ANNALES
POLONICI MATHEMATICI
XXXIX (1981)

On a minimum principle in several complex variables

by H. ALEXANDER (Chicago, Ill.)

Stefan Bergman in memoriam

Abstract. A domain Ω in C^n ($n \geqslant 2$) is said to satisfy the minimum principle if every bounded holomorphic function on Ω whose boundary values are in modulus essentially bounded below by a positive quantity is itself bounded below in modulus on Ω. The polydisc is shown to satisfy this minimum principle.

1. Let Ω be a bounded domain in C^n ($n > 1$) with a piecewise smooth boundary $\partial\Omega$. Denote by $H^\infty(\Omega)$ the space of bounded holomorphic functions f on Ω and let μ be surface (i.e. Hausdorff $2n-1$) measure on $\partial\Omega$. Every f in $H^\infty(\Omega)$ has non-tangential boundary values μ-almost everywhere on $\partial\Omega$. Let f^* denote the $L^\infty(\mu)$ boundary value function on $\partial\Omega$. We say that the minimum principle holds on Ω if, whenever $f \in H^\infty(\Omega)$ and $|f^*| \geqslant \delta > 0$, μ-almost everywhere on $\partial\Omega$, it follows that $|f| \geqslant \delta$ on Ω. Of course, the minimum principle never holds for a domain Ω in C^1. If f is continuous on the closure of Ω, then, as $n > 1$, each of the level sets $L_p = \{z \in \Omega\colon f(z) = f(p)\}$ for $p \in \Omega$ is non-compact and the closure of L_p meets $\partial\Omega$. Thus, for continuous functions f, $\inf_{z\in\Omega}|f(z)| = \inf_{z\in\partial\Omega}|f^*(z)|$. The minimum principle asserts the same relation, with "inf" replaced by "essential inf" on $\partial\Omega$, for a general function in $H^\infty(\Omega)$. This argument shows, for continuous f on $\bar{\Omega}$, that $f(\Omega) = f(\partial\Omega)$. Rudin has given an equivalent formulation of the minimum principle which reduces to this in the continuous case; namely, for $f \in H^\infty(\Omega)$, $f(\Omega) = \text{ess range } f^*$.

It is not known for what domains Ω the minimum principle holds. The modest purpose of this note is to show that it does hold for the polydisc. It would be interesting to decide this question for strictly pseudoconvex domains and, in particular, for the ball. Our verification for the polydisc employs the fact that the Bergman–Shilov boundary is, in this case, the torus and sheds no light on the problem for the ball.

Recall that $f \in H^\infty(\Omega)$ is an *inner function* if $|f^*| = 1$, μ a.e. Whenever the minimum principle holds in Ω, it follows that every inner function in Ω is constant. For the ball ($n > 1$) this also is not known. For the polydisc, an alternate proof was given in [1].

2. To fix some notation, let U be the open unit disc in the complex plane and let T be its boundary. Then U^n is the polydisc in C^n and T^n, the n-torus, is the Bergman–Shilov boundary of U^n. Let μ be "surface" measure on ∂U^n and let σ be normalized Haar measure on T^n. If $f \in H^\infty(U^n)$, then f^* is its μ almost everywhere defined boundary value function on ∂U^n and f^{**} will denote its σ almost everywhere defined boundary value function on T^n. Then $f^{**}(p) = \lim_{r\to 1} f(rp)$ for σ almost all $p \in T^n$; see [2]. Let $P_r(\theta) = (1-r^2)/(1-2r\cos\theta + r^2)$ be the Poisson kernel on the unit disc.

THEOREM. *The minimum principle holds on U^n for $n > 1$; i.e., if $f \in H^\infty(U^n)$ and $|f^*| \geqslant \delta > 0$, μ a.e., then $|f| \geqslant \delta$ on U^n.*

By a Fubini argument, it is enough to verify this for $n = 2$.

LEMMA 1. *For almost all θ, $f^*(e^{i\theta}, se^{i\varphi})$ exists for all s, $0 \leqslant s < 1$, and all φ, $0 \leqslant \varphi < 2\pi$, is the $\lim_{r\to 1} f(re^{i\theta}, se^{i\varphi})$ and, moreover,*

$$f^*(e^{i\theta}, se^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} f^{**}(e^{i\theta}, e^{it}) P_s(\varphi - t)\, dt. \tag{1}$$

Proof. For all θ, φ and $s < r < 1$ we have the Poisson integral formula

$$f(re^{i\theta}, se^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta}, re^{it}) \frac{r^2 - s^2}{|r - se^{i(\varphi - t)}|^2}\, dt. \tag{2}$$

By Fubini's theorem, for almost all θ,

$$f^{**}(e^{i\theta}, e^{it}) = \lim_{r\to 1} f(re^{i\theta}, re^{it}) \quad \text{for almost all } t.$$

Hence the Lebesgue bounded convergence theorem applied to the right-hand side of (2) yields the desired conclusions. $\square$

Thus, for almost all θ, there is a well-defined bounded measurable function $f_\theta(w) \equiv f^*(e^{i\theta}, w)$ defined for all $|w| < 1$.

LEMMA 2. *$f_\theta(w)$ is a holomorphic function of $w \in U$.*

Proof. For a fixed θ for which f_θ is defined, put $g_n(w) = f((1-1/n)e^{i\theta}, w)$ for $|w| < 1$. Then $\{g_n\}$ is a normal family of holomorphic functions in U and $g_n \to f_\theta$ pointwise. Hence f_θ is holomorphic. $\square$

By hypothesis $|f^*| \geqslant \delta$, μ a.e. Hence, for almost all θ, $|f_\theta(w)| \geqslant \delta$ for almost all $w \in U$ with respect to planar measure on U. As f_θ is holomorphic, we conclude that for almost all θ, $|f_\theta(w)| \geqslant \delta$ for *all* $|w| < 1$. For such θ, let $h_\theta = 1/f_\theta$; h_θ is a bounded holomorphic function on U.

LEMMA 3. *$|f^{**}| \geqslant \delta$, σ a.e. on T^2.*

Proof. By (1), for almost all θ, $s < 1$,

$$f_\theta(se^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} f^{**}(e^{i\theta}, e^{it}) P_s(\varphi - t)\, dt.$$

Hence, for almost all θ, $f^{**}(e^{i\theta}, e^{i\varphi}) = \lim_{s\to 1} f_\theta(se^{i\varphi})$ for almost all φ. Since $|f_\theta(w)| \geqslant \delta$ for almost all θ and for all w, we conclude, by Fubini, that $|f^{**}| \geqslant \delta$, σ a.e. □

Define $g = 1/f^{**} \in L^\infty(T^2)$; $\|g\|_\infty \leqslant 1/\delta$.

LEMMA 4. $g \in H^\infty(T^2)$.

Proof. It suffices to show, for every pair of integers (m, n) with either $m > 0$ or $n > 0$, that

$$I \equiv \int_{T^2} g \cdot e^{im\theta} \cdot e^{in\varphi}\, d\theta\, d\varphi = 0; \quad \text{see [2]}.$$

Without loss of generality, suppose $n > 0$. Then, by Fubini, $I = \int e^{im\theta}\left(\int e^{in\varphi} g\, d\varphi\right) d\theta$. Now, for almost all θ, $h_\theta(w) = 1/f_\theta(w)$ for $w \in U$ and so, for almost all θ, $g(e^{i\theta}, e^{i\varphi}) = \lim_{r\to 1} h_\theta(re^{i\varphi})$ for almost all φ. Hence, for almost all θ,

$$\int e^{in\varphi} g(e^{i\theta}, e^{i\varphi})\, d\varphi = \lim_{r\to 1} \int e^{in\varphi} h_\theta(re^{i\varphi})\, d\varphi = 0$$

(the last integral being zero because $n > 0$ and $h_\theta \in H^\infty(U)$). Thus $I = 0$. □

Proof of the Theorem. Let $G \in H^\infty(U^2)$ be the holomorphic extension of g; $|G(z)| \leqslant \|g\|_\infty \leqslant 1/\delta$ for $z \in U^2$. Then, as $(Gf)^{**} = gf^{**} = 1, \sigma$ a.e., we have $Gf \equiv 1$ on U^2. Hence $|f(z)| = 1/|G(z)| \geqslant \delta$ for $z \in U^2$. Q.E.D.

Remark. One might ask whether this result can be made quantitative by showing that there is a constant $c > 0$ such that if $F \in H^\infty(U^2)$, $\|F\|_\infty = 1$, and $F(0) = 0$, then $\mu\{z \in \partial U^2 \colon |F^*(z)| > \frac{1}{2}\} \geqslant c$. This, however, is false: Consider $F_n(z, w) = \frac{1}{2}(z^n + w^n)$. Since $|F_n| \to \frac{1}{2}$ on $\partial U^2 \backslash T^2$ as $n \to \infty$, it follows that $\mu\{z \in \partial U^2 \colon |F_n^*(z)| > \frac{1}{2}\} \to 0$.

References

[1] H. Alexander, *Holomorphic mappings from the ball and polydisc*, Math. Ann. 209 (1974), p. 249–256.

[2] W. Rudin, *Function theory in polydiscs*, New York, Benjamin 1969.

Reçu par la Rédaction le 31. 5. 1978