

INTEGRABILITY THEOREMS FOR POWER SERIES

BY

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1. Introduction and statements of the results. Let f be a real function on $(0, 1)$ defined by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and put

$$S_n = \sum_{k=0}^n a_k.$$

Askey [1] established the following theorem connecting the integrability of f with the behaviour of the coefficients a_n :

THEOREM A. *Let $a_n \geq 0$. Then, for $1 \leq p \leq \infty$,*

$$(1.1) \quad \left[\int_0^1 [f(x)]^p dx \right]^{1/p} < \infty$$

if and only if

$$(1.2) \quad \left[\sum_{n=1}^{\infty} n^{-2} S_n^p \right]^{1/p} < \infty.$$

Theorem A has been subsequently generalized by Khan [7] in the following form (cf. also Hardy and Littlewood [5], Theorem 3):

THEOREM B. *Let $a_n \geq 0$ and let $r < 1$. Then, for $1 \leq p \leq \infty$,*

$$(1.3) \quad \left[\int_0^1 (1-x)^{-r} [f(x)]^p dx \right]^{1/p} \leq K \left[\sum_{n=1}^{\infty} n^{r-2} S_n^p \right]^{1/p}$$

and

$$(1.4) \quad \left[\sum_{n=1}^{\infty} n^{r-2} S_n^p \right]^{1/p} \leq K \left[\int_0^1 (1-x)^{-r} [f(x)]^p dx \right]^{1/p}.$$

(Here, and in what follows, K denotes a constant not necessarily the same at each occurrence.)

Recently, Askey and Karlin [2] generalized Theorem A in the form of the following theorems:

THEOREM C. *Let Φ be a positive increasing convex function defined on $[0, \infty)$. Then*

$$(1.5) \quad \int_0^1 \Phi[|f(x)|] dx \leq \sum_{n=0}^{\infty} [(n+1)(n+2)]^{-1} \Phi(|S_n|).$$

THEOREM D. *Under the same assumptions as in Theorem C, we have*

$$(1.6) \quad \int_0^1 (1-x)^\beta \Phi(|f(x)|) dx \leq \sum_{n=0}^{\infty} \Phi(|S_n|) \frac{\Gamma(\beta+2)\Gamma(n+1)}{\Gamma(n+\beta+3)} \\ \leq K_\beta \sum_{n=0}^{\infty} \Phi(|S_n|) (n+1)^{-2-\beta}$$

for $\beta > -2$.

THEOREM E. *Under the same assumptions as in Theorem C, we have*

$$(1.7) \quad \int_0^1 \frac{\Phi(|f(x)|)}{(1-x)^2} dx \leq \sum_{n=0}^{\infty} \Phi\left(\left|\frac{S_n^*}{n+1}\right|\right),$$

where $S_n^* = \sum_{k=0}^n S_k$.

It is worth mentioning here that, using the method of asymptotic approximations, Chen [3] also generalized the Hardy-Littlewood theorem [5].

The aim of this paper is to prove some more general theorems which contain the first part of Theorem B as a special case and also generalize Theorems C, D and E of Askey and Karlin. The case $0 \leq r < 1$ of (1.3) is included in the following theorem:

THEOREM 1. *Let f and Φ satisfy the assumptions of Theorem C. Suppose further that Ψ is a non-negative non-decreasing function in $L(0, 1)$. Then*

$$(1.8) \quad \int_0^1 \Psi(x) \Phi[|f(x)|] dx \leq K \sum_{n=0}^{\infty} \Phi(|S_n|) \alpha_n^{(0)},$$

where

$$\alpha_n^{(0)} = \int_{1-1/n}^1 (1-x) \Psi(x) dx.$$

A generalization of (1.3) for $r < 0$ is included in Theorem 2 in which we have to impose some more restrictions on Ψ . Here the assumption $\alpha_n \geq 0$ can also be dropped.

THEOREM 2. *Let f and Φ satisfy the assumptions of Theorem C. Suppose that there is an integer $p \geq 1$ such that $\Psi, \Psi', \dots, \Psi^{(p-1)}$ are absolutely continuous on $[0, 1)$ and that*

$$\Psi(1) = \Psi'(1) = \dots = \Psi^{(p-1)}(1) = 0.$$

Furthermore, suppose that $\Psi^{(p)}$ has a constant sign and $|\Psi^{(p)}|$ is non-decreasing in the set $\{x \in (0, 1): |\Psi^{(p)}(x)| \text{ exists}\}$. Then

$$(1.9) \quad \int_0^1 \Psi(x) \Phi(|f(x)|) dx \leq K \sum_{n=0}^{\infty} \Phi(|S_n|) n^{-p-1} |\Psi^{(p-1)}(1 - 1/n)|.$$

We obtain (1.3) of Theorem B for $r < 0$ by putting $\Psi(x) = (1-x)^{-r}$ and $p = -[r]$, where $[r]$ denotes the largest integer equal to or less than r .

By a similar argument as that used in the proof of Theorem 2 (see Section 4), we can also obtain

THEOREM 3. *Let f, Φ and Ψ satisfy the assumptions of Theorem 2. Then*

$$(1.10) \quad \int_0^1 \Psi(x) \Phi(|f(x)|) dx \leq K \sum_{n=0}^{\infty} \Phi\left(\left|\frac{S_n^*}{n+1}\right|\right) n^{-1-p} |\Psi^{(p-1)}(1 - 1/p)|,$$

where

$$S_n^* = \sum_{k=0}^n S_k.$$

Theorem E of Askey and Karlin follows from Theorem 3 by putting $\Psi(x) = (1-x)^{-2}$ and $p = 2$.

2. Lemmas. The proofs of our theorems are based upon the following lemmas.

LEMMA 1. *If a function $\Psi \in L(0, 1)$ is non-negative and non-decreasing on a set E of measure 1 in $(0, 1)$, then*

$$\int_0^1 x^{n+p} \Psi(x) dx \asymp \int_{1-1/n}^1 \Psi(x) dx \quad (n \rightarrow \infty)$$

for any fixed p .

(By $g(n) \asymp h(n)$ ($n \rightarrow \infty$) is meant that there exist positive numbers K_1 and K_2 , independent of x , such that $K_1 h(n) < g(n) < K_2 h(n)$ for all sufficiently large values of n).

This lemma is due to Kennedy [6].

LEMMA 2. *Under the same assumptions as in Lemma 1, we have*

$$\int_0^1 x^{n+p} (1-x) \Psi(x) dx \asymp \int_{1-1/n}^1 (1-x) \Psi(x) dx$$

as $n \rightarrow \infty$.

Proof. Let us put $\tilde{\Psi}(x) = \Psi(x)$ for $x \in E$ and $\tilde{\Psi}(x) = \lim_{y \rightarrow x^-} \Psi(y)$ for $x \in (0, 1) \setminus E$. Clearly, $\tilde{\Psi}$ satisfies the assumptions of the lemma with $E = (0, 1)$ and $\tilde{\Psi} = \Psi$ almost everywhere (cf. [6], p. 59). Hence we can additionally assume that $E = (0, 1)$. Then, for n sufficiently large, we have

$$\begin{aligned}
 (2.1) \quad & \int_0^1 \Psi(x) x^{n+p} (1-x) dx \\
 &= \int_0^{1-1/n} \Psi(x) x^{n+p} (1-x) dx + \int_{1-1/n}^1 \Psi(x) x^{n+p} (1-x) dx \\
 &\leq \Psi(1-1/n) \int_0^{1-1/n} x^{n+p} (1-x) dx + \int_{1-1/n}^1 \Psi(x) (1-x) dx \\
 &= \Psi(1-1/n) \left[\int_0^{1-1/n} x^{n+p} dx - \int_0^{1-1/n} x^{n+p+1} dx \right] + \int_{1-1/n}^1 \Psi(x) (1-x) dx \\
 &= \Psi(1-1/n) (1-1/n)^{n+p+1} \frac{2n+p+1}{n(n+p+1)(n+p+2)} + \int_{1-1/n}^1 \Psi(x) (1-x) dx \\
 &\leq \Psi(1-1/n) \frac{2n+p+1}{n(n+p+1)(n+p+2)} + \int_{1-1/n}^1 \Psi(x) (1-x) dx.
 \end{aligned}$$

But, Ψ being non-decreasing, we have

$$\begin{aligned}
 (2.2) \quad & \int_{1-1/n}^1 \Psi(x) (1-x) dx \geq \Psi(1-1/n) \int_{1-1/n}^1 (1-x) dx \\
 &= \frac{1}{2} n^{-2} \Psi(1-1/n).
 \end{aligned}$$

Combining estimates (2.1) and (2.2), we obtain

$$(2.3) \quad \int_0^1 \Psi(x) x^{n+p} (1-x) dx = O \left(\int_{1-1/n}^1 \Psi(x) (1-x) dx \right)$$

as $n \rightarrow \infty$.

Conversely,

$$\begin{aligned}
 (2.4) \quad & \int_0^1 \Psi(x) x^{n+p} (1-x) dx \geq (1-1/n)^{n+p} \int_{1-1/n}^1 \Psi(x) (1-x) dx \\
 &\sim e^{-1} \int_{1-1/n}^1 \Psi(x) (1-x) dx
 \end{aligned}$$

as $n \rightarrow \infty$.

Hence, by virtue of (2.3) and (2.4), we have

$$\int_0^1 x^{n+p}(1-x)\Psi(x)dx \asymp \int_{1-1/n}^1 (1-x)\Psi(x)dx$$

as $n \rightarrow \infty$.

This completes the proof of Lemma 2.

3. Proof of Theorem 1. Summation by parts yields

$$f(x) = \sum_{n=0}^{\infty} S_n x^n (1-x).$$

Since

$$(1-x) \sum_{n=0}^{\infty} x^n = 1, \quad 0 \leq x < 1,$$

using Jensen's inequality we have

$$\Phi(|f(x)|) \leq \sum_{n=0}^{\infty} \Phi(|S_n|) x^n (1-x).$$

Then, integrating term by term, we obtain

$$\int_0^1 \Psi(x) \Phi(|f(x)|) dx \leq \sum_{n=0}^{\infty} \Phi(|S_n|) \int_0^1 \Psi(x) x^n (1-x) dx.$$

Hence, using Lemma 2 with $p = 0$, we have

$$\begin{aligned} \int_0^1 \Psi(x) \Phi(|f(x)|) dx &\leq K \sum_{n=0}^{\infty} \Phi(|S_n|) \int_{1-1/n}^1 \Psi(x) (1-x) dx \\ &= K \sum_{n=0}^{\infty} \Phi(|S_n|) \alpha_n^{(0)}. \end{aligned}$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2. As in the proof of Theorem 1, we have

$$(4.1) \quad \Phi(|f(x)|) \leq \sum_{n=0}^{\infty} \Phi(|S_n|) x^n (1-x).$$

The conditions imposed on Ψ imply that it has the constant sign on $(0, 1)$. So, integrating term by term, we obtain

$$(4.2) \quad \int_0^1 \Psi(x) \Phi(|f(x)|) dx \leq \sum_{n=0}^{\infty} \Phi(|S_n|) \left| \int_0^1 \Psi(x) x^n (1-x) dx \right|.$$

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