AGASSIZ SUM OF ALGEBRAS

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A colimit-type construction of an algebra over a system of algebras without nullary operations indexed by a semilattice was introduced by Płonka in [2]. Lakser, Padmanabhan and Platt generalized it to the concept of Płonka sum described in [1]; Płonka sum applies also to algebras having nullaries. Both constructions are very natural and have already found numerous applications.

In the present note* we would like to generalize these concepts still further. The new concept of Agassiz sum does not impose any restrictions on the indexing algebra and a corresponding extension of the principal result of [2] remains valid also in the case of Agassiz sums.

Let $K$ be a class of algebras of type $\tau$, and let $I$ ($I$ for Indexing Algebras) be a class of algebras of type $\varrho$. The only assumption we make is that algebras of $I$ have a nullary operation whenever algebras of $K$ do.

To every polynomial symbol $p$ of type $\tau$ we assign a polynomial symbol $N(p)$ of type $\varrho$ ($N$ for the Name of the Polynomial) satisfying

(i) the variables of $p$ and $N(p)$ are the same;
(ii) $N$ preserves composition, that is,

$$N(p(q_1, \ldots, q_k)) = N(p)(N(q_1), \ldots, N(q_k))$$

is an identity in $I$.

These two conditions say that $N$ is a product-preserving functor from the theory of $K$ into the theory of $I$.

Let $K$, $I$ and $N$ be given and let $B$ be an algebra of the indexing class $I$. Let $R \subseteq B^2$ be a transitive relation on the underlying set $B$ of the algebra $B$ such that

(a) if $j = h(b_1, \ldots, i, \ldots, b_n)$ for some algebraic operation $h$ of $B$ and for some $b_1, \ldots, b_n \in B$, then $\langle i, j \rangle \in R$.

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An Agassiz system $S$ of algebras over $B$ is a family $(A_i : i \in B)$ of algebras from $K$ together with a family $(f_{ij} : \langle i, j \rangle \in R)$ of homomorphisms $f_{ij} : A_i \to A_j$ such that

(b) $f_{jk} \circ f_{ij} = f_{ik}$ whenever $\langle i, j \rangle, \langle j, k \rangle \in R$.

Form the disjoint union $A = \bigcup (A_i : i \in B)$ and define an algebra of type $\tau$ on $A$ as follows. Let $p$ be an $n$-ary polynomial symbol and $n \geq 1$. For $a_1, \ldots, a_n \in A$, let $b_1, \ldots, b_n$ be the (uniquely determined) elements of $B$ such that $a_i \in A_{b_i}$, and let $b = N(p)(b_1, \ldots, b_n)$. By (a), $\langle b_i, b \rangle \in R$, so we can set $a_i^* = f_{b_i,b}(a_i)$. All $a_i^*$ are in $A_b$; we write

$$p(a_1, \ldots, a_n) = p_{A_b}(a_1^*, \ldots, a_n^*).$$

If $p$ is nullary, then $N(p) = \mathbb{N}(p)$ is nullary and we define $p$ on $A$ to be the value of $p_{A_{\mathbb{N}(p)}}$. Let $A$ be the algebra on the set $A$ whose operations have just been defined. The algebra $A$ is called the Agassiz sum of $S$ and denoted by $A = \lim_N(S)$.

Example 1. Let the algebras of $K$ have no nullary operations and let $I$ be the class of semilattices. For a polynomial symbol $p$, set

$$N(p) = ((x_1 \vee x_2) \vee \ldots) \vee x_k,$$

where $x_1, \ldots, x_k$ are all the variables of $p$. If the relation $R$ is the partial ordering of a semilattice $B$, then the Agassiz sum of the corresponding Agassiz system is the sum described in [2].

Example 2. If nullary operations are permitted to appear in the algebras of Example 1, the Plonka sum of [1] is obtained.

Example 3. The direct product $A \times B$ of two algebras of the same type is obtained as the Agassiz sum with the naming functor $N(p) = p$ of an Agassiz system consisting of $|B|$ copies of the algebra $A$ and of all the canonical isomorphisms between them.

A large variety of examples can be constructed from

**Proposition.** Let $I$ be the class of semigroups and let $I_0$ be the class of semigroups with 0. For a polynomial symbol $p$ of type $\tau$, write $N(p) = ((x_1 \cdot x_2) \cdot \ldots) \cdot x_k$, where $x_1, \ldots, x_k$ lists all variables of $p$ (with repetition) in the order of their occurrence. If $p$ is nullary, set $N(p) = 0$. Then $N$ satisfies (i) and (ii) for any class $K$.

Let $\lim_N(K, I)$ denote the class of all isomorphic copies of all Agassiz sums with given $K$, $I$ and $N$. Let $\text{Id}(K)$ be the set of all identities that hold in $K$. An identity $p = q$ in $\text{Id}(K)$ is $N$-regular if $N(p) = N(q)$ holds in $I$. Let $\text{Id}_N(K)$ be the set all $N$-regular identities.

**Theorem.** $\text{Id}(\lim_N(K, I)) = \text{Id}_N(K)$.

If this theorem is specialized to the case described in Example 1, it becomes the main result of [2]. A result of [1] is obtained if that theorem is applied to Example 2.
Observe that \( \text{Id}_N(K) = \text{Id}(K) \cap \text{Id}(I) = \text{Id}(K \cup I) \) whenever \( K \) and \( I \) are of the same type and the functor \( N \) is trivial. Let \( K \) and \( I \) be equational classes of the same type. It is natural to ask under what conditions can every algebra of \( K \vee I \) be represented as an Agassiz sum. S. M. Lee has shown that this happens in several cases of pairs of equational classes of idempotent semigroups.

**Problem 1.** Let \( K \) and \( L \) be equational classes of algebras of the same type and let \( K \subseteq L \). What conditions are sufficient for the existence of an equational class \( I \) and a naming functor \( N \) with \( L = \text{lim}_N(K, I) \)? (P 892)

An identity \( p = q \) is *regular* if the same variables occur on both its sides. Thus, in Example 1 we always get \( \text{Id}(\text{lim}_N(K, I)) \) as a well-defined subset of \( \text{Id}(K) \).

**Problem 2.** Under what conditions can \( \Sigma \subseteq \text{Id}(K) \) be represented in the form \( \Sigma = \text{Id}(\text{lim}_N(K, I)) \) for some \( I \) and \( N \)? (P 893)

**REFERENCES**


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