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On a functional equation with divergent solutions

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In our preceding paper [1] we have dealt with the linear functional equation

(1) $$\varphi[f(z)]-g(z)\varphi(z) = h(z)$$

under the assumption that the functions $f(z)$, $g(z)$ and $h(z)$ are analytic in a neighbourhood of the origin, $f(0) = 0$, $g(0) = 0$ and $f'(0) = 0$. We have proved that every formal solution (1) of equation (1) is actual, i.e. has a positive radius of convergence. In the present note we are going to show that this conclusion is not longer valid if we drop the assumption $f'(0) = 0$.

We shall confine our attention to the particular case of (1)

(2) $$\varphi(sz)-z^q\varphi(z) = h(z) .$$

In the sequel we are always assuming that $h(z)$ is an analytic function in a neighbourhood of the origin, with an expansion

(3) $$h(z) = \sum_{n=0}^{\infty} a_n z^n ,$$

$q \geqslant 1$ is a positive integer, and

(4) $$0 < |s| < 1 .$$

At first let us note the following

THEOREM 1. *Equation* (2) *has a unique formal solution.*

Proof. Let us write

(5) $$\varphi(z) = \sum_{n=0}^{\infty} c_n z^n .$$

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(1) *A formal solution* of equation (1) is a formal power series that inserted into the equation satisfies it formally.

Inserting (3) and (5) into (2) we obtain

$$\sum_{n=0}^{\infty} c_n s^n z^n - \sum_{n=q}^{\infty} c_{n-q} z^n = \sum_{n=0}^{\infty} a_n z^n ,$$

whence

$$c_n = s^{-n} a_n \qquad \text{for } n = 0, \dots, q-1 , \tag{6}$$

$$c_n = s^{-n}(a_n + c_{n-q}) \qquad \text{for } n \geqslant q . \tag{7}$$

Relations (6) and (7) allow us to determine the sequence c_n uniquely.

However, series (5) may have the radius of convergence equal to zero. Below (Theorems 2 and 3) we shall exhibit two examples of functional equations of form (2) with divergent formal solutions.

Let us write

$$n = kq + l , \tag{8}$$

where the integers $k \geqslant 0$ and $0 \leqslant l \leqslant q-1$ are uniquely determined by n. By the successive use of (6) and (7) we obtain

$$c_{kq+l} = \sum_{i=0}^{k} \Big(\prod_{j=i}^{k} s^{-jq-l}\Big) a_{iq+l} . \tag{9}$$

Writing

$$m_{ni} = \sum_{j=i}^{k} (jq+l) = (k-i+1)\Big(\frac{k+i}{2} q + l\Big) , \tag{10}$$

we obtain from (9)

$$c_n = c_{kq+l} = \sum_{i=0}^{k} s^{-m_{ni}} a_{iq+l} . \tag{11}$$

For every fixed n the exponents m_{ni} form a decreasing sequence

$$m_{n0} \geqslant m_{n1} \geqslant \dots \geqslant m_{nk} . \tag{12}$$

It follows from (8) and (10) that

$$\lim_{n\to\infty} \frac{m_{ni}}{n} = \infty \tag{13}$$

whenever i is fixed (independent of n).

THEOREM 2. *If s is real, $0 < s < 1$, $h(z) \not\equiv 0$ and*

$$\operatorname{Arg} a_n = \theta = \text{const}, \quad n = 0, 1, 2, \ldots, \tag{14}$$

then the formal solution of equation (2) *has the radius of convergence equal to zero.*

Proof. Let a_{Kq+L} be the first non-zero coefficient in series (3). In view of (14) we may write (11) as

$$c_n = c_{kq+l} = \varepsilon \sum_{i=0}^{k} s^{-m_{ni}} |a_{iq+l}|, \tag{15}$$

where ε is the constant with $|\varepsilon| = 1$, $\operatorname{Arg} \varepsilon = \theta$. Since all summands occurring under the $\sum$ sign in (15) are real and non-negative, we have the estimation for $l = L$:

$$|c_{kq+L}| \geqslant s^{-m_{nK}} |a_{Kq+L}| \quad \text{for } k \geqslant K$$

($n = kq + L$), whence in view of (13)

$$\lim_{k\to\infty} \sqrt[kq+L]{|c_{kq+L}|} = \infty$$

and

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \infty,$$

which was to be proved.

In particular, for $\theta = 0$ or $\theta = \pi$, we have the following

COROLLARY. *If s is real, $0 < s < 1$, $h(z) \not\equiv 0$, and all a_n are real and of a constant sign, then the formal solution of equation* (2) *has the radius of convergence equal to zero.*

However, in Theorem 2 it is not enough to assume that (14) holds for n sufficiently large (cf. the example in the end of the paper).

THEOREM 3. *Let A and $M \leqslant 1$ be positive constants and suppose that*

$$M < \max\left(\sqrt[q]{\frac{1}{|s|^{q-1}+1}}, \sqrt[q]{\frac{1-|s|}{|s|^{q-1}}}\right). \tag{16}$$

If

$$|a_n| = AM^n \quad \text{for } n = 0, 1, 2, \ldots, \tag{17}$$

then the formal solution of equation (2) *has the radius of convergence equal to zero.*

Proof. Condition (17) means that $a_n = \varepsilon_n A M^n$, where $|\varepsilon_n| = 1$. Condition (16) implies that either

$$M^q < \frac{1}{|s|^{q-1}+1} < 1\,, \tag{18}$$

or

$$M^q < \frac{1-|s|}{|s|^{q-1}}\,. \tag{19}$$

Formula (11) may be written as

$$c_n = c_{kq+l} = A \sum_{i=0}^{k} s^{-m_{ni}} M^{iq+l} \varepsilon_{iq+l} = A M^l s^{-m_{n0}} \Big(\varepsilon_l + \sum_{i=1}^{k} s^{m_{n0}-m_{ni}} M^{iq} \varepsilon_{iq+l}\Big)\,.$$

Now,

$$\Big|\sum_{i=1}^{k} s^{m_{n0}-m_{ni}} M^{iq} \varepsilon_{iq+l}\Big| \leqslant \sum_{i=1}^{k} |s|^{m_{n0}-m_{ni}} M^{iq}\,.$$

If condition (18) is fulfilled, then we have according to (4) and (12)

$$\sum_{i=1}^{k} |s|^{m_{n0}-m_{ni}} M^{iq} \leqslant |s|^{m_{n0}-m_{n1}} \sum_{i=1}^{k} M^{iq}$$

$$\leqslant |s|^{m_{n0}-m_{n1}} \sum_{i=1}^{\infty} M^{iq} = |s|^{m_{n0}-m_{n1}} \frac{M^q}{1-M^q}\,.$$

Taking $l = q-1$, we get by (10)

$$m_{n0} - m_{n1} = q-1\,.$$

But

$$|s|^{q-1} \frac{M^q}{1-M^q} \overset{\mathrm{df}}{=} \vartheta < 1$$

in view of (18). Hence

$$\Big|\sum_{i=1}^{k} s^{m_{n0}-m_{ni}} M^{iq} \varepsilon_{iq+q-1}\Big| \leqslant \vartheta < 1\,. \tag{20}$$

If condition (19) is fulfilled, then

$$\sum_{i=1}^{k} |s|^{m_{n0}-m_{ni}} M^{iq} \leqslant M^q \sum_{i=1}^{k} |s|^{m_{n0}-m_{ni}} \leqslant M^q \sum_{j=m_{n0}-m_{n1}}^{\infty} |s|^j = M^q \frac{|s|^{m_{n0}-m_{n1}}}{1-|s|}\,.$$

We take again $l = q-1$. By (19)

$$M^q \frac{|s|^{q-1}}{1-|s|} \overset{\text{df}}{=} \vartheta < 1 ,$$

and we arrive again at estimation (20).

Thus we have for $n = kq+q-1$

$$\begin{aligned} |c_n| &= |c_{kq+q-1}| \\ &= AM^{q-1}|s|^{-m_{n0}}\Big|\varepsilon_{q-1}+\sum_{i=1}^{k} s^{m_{n0}-m_{ni}} M^{iq}\varepsilon_{iq+q-1}\Big| \\ &\geqslant AM^{q-1}|s|^{-m_{n0}}\Big(1-\Big|\sum_{i=1}^{k} s^{m_{n0}-m_{ni}} M^{iq}\varepsilon_{iq+q-1}\Big|\Big) \end{aligned}$$

and by (20)

$$|c_n| = |c_{kq+q-1}| \geqslant AM^{q-1}|s|^{-m_{n0}}(1-\vartheta) .$$

Hence we obtain according to (13)

$$\lim_{k\to\infty} \sqrt[kq+q-1]{|c_{kq+q-1}|} = \infty ,$$

i.e.

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \infty ,$$

which was to be proved.

The example of the function

$$h(z) = M+\sum_{n=1}^{\infty} (M^{n+1}s^n - M^n)z^n , \qquad M > 0 , \tag{21}$$

shows that in Theorem 3 condition (17) cannot be replaced by the asymptotic relation

$$|a_n| \sim AM^n , \qquad n\to\infty .$$

Here we have

$$\lim_{n\to\infty} \frac{|M^{n+1}s^n - M^n|}{M^n} = \lim_{n\to\infty} |Ms^n - 1| = 1 ,$$

but the equation

$$\varphi(sz) - z\varphi(z) = h(z) \tag{22}$$

with the right-hand side given by (21) has the convergent solution

$$\varphi(z) = \sum_{n=0}^{\infty} M^{n+1}z^n . \tag{23}$$

The same example (with s real, $0 < s < 1$) shows that in Theorem 2 it is not enough to assume that (14) holds from some term on. Here we have $M^{n+1} s^n - M^n = M^n(Ms^n - 1) < 0$ for n sufficiently large, and nevertheless equation (22) with the right-hand side given by (21) has convergent solution (23).

Finally let us note that condition (4) is essential in Theorems 2 and 3. If $|s| \geqslant 1$, then we may write equation (2) in the form

$$\varphi(z) = s^{-q} z^q \varphi(s^{-1}z) + h(s^{-1}z) , \tag{24}$$

where now $|s^{-1}| \leqslant 1$. According to results of Smajdor [2] and [3], the formal solution of equation (24) has a positive radius of convergence.

References

[1] M. Kuczma, *Analytic solutions of a linear functional equation*, Ann. Polon. Math. 21 (1969), pp. 297-303.

[2] W. Smajdor, *On the existence and uniqueness of analytic solutions of the functional equation* $\varphi(z) = h(z, \varphi[f(z)])$, ibidem 19 (1967), pp. 37-45.

[3] — *Analytic solutions of the functional equation* $\varphi(z) = h(z, \varphi[f(z)])$ *with right-side contracting*, Aequationes Math. 2 (1969), pp. 30-38.

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