ldots$, i.e. the assumptions of Theorem 2 are fulfilled, consequently the corresponding sequence y_n oscillates between zero and infinity.

References

[1] H. J. Hamilton, *Roots of equations by functional iteration*, Duke Math. J 13 (1946), p. 113-121.

[2] M. Kuczma, *Functional equations in a single variable*, Warszawa 1968.

[3] — *On the convergence of iterates*, Ann. Polon. Math. 20 (1968), p. 195-198.

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