COLLOQUIUM MATHEMATICUM

VOL. XLIX 1985 FASC. 2

LIFT SPACES AND GENERALIZED CONNECTIONS

BY

ALEKSANDER CAŁKA (WROCŁAW)

0. Introduction. Let M be a connected, paracompact and smooth manifold with a left transitive action on M of a connected Lie group G. If $p, q \in M$, then there exists an element $g \in G$ such that $q = gp$. We shall describe a method of determining such a g.

Let Q be any continuous map from the tangent bundle of M into the Lie algebra of G (identified with the tangent space at the unit e of G) such that

$$\frac{d}{dt}\Big|_{t=0} \exp\big(tQ(v)\big)x = v$$

for any $x \in M$ and $v \in T_x M$. Let γ_t, $t \in [0, 1]$, be a piecewise C^1-path in M joining p to q. If $\bar{\gamma}_t$, $t \in [0, 1]$, is a piecewise C^1-path in G such that

$$\dot{\bar{\gamma}}_t = Q(\dot{\gamma}_t)\bar{\gamma}_t, \ t \in [0, 1], \quad \bar{\gamma}_0 = e,$$

then $\bar{\gamma}_1 q = p$.

It is natural to ask the following question: when is the point $\bar{\gamma}_1$ independent of the path γ_t, $t \in [0, 1]$, such that $\bar{\gamma}_0 = p$ and $\bar{\gamma}_1 = q$? (Similarly, when does it depend only on the homotopy class of γ?)

The purpose of this paper is to give an answer to this question. In the sequel we shall introduce the concept of lift spaces and their holonomy groups. Using standard techniques of holonomy theory we shall prove (Proposition 1) that these groups are Lie groups and the answer to our question is affirmative if and only if the corresponding holonomy group is trivial (in the local case – if and only if it is discrete). In Section 2 we construct the holonomy bundles of lift spaces and observe (Remark 4) that there exists a natural one-to-one correspondence between linear lift spaces over M with group G and connections in the trivial principal bundle $M \times G \to M$. In the case of a nonlinear lift space, this gives a general construction of a nonlinear connection in $M \times G \to M$. Nonlinear connections have been studied by Ehresmann.

Next we give (Propositions 3 and 4) an interpretation of the holonomy groups and the holonomy bundles and prove (Corollary 5) that the holonomy bundle of any continuous lift space is differentiable. Section 3 contains some property for flatness and local flatness. In Section 4 the results of Sections 1-3 are applied to the bundle of a homogeneous space.

Throughout this paper we assume that all manifolds are Hausdorff, paracompact, and smooth and that Lie groups have a countable basis of neighbourhoods (i.e., countably many components). The Lie algebra of a Lie group G will always be denoted by $\mathbf{G}$. For a left Lie group action $G \times M \to M$, we use the notation

$$Xp = \frac{d}{dt}\Big|_{t=0} \exp(tX)p$$

for $p \in M$ and $X \in \mathbf{G}$. A similar convention is adopted for the right action. The universal covering of a manifold M will be denoted by $c_M \colon \tilde{M} \to M$, while the tangent bundle projection $TM \to M$ by π_M.

The author wishes to express his best thanks to A. Derdziński and W. Roter for their encouragement and advice given during the preparation of this paper.

1. Lift spaces and their holonomy groups. A triple (M, G, Q) is called a *lift space* (over M with group G) if

(i) M is a connected manifold (called the *base*),

(ii) G is a Lie group,

(iii) Q is a continuous map $TM \to \mathbf{G}$ (called the *lift map*).

The lift space (M, G, Q) is called *linear* (resp. *homogeneous*) if Q is linear (resp. homogeneous) on each fibre of TM. If Q is a C^k-map, $k \geqslant 0$, then (M, G, Q) is said to be *of class* C^k.

Let (M, G, Q) be a lift space. For any piecewise C^1-path $\gamma\colon [a, b] \to M$ we define the *lift of* γ *with origin* $g \in G$ to be any piecewise C^1-path $\bar{\gamma}\colon [a, b] \to G$ such that

(1) $$\dot{\bar{\gamma}}_t = Q(\gamma_t, \dot{\gamma}_t)\bar{\gamma}_t \text{ for all } t \in [a, b], \quad \bar{\gamma}_a = g.$$

Given a lift space (M, G, Q), for any piecewise C^1-path $\gamma\colon [a, b] \to M$ and for any $g \in G$ there exists a unique lift $\bar{\gamma}(g)$ *of* γ with origin g (see [3], p. 69, the Lemma). Its value for the parameter $t \in [a, b]$ will be denoted by $\bar{\gamma}(g)(t) = \bar{\gamma}(g, t)$.

Example 1. Let $(M, G, Q) = (R^n, R^m, Q)$, where $n, m \geqslant 1$ and $Q\colon R^{2n} \to R^n$ is an arbitrary continuous map. If $\gamma\colon [a, b] \to M$ is a piecewise C^1-path, then its lift with origin $x_0 \in R^m$ is given by

(2) $$\bar{\gamma}(x_0, t) = \int_a^t Q\left(\gamma_s, \frac{d}{ds}\gamma_s\right)ds + x_0.$$

Example 2. Let $(M, G, Q) = (R^n, \mathrm{GL}(m, R)_0, Q)$, where n, $m \geqslant 1$ and Q: $R^{2n} \to \mathrm{gl}(m, R)$ is a continuous map. For any piecewise C^1-path γ: $[a, b] \to M$, the lift of γ with origin $A \in \mathrm{GL}(m, R)_0$ is the solution of the ordinary linear differential equation

$$\text{(3)} \qquad \dot{A}_t = Q_t A_t \text{ for all } t \in [a, b], \quad A_a = A,$$

where

$$Q_t = Q\left(\gamma_t, \frac{d}{dt}\gamma_t\right).$$

Given a lift space (M_2, G_2, Q_2), a Lie group G_1, and a manifold M_1 together with a C^1-map f: $M_1 \to M_2$ and a Lie group homomorphism h: $G_2 \to G_1$, we define the *induced* lift spaces $(M_1, G_2, Q \circ f_*)$, $(M_1, G_1, h_* \circ Q \circ f_*)$, and $(M_2, G_1, h_* \circ Q)$ so that the pairs

$$(\mathrm{id}_{M_1}, h)\colon (M_1, G_2, Q \circ f_*) \to (M_1, G_1, h_* \circ Q \circ f_*),$$

$$(f, h)\colon (M_1, G_2, Q \circ f_*) \to (M_2, G_2, h_* \circ Q)$$

are lift space morphisms in the sense given by

Definition 1. Let (M_i, G_i, Q_i), $i = 1, 2$, be lift spaces of class C^k, $k \geqslant 0$, f: $M_1 \to M_2$ a C^{k+1}-map, h: $G_1 \to G_2$ a Lie group homomorphism. The pair (f, h) is called a *lift space morphism* $(M_1, G_1, Q_1) \to (M_2, G_2, Q_2)$ if the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\ h\ } & G_2 \\ \uparrow{\scriptstyle Q_1} & & \uparrow{\scriptstyle Q_2} \\ TM_1 & \xrightarrow{\ f\ } & TM_2 \end{array}$$

commutes.

Lift spaces of class C^k, $k \geqslant 0$, and their morphisms form a category.

The following lemma is immediate:

Lemma 1. *Let (M, G, Q) be a lift space and let γ: $[a, b] \to M$ and η: $[c, d] \to M$ be piecewise C^1-paths in M. Then*:

(i) *The lifts of γ are invariant under right translations in G: $\bar{\gamma}(g_1)g_2 = \bar{\gamma}(g_1 g_2)$ for all g_1, $g_2 \in G$; in particular, $\bar{\gamma}(g) = \bar{\gamma}(e)g$ for all $g \in G$.*

(ii) *Lifting of paths commutes with the translations T: $R \to R$ of the parameter: $\overline{\gamma \circ T}(g) = \bar{\gamma}(g) \circ T$ for all $g \in G$.*

(iii) *If $\gamma(b) = \eta(c)$, then the composite path $\eta \circ \gamma$ defined on $[a, d-c+b]$ by $\eta \circ \gamma(t) = \gamma(t)$ for $t \leqslant b$, $\eta \circ \gamma(t) = \eta(t+c-b)$ for $t \geqslant b$ satisfies $\overline{\eta \circ \gamma}(g) = \bar{\eta}(\bar{\gamma}(g, b)) \circ \bar{\gamma}(g)$ for all $g \in G$.*

(iv) *If (f, h): $(M, G, Q) \to (M_1, G_1, Q_1)$ is a lift space morphism, then $\overline{f \circ \gamma}(e_1) = h \circ \bar{\gamma}(e)$, e, e_1 being the identities of G, G_1, respectively.*

(v) *If (M, G, Q) is homogeneous, then lifting of paths commutes with* *piecewise* C^1 *changes of the parameter* $s\colon R \to R$: $\overline{\gamma \circ s}(g) = \bar{\gamma}(g) \circ s$ *for all* $g \in G$. *In particular,* $\overline{(-\gamma)}(g) = -\bar{\gamma}(g)$, *where* $-\gamma\colon [a, b] \to M$ *is given by* $(-\gamma)(t) = \gamma(b+a-t)$.

Let (M, G, Q) be a lift space. Given any piecewise C^1-path in M, the fact whether or not its lift with origin $g \in G$ is closed is independent of the choice of g. It would be interesting to know what conditions the lift space should satisfy in order that the lift of any closed path be closed again. To be precise, we formulate

Definition 2. A lift space is called *flat* (resp. *locally flat*) if the lift with origin e of any closed piecewise C^1-path (resp. of any closed piecewise C^1-path homotopic to zero) is closed.

Remark 1. It is easy to see that the lift space (M, G, Q) is locally flat if and only if for any point $p \in M$ there exists a neighbourhood U of p such that the induced lift space $(U, G, Q|_U)$ is flat. If M is simply connected, then flatness and local flatness of (M, G, Q) are equivalent.

Now we proceed to define, for a lift space (M, G, Q), its holonomy semigroups and holonomy groups which will measure its deviation from flatness. Let $p \in M$ and let Ω_p (resp. Ω_p^0) denote the set of all closed (resp. closed and homotopic to zero) piecewise C^1-paths starting from p. Let $h_p\colon \Omega_p \to G$ be the map defined as follows: $h_p(\gamma) = \bar{\gamma}(e, b)$, where $\gamma\colon [a, b] \to M$ is any loop (closed path) at p. Both sets Ω_p and Ω_p^0 are semigroups with respect to the path composition operation, and h_p is a semigroup homomorphism (cf. (iii) of Lemma 1).

By the *holonomy semigroup* $K_{Q,p}$ (resp. *restricted holonomy semigroup* $K_{Q,p}^0$) of a lift space (M, G, Q) at $p \in M$ we mean the image of the set Ω_p (resp. Ω_p^0) in the map h_p. The *holonomy group* $|K_{Q,p}|$ (resp. *restricted holonomy group* $|K_{Q,p}^0|$) of the lift space (M, G, Q) at $p \in M$ is defined to be the subgroup of G generated (algebraically) by $K_{Q,p}$ (resp. by $K_{Q,p}^0$).

LEMMA 2. *Let* $p \in M$. *Then*:

(i) $|K_{Q,p}^0|$ *is a normal subgroup of* $|K_{Q,p}|$.

(ii) *There exists a unique group homomorphism*

$$h_p^*\colon \pi_1(M, p) \to |K_{Q,p}|/|K_{Q,p}^0|$$

which is surjective and for which the diagram

$$\begin{array}{ccccc} \Omega_p & \xrightarrow{h_p} & K_{Q,p} & \xrightarrow{i} & |K_{Q,p}| \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ \pi_1(M,p) & & \xrightarrow{h_p^*} & & |K_{Q,p}|/|K_{Q,p}^0| \end{array}$$

commutes (pr_1 *and* pr_2 *are quotient projections and* i *is the natural embedding*).

Proof. (i) By our definitions it suffices to prove that $gg_0g^{-1}\in|K^0_{Q,p}|$ whenever $g_0\in K^0_{Q,p}$ and $g\in K_{Q,p}$. Let $g=h_p(\gamma)$, $g_0=h_p(\eta)$ for some $\gamma\in\Omega_p$, $\eta\in\Omega^0_p$ and put $g_1=h_p(-\eta)$. Clearly, $\gamma\circ\eta\circ(-\gamma)$, $\gamma\circ(-\gamma)\in\Omega^0_p$, so that

$$h_p(\gamma\circ\eta\circ(-\gamma))=gg_0g_1, \quad h_p(\gamma\circ(-\gamma))=gg_1\in K^0_{Q,p}.$$

Thus $g_1^{-1}g^{-1}\in|K^0_{Q,p}|$ and we obtain $gg_0g^{-1}=(gg_0g_1)(g_1^{-1}g^{-1})\in|K^0_{Q,p}|$.

(ii) If $\mathrm{pr}_1(\gamma_1)=\mathrm{pr}_1(\gamma_2)$ for $\gamma_1,\gamma_2\in\Omega_p$, then $\gamma_1\circ(-\gamma_2)$ and $\gamma_2\circ(-\gamma_2)$ belong to Ω^0_p and we obtain

$$h_p(\gamma_1)(h_p(\gamma_2))^{-1}=h_p(\gamma_1\circ(-\gamma_2))(h_p(\gamma_2\circ(-\gamma_2)))^{-1}\in|K^0_{Q,p}|.$$

Thus the map h^*_p: $\mathrm{pr}_1(\gamma)\to\mathrm{pr}_2h_p(\gamma)$ is a well-defined group homomorphism. It is surjective since the set $\mathrm{pr}_2K_{Q,p}$ generates the group $|K_{Q,p}|/|K^0_{Q,p}|$ and $\mathrm{pr}_2K_{Q,p}=h^*_p\pi_1(M,p)$. This completes the proof.

The quotient group $\pi_{Q,p}=|K_{Q,p}|/|K^0_{Q,p}|$ is called the *fundamental holonomy group of* (M, G, Q) *at* $p\in M$. Using the fact that the fundamental group of any paracompact manifold is countable, we obtain

COROLLARY 1. *The fundamental holonomy group* $\pi_{Q,p}$ *of a lift space* (M, G, Q) *at any point* $p\in M$ *is countable.*

We are now going to prove that the holonomy groups and restricted holonomy groups of any lift space (M, G, Q) are Lie subgroups of G. First we show that lift spaces have properties analogous to fibrations:

LEMMA 3. *Let* H: $[a,b]\times[0,1]\to M$ *be a piecewise* C^1*-homotopy (with loose ends) between the paths* γ,η: $[a,b]\to M$. *Let the map* F: $[a,b]\times[0,1]\to M$ *be defined as follows*: $F(t,s)=H^s(e,t)$, *where* $H^s(t)=H(t,s)$. *Then* F *is a homotopy with fixed origin between the lifts* $\bar\gamma(e)$ *and* $\bar\eta(e)$. *If* Q *is of class* C^k, $k\geqslant 1$, *then* F *is piecewise* C^k.

Proof. Our assertion follows immediately from the dependence on the parameter theorem for ordinary differential equations.

LEMMA 4. *For any lift space* (M, G, Q) *and* $p\in M$, *the restricted holonomy semigroup at* p *is a pathwise connected subsemigroup of* G.

Proof. Since any two paths belonging to Ω^0_p are homotopic to zero, they are homotopic. The homotopy may be chosen to be piecewise C^1 (cf. [3], p. 284). It is defined on some tetragon (not a rectangle!), but it may be easily extended to a rectangular piecewise C^1-homotopy. Our assertion follows now from Lemma 3.

We shall use the following obvious

LEMMA 5. *If* A *is a pathwise connected subset of a topological group* G *which contains the identity, then the subgroup generated by* A *is pathwise connected.*

PROPOSITION 1. *Let (M, G, Q) be a lift space. Then for any point $p\in M$ the holonomy group $|K_{Q,p}|$ is a Lie subgroup of G and the restricted holonomy group $|K^0_{Q,p}|$ is its identity component.*

Proof. By Lemmas 4 and 5 the group $|K^0_{Q,p}|$ is pathwise connected. By the Kuranishi-Yamabe theorem (cf.[5]) it is a Lie subgroup of G. Since $|K^0_{Q,p}|$ is a normal subgroup of $|K_{Q,p}|$ (cf. (i) of Lemma 1), our assertion follows from Corollary 1.

Let (M, G, Q) be a lift space. Let $p, q\in M$ and $\gamma\colon [a, b]\to M$ be a piecewise C^1-path in M joining p to q and $g_0 = \bar{\gamma}(e, b)$.

LEMMA 6. *The inner automorphism* $\operatorname{ad}(g_0)$ *restricted to $|K_{Q,p}|$ (resp. to $|K^0_{Q,p}|$) is a Lie group isomorphism $|K_{Q,p}|\to|K_{Q,q}|$ (resp. $|K^0_{Q,p}|\to|K^0_{Q,q}|$).*

Proof. Suppose that $\operatorname{ad}(g_0)K_{Q,p}\subset|K_{Q,q}|$ and $\operatorname{ad}(g_0^{-1})K_{Q,q}\subset|K_{Q,p}|$. Then $\operatorname{ad}(g_0)$ induces a Lie group homomorphism $|K_{Q,p}|\to|K_{Q,q}|$ (regarded as Lie subgroups of G). By symmetry, it is a Lie group isomorphism and it induces an isomorphism of the identity components $|K^0_{Q,p}|\to|K^0_{Q,p}|$. Let $\eta\colon [c, d]\to M$ be a piecewise C^1-path in M joining q to p and $g_1 = \bar{\eta}(e, d)$. For every $g\in K_{Q,p}$ we have $g = h_p(\alpha)$, $\alpha\in\Omega_p$ and $g_0 g g_1 = h_q(\gamma\circ\alpha\circ\eta)$, $g_0 g_1 = h_q(\gamma\circ\eta)\in K_{Q,q}$. Thus,

$$g_1^{-1}g_0^{-1}\in|K_{Q,q}| \quad\text{and}\quad \operatorname{ad}(g_0)(g) = (g_0 g g_1)(g_1^{-1}g_0^{-1})\in|K_{Q,q}|.$$

The proof for $\operatorname{ad}(g_0^{-1})K_{Q,q}\subset|K_{Q,p}|$ is similar. This proves our assertion.

If there exist a piecewise C^1-paths $\gamma\colon [a, b]\to M$ and $\eta\colon [c, d]\to M$ such that $\gamma_a = \eta_d = p$, $\gamma_b = \eta_c = q$, and $\eta\circ\gamma\in\Omega^0_p$, then the points p and q are called *conjugate* in the lift space (M, G, Q). In this case, by the proof of Lemma 6, $\operatorname{ad}(\bar{\gamma}(e, b))$ takes $K_{Q,p}$ (resp. $K^0_{Q,p}$) onto $K_{Q,q}$ (resp. onto $K^0_{Q,q}$).

We say that a lift space (M, G, Q) is *uniform* if any two points in M are conjugate. Thus we have

COROLLARY 2. *Let (M, G, Q) be a lift space. Then the holonomy groups (resp. restricted holonomy groups) at all points of M are mutually isomorphic as Lie groups. If the lift space (M, G, Q) is uniform, then the holonomy semigroups (resp. restricted holonomy semigroups) at all points of M are also mutually isomorphic.*

It is easy to see that if for every $p\in M$ we have $K^0_{Q,p} = |K^0_{Q,p}|$, then the lift space (M, G, Q) is uniform. In fact, given $p, q\in M$ and a piecewise C^1-path $\gamma\colon [a, b]\to M$ joining p to q, let $g_0 = \bar{\gamma}(e, b)$ and $g_1^{-1} = \overline{(-\gamma)}(e, b)$. We have $g_1^{-1}g_0\in K^0_{Q,p}$, so $g_0^{-1}g_1\in|K^0_{Q,p}| = K^0_{Q,p}$ and there exists a piecewise C^1-loop α at p, homotopic to zero, $\alpha\colon [a_1, b_1]\to M$, such that $\bar{\alpha}(e, b_1) = g_0^{-1}g_1$. Define a path η in M by $\eta = \alpha\circ(-\gamma)$. Then $h_p(\eta\circ\gamma) = e$ and γ, $-\eta$ are homotopic, which shows that p and q are conjugate.

Example 3. In Example 1, let Q be of class C^1 and linear on fibres. Fix a basis $(e_i)_{i=1}^n$ of R^n and let $Q_i(x) = Q(x, e_i)$ for $x\in R^n$, $1\leqslant i\leqslant n$. By (2) our lift space is flat if and only if

$$\frac{\partial}{\partial x^i} Q_k = \frac{\partial}{\partial x^k} Q_i \quad \text{for all } i, k \leqslant n.$$

Thus, every homogeneous lift space of the form (R, R, Q) is flat.

Example 4. Let (M, G, Q) be a homogeneous lift space. Then, by (v) of Lemma 1, we have $K_{Q,q} = |K_{Q,q}|$, $K^0_{Q,q} = |K^0_{Q,q}|$, $q \in M$.

Example 5. In Example 1, let $n = m = 1$ and $Q(x, y) = f(y)$, where $f\colon R \to R$ is a continuous function such that $f \geqslant 0$, $f(x) = 0$ for $x \leqslant -1$, and $f(x) = 1 + x^2$ for $x \geqslant 0$. Then the lift of any piecewise C^1-path $\gamma\colon [a, b] \to M$ is given by

$$\bar{\gamma}(0, t) = \int_a^t f(\gamma'(s))\,ds.$$

Hence $K_{Q,1} = K^0_{Q,1} = R_+$, $K_{Q,-1} = K^0_{Q,-1} = R_+ \cup \{0\}$, and so the semigroups $K_{Q,1}$ and $K_{Q,-1}$ are not isomorphic. Therefore, this lift space is not uniform.

Remark 2. Holonomy groups measure the deviation from flatness (resp. local flatness) of lift spaces in the following sense:

The lift space is flat (resp. locally flat) if and only if its holonomy group at some point is trivial (resp. discrete).

Remark 3. By (iv) of Lemma 1 it is clear that any lift space morphism $(f, h)\colon (M, G, Q) \to (M_1, G_1, Q_1)$ induces semigroup homomorphism $K_{Q,p} \to K_{Q_1,f(p)}$ and $K^0_{Q,p} \to K^0_{Q_1,f(p)}$ and, by Proposition 1, Lie group homomorphisms $|K_{Q,p}| \to |K_{Q_1,f(p)}|$ and $|K^0_{Q,p}| \to |K^0_{Q_1,f(p)}|$ for each $p \in M$.

2. Holonomy bundle of a lift space. Let (M, G, Q) be a lift space of class C^k, $k \geqslant 0$, and fix a point $p \in M$. For every $q \in M$ we denote by S^p_q the set of all end-points of the lifts (with origin $e \in G$) of all piecewise C^1-paths joining p to q. It is easy to see that the set $g|K_{Q,p}|$ is independent of the choice of $g \in S^p_q$. This set will be called the *fibre over* q of the lift space (M, G, Q) and denoted by $|S^p_q|$. We have

$$(4) \qquad gK_{Q,p} \subset S^p_q \subset |S^p_q| \quad \text{for all } g \in S^p_q.$$

In fact, let $g_i \in S^p_q$, $i = 1, 2$. Thus, there exist piecewise C^1-paths $\gamma_i\colon [a_i, b_i] \to M$ such that $\gamma_i(a_i) = p$, $\gamma_i(b_i) = q$, and $\gamma_i(e, b_i) = g_i$. We shall show that $g_1^{-1} g_2 \in |K_{Q,p}|$. Let $g = \overline{(-\gamma_1)}(e, b_1)$. Then $g_1^{-1} g_2 = (gg_1)^{-1}(gg_2)$ and

$$gg_i = \overline{(-\gamma_1) \circ \gamma_i}(e, b_i + b_1 - a_1) \in K_{Q,p}, \quad i = 1, 2.$$

We shall now show that the disjoint union of all fibres of a lift space (M, G, Q) admits a natural principal fibre bundle structure. We denote the set

$$\bigcup_{q\in M} \{q\} \times |S_q^p| \subset M \times G$$

by $Q(M, p)$ and the restriction of the projection pr_M: $M \times G \to M$ (resp. pr_G: $M \times G \to G$) to $Q(M, p)$ by pr_M^p (resp. pr_G^p).

Let $\{U_l, \varphi_l\}$ be an atlas on the manifold M such that

$$\varphi_l(U_l) = \{x \in R^{\dim M}: \|x\| < 1\} \quad \text{for all } l.$$

Let $q_l = \varphi_l^{-1}(0)$. For every l and $q \in U_l$ the path $\gamma^{q,l}$: $[0, 1] \to M$ is defined by $\gamma^{q,l}(t) = \varphi_l^{-1}(t\varphi_l(q))$. Let us pick some collection of piecewise C^1-paths γ^l: $[-1, 0] \to M$ joining points p and q_l for every l and let γ_q^l denote the composite path $\gamma^{q,l} \circ \gamma^l$. For every l we define the map Λ_l: $U_l \to U_k \times G$ as follows:

$$\Lambda_l(q) = (q, \bar{\gamma}_q^l(e, 1)) \quad \text{for } q \in U_l. \tag{5}$$

By our definitions we have $\operatorname{im} \Lambda_l \subset Q(M, p)$, $\mathrm{pr}_M^p \circ \Lambda_l = \mathrm{id}_{U_l}$ and, clearly, these maps are of class C^k (cf. Lemma 3). By the subbundle theorem (cf. [3], p. 84), $Q(M, p)$ is a reduced subbundle, of class C^k (as will be seen in Corollary 5, of class C^{k+1}), of the trivial principal fibre bundle $M \times G$, on which the structure group $|K_{Q,p}|$ acts on the right and the maps Λ_l are local cross-sections. This bundle will be called the *holonomy bundle* of the lift space (M, G, Q) with base point $p \in M$.

As an immediate consequence of Remark 2 we obtain

PROPOSITION 2. *The lift space* (M, G, Q) *is flat* (*resp. locally flat*) *if and only if for some point* $p \in M$ *the bundle projection* pr_M^p: $Q(M, p) \to M$ *is a homeomorphism* (*resp. a covering*).

Let (M, G, Q) with base point $p \in M$ be of class C^k, $k \geq 0$. Let $p_1 \in M$ and $g_0 \in |S_{p_1}^p|$. Let R denote the restriction of the map $M \times G \ni$ $\ni (q, g) \mapsto (q, gg_0^{-1}) \in M \times G$ to $Q(M, p)$ and let α be the restriction of the inner automorphism $\mathrm{ad}(g_0)$: $G \to G$ to $|K_{Q,p}|$.

LEMMA 7. *The pair* (R, α) *is a principal bundle isomorphism* $Q(M, p) \to Q(M, p_1)$. *In particular, for any piecewise* C^1-*path* γ: $[a, b] \to M$ *joining* p *to* p_1 *we have*

$$Q(M, p) = Q(M, p_1)\bar{\gamma}(e, b).$$

Proof. We have $g_0 = g_1 g_2$ for some $g_1 \in S_{p_1}^p$, $g_2 \in |K_{Q,p}|$ and, by Lemma 6,

$$\mathrm{ad}(g_0)|K_{Q,p}| = \mathrm{ad}(g_1)\,\mathrm{ad}(g_2)|K_{Q,p}| = \mathrm{ad}(g_1)|K_{Q,p}| = |K_{Q,p_1}|.$$

For any $q \in M$ and $g \in S_q^{p_1}$ we have $|S_q^{p_1}| = g|K_{Q,p_1}|$, $gg_0 \in S_q^p$, so

$$|S_q^{p_1}|g_0 = gg_0(\mathrm{ad}(g_0^{-1})|K_{Q,p_1}|) = gg_0|K_{Q,p}| = |S_q^p|.$$

Therefore, $R(Q(M, p)) = Q(M, p_1)$. Clearly, $R(xg) = R(x)\alpha(g)$ for each $x \in Q(M, p)$ and $g \in |K_{Q,p}|$, which completes the proof.

Thus we obtain

COROLLARY 3. *Let (M, G, Q) be a lift space. Then the holonomy bundles at all points of M are mutually isomorphic as principal fibre bundles.*

Given a lift space (M, G, Q) with base point $p \in M$ for any piecewise C^1-path γ: $[a, b] \to M$ we define its *horizontal lift* $\hat{\gamma}(g)$ in the bundle $M \times G$ with origin $g \in G$ as follows:

$$\hat{\gamma}(g)_t = (\gamma_t, \bar{\gamma}(g, t)) \quad \text{for } t \in [a, b]. \tag{6}$$

Clearly, if $g \in |S^p_{\gamma_a}|$, then the entire path $\hat{\gamma}(g)_t$ belongs to $Q(M, p)$. A path in $M \times G$ (resp. in $Q(M, p)$) is called *horizontal in $M \times G$* (resp. *in $Q(M, p)$*) if it is the horizontal lift of some piecewise C^1-path γ: $[a, b] \to M$ with origin $g \in G$ (resp. $g \in |S^p_{\gamma_a}|$). A vector tangent to $M \times G$ is called *horizontal in $M \times G$* (resp. *in $Q(M, p)$*) if it is tangent to a horizontal path in $M \times G$ (resp. in $Q(M, p)$). The set of all horizontal vectors in $M \times G$ (resp. in $Q(M, p)$), denoted by $\bar{T}^Q$ (resp. by T^Q), in view of (1) and (6) is given by

$$T^Q_{(q,g)} = \{(v, Q(v)g) \colon v \in T_q M\} \tag{7}$$

for $(q, g) \in M \times G$ (resp. for $(q, g) \in Q(M, p)$).

Note that the sets of all horizontal paths in $M \times G$ and $\bar{T}^Q$ (resp. in $Q(M, p)$ and T^Q) are G-invariant (resp. $|K_{Q,p}|$-invariant).

Remark 4. If the lift space (M, G, Q) of class C^k, $k \geq 0$, is linear, then $\bar{T}^Q$ is a connection of class C^k in the trivial principal bundle $M \times G$ (i.e., G-invariant distribution of class C^k such that $T(M \times G) = \bar{T}^Q \oplus Tv(M \times G)$, where $Tv(M \times G) = TG$ is the vertical tangent bundle). If $p \in M$, then $|K_{Q,p}|$, $|K^0_{Q,p}|$, and $Q(M, p)$ are exactly its holonomy group, restricted holonomy group, and holonomy bundle at the point $(p, e) \in M \times G$, respectively. In this case T^Q is the reduction of the connection $\bar{T}^Q$ to the holonomy bundle through the point (p, e), and horizontal paths in $M \times G$ (resp. in $Q(M, p)$) are exactly the horizontal paths with respect to the connection $\bar{T}^Q$ (resp. T^Q).

Conversely, given any connection T of class C^k, $k \geq 0$, in the trivial principal bundle $M \times G$, the map

$$Q(v) = X^v, \quad \text{where } (v, X^v) \in T_{(q,e)} \text{ for } q \in M \text{ and } v \in T_q M, \tag{8}$$

defines a linear lift space (M, G, Q) of class C^k such that $T = \bar{T}^Q$.

Consequently: a linear lift space (M, G, Q) of class C^2 is locally flat if and only if $\bar{T}^Q$ is involutive (cf. [3], p. 14).

Now we shall give an interpretation of the holonomy group and the holonomy bundle of the lift space (M, G, Q). For each $p \in M$, $H_{Q,p}$ will denote the set of all (left) cosets $g|K_{Q,p}|$, $g \in G$, and the principal bundle projection $G \ni g \mapsto g|K_{Q,p}| \in H_{Q,p}$ will be denoted by $\varphi_{Q,p}$.

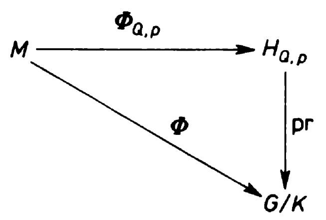
PROPOSITION 3. *Let (M, G, Q) be a lift space of class C^k, $k \geqslant 0$, with base point $p \in M$. Then*:

(i) *There exists a unique map $\Phi_{Q,p}$: $M \to H_{Q,p}$ of class C^{k+1} such that*

$$(\Phi_{Q,p})_* = Q\cdot(\Phi_{Q,p}\circ\pi_M), \qquad \Phi_{Q,p}(p) = \varphi_{Q,p}(p) \tag{9}$$

(*i.e.*, $(\Phi_{Q,p})_*(v) = Q(v)\Phi_{Q,p}(q)$ *for* $q \in M$ *and* $v \in T_q M$).

(ii) *Let $K \subset G$ be a Lie subgroup of G with the bundle projection φ: $G \to G/K$. If there exists a differentiable map Φ: $M \to G/K$ which satisfies condition* (9), *then $|K_{Q,p}| \subset K$ and the diagram*



commutes (*where* pr *is the induced projection*).

Proof. (i) We have only to prove the existence of $\Phi_{Q,p}$, since the uniqueness is evident from (ii). Let $q \in M$ and γ_i: $[a_i, b_i] \to M$, $i = 1, 2$, be piecewise C^1-paths joining p to q. By (3), $\overline{\gamma_1}(e, b_1)^{-1}\overline{\gamma_2}(e, b_2) \in |K_{Q,p}|$. Thus the map $\Phi_{Q,p}(q) = \varphi_{Q,p}(\overline{\gamma^q}(e, 1))$, where, for any $q \in M$, γ^q: $[0, 1] \to M$ is some piecewise C^1-path joining p to q, is well defined. For any piecewise C^1-path γ: $[0, 1] \to M$ starting from p, we have

$$\Phi_{Q,p}(\gamma_t) = \varphi_{Q,p}(\overline{\gamma}(e, t))$$

and

$$\frac{d}{dt}\Phi_{Q,p}(\gamma_t) = (\varphi_{Q,p})_*\big(Q(\dot{\gamma}_t)\overline{\gamma}(e, t)\big) = Q(\dot{\gamma}_t)\Phi_{Q,p}(\gamma_t)$$

for $t \in [0, 1]$. Therefore, (9) holds and $\Phi_{Q,p}$ is of class C^{k+1}.

(ii) Let φ: $M \to G/K$ satisfy condition (9). First we prove that for every piecewise C^1-path γ: $[0, 1] \to M$ starting from p we have $\Phi(\gamma_t) = \varphi(\overline{\gamma}(e, t))$, $t \in [0, 1]$. In fact, consider the ordinary differential equation on the manifold G/K, given by

$$\dot{\xi}_t = X(t, \xi_t), \qquad \xi_0 = \varphi(e), \tag{10}$$

where $X(t, y) = Q(\dot{\gamma}_t)y$ for $t \in [0, 1]$, $y \in G/K$. By (1), we have

$$\frac{d}{dt}\varphi(\overline{\gamma}(e, t)) = \varphi_*\big(Q(\dot{\gamma}_t)\overline{\gamma}(e, t)\big) = Q(\dot{\gamma}_t)\varphi(\overline{\gamma}(e, t)) = X\big(t, \varphi(\overline{\gamma}(e, t))\big)$$

and, by (9),

$$\frac{d}{dt}\Phi(\gamma_t) = \Phi_*(\dot{\gamma}_t) = Q(\dot{\gamma}_t)\Phi(\gamma_t) = X\big(t, \Phi(\gamma_t)\big).$$

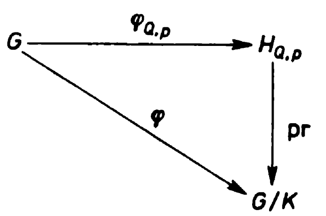
Since $\Phi(\gamma_0) = \varphi(\bar{\gamma}(e, 0)) = e$, our assertion is implied immediately by the following

LEMMA 8. *Let* α: $[0, 1] \to G$ *be a piecewise* C^1*-path and* $X_t = \dot{\alpha}_t \alpha_t^{-1}$. *Then the solution of equation* (10) *with* $X(t, y) = X_t y$ *for* $t \in [0, 1]$ *and* $y \in G/K$ *is unique.*

Proof. Let $\varphi(e) = p_1$. Clearly, the path $\xi_t = \alpha_t p_1$ is a solution. Let η_t be any other solution. By the local triviality of the bundle $G \to G/K$ there exist $\varepsilon > 0$ and a C^1-path β: $[0, \varepsilon] \to G$ such that $\eta_t = \beta_t p_1$. Assume that α is of class C^1 on $[0, \varepsilon]$; then $\beta_t = \alpha_t s_t$ for some C^1-path s: $[0, \varepsilon] \to G$, so that $\dot{\beta}_t = X_t \beta_t + \alpha_t \dot{s}_t$ and $(\alpha_t \dot{s}_t) p_1 = 0$. Thus $\alpha_t \dot{s}_t \in \alpha_t s_t (T_e K)$ and there exists a continuous path Y: $[0, \varepsilon] \to T_e K$ such that $\dot{s}_t = s_t Y_t$. Since $s_0 \in K$, we obtain $s_t \in K$ (cf. [3], p. 69) and $\eta_t = \xi_t$ for $0 \leqslant t \leqslant \varepsilon$. The standard continuation argument concludes the proof of Lemma 8.

Now we complete the proof of Proposition 3. Let $g \in K_{Q,p}$; then there exists a piecewise C^1-loop γ: $[a, b] \to M$ at p such that $\bar{\gamma}(e, b) = g$; so $\varphi(e) = \Phi(p) = \varphi(\bar{\gamma}(e, b)) = \varphi(g)$ and $g \in K$. Since $K_{Q,p}$ generates the group $|K_{Q,p}|$, we obtain $|K_{Q,p}| \subset K$ and $\varphi = \mathrm{pr} \circ \varphi_{Q,p}$. Thus $\Phi(\gamma_t) = \mathrm{pr} \circ \varphi_{Q,p}(\bar{\gamma}(e, t)) = \mathrm{pr} \circ \Phi_{Q,p}(\gamma_t)$ for any piecewise C^1-path γ starting from p; so $\Phi = \mathrm{pr} \circ \Phi_{Q,p}$. This completes the proof.

COROLLARY 4. *Let* $K \subset G$ *be a Lie subgroup of* G *such that* $|K_{Q,p}| \subset K$ *and let* φ, pr *denote the natural projections*:



Let $\Phi_K = \mathrm{pr} \circ \Phi_{Q,p}$. *Then for any piecewise* C^1 *path* γ: $[a, b] \to M$ *starting from* p *we have*

$$\Phi_K(\gamma_t) = \varphi(\bar{\gamma}(e, t)) \quad \text{for each } t \in [a, b]. \tag{11}$$

In particular, $\Phi_{Q,p}(\gamma_t) = \varphi_{Q,p}(\bar{\gamma}(e, t))$.

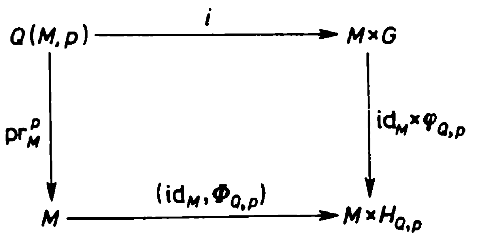
The principal fibre bundle $\varphi_{Q,p}$: $G \to H_{Q,p}$ is analytic; hence the induced principal bundle $\mathrm{id}_M \times \varphi_{Q,p}$: $M \times G \to M \times H_{Q,p}$ (with structure group $|K_{Q,p}|$) is of class C^∞, and will be denoted by $M \times G(M \times H_{Q,p})$. Since the map $\Phi_{Q,p}$ is of class C^{k+1}, we have the induced embedding of class C^{k+1}, $(\mathrm{id}_M, \Phi_{Q,p})$: $M \to M \times H_{Q,p}$, whose image, denoted by $\mathrm{gr}\,\Phi_{Q,p}$, is the graph of $\Phi_{Q,p}$. Clearly, $\mathrm{gr}\,\Phi_{Q,p} \subset M \times H_{Q,p}$ is a closed submanifold of class C^{k+1}.

PROPOSITION 4. *Let* (M, G, Q) *be a lift space with base point* $p \in M$. *Then the holonomy bundle* $Q(M, p)$ *is isomorphic to*

(i) *the induced principal bundle by the map* $\Phi_{Q,p}$ *from the bundle* $\varphi_{Q,p}\colon G \to H_{Q,p}$;

(ii) *the restriction of the bundle* $M \times G (M \times H_{Q,p})$ *to* $\operatorname{gr} \Phi_{Q,p}$.

Proof. The diagram



commutes (where $i = (\mathrm{pr}^p_M, \mathrm{pr}^p_G)$ is the natural embedding). In fact, by Corollary 4 we obtain $\varphi^{-1}_{Q,p}(\Phi_{Q,p}(\gamma_t)) = \bar{\gamma}(e, t)|K_{Q,p}| = |S_{\gamma_t}|$ for any piecewise C^1-path $\gamma\colon [a, b] \to M$ started from p and for each $t \in [a, b]$. Thus for any $q \in M$ we have $\varphi^{-1}_{Q,p}(\Phi_{Q,p}(q)) = |S_q|$. Clearly, $i(xg) = i(x)g$ for $x \in Q(M, p)$, $g \in |K_{Q,p}|$, which implies easily that $Q(M, p)$ is isomorphic to (i) and (ii).

Since the map $(\mathrm{id}_M, \Phi_{Q,p})$ is of class C^{k+1}, we obtain

COROLLARY 5. *Let the lift space* (M, G, Q) *with base point* $p \in M$ *be of class* C^k, $k \geqslant 0$. *Then the holonomy bundle* $Q(M, p)$ *admits a natural structure of a principal fibre subbundle of class* C^{k+1}, *of the trivial bundle* $M \times G \to M$.

Remark 5. If the lift space (M, G, Q) is of class C^k, $k \geqslant 0$, then the local cross-sections of $Q(M, p)$, $p \in M$, defined by (5) are not of class C^{k+1} in general (cf. Example 7 below). However, given any open cover $\{V_l\}$ of $H_{Q,p}$ with local cross-sections $\Pi_l\colon V_l \to \varphi^{-1}_{Q,p}(V_l) \subset G$ of class C^∞, by Proposition 5 below the local cross-sections of the bundle $Q(M, p)$, i.e., $\pi_l\colon U_l \to Q(M, p)$, where $U_l = \Phi^{-1}_{Q,p}(V_l)$, are given by $\pi_l(q) = (q, \Pi_l \Phi_{Q,p}(q))$ for $q \in U_l$. Clearly, they are of class C^{k+1}.

The Lie subalgebra $T_e|K_{Q,p}| \subset G$ (denoted by $k_{Q,p}$) is called the *holonomy algebra* of the lift space (M, G, Q) at $p \in M$. The disjoint union

$$k_Q(M) = \dot{\bigcup_{q \in M}} k_{Q,p}$$

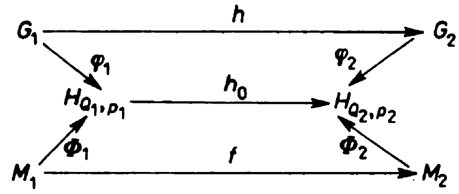
will be called the *holonomy algebra bundle* of (M, G, Q).

COROLLARY 6. *Let the lift space* (M, G, Q) *be of class* C^k, $k \geqslant 0$. *Then the holonomy algebra bundle* $k_Q(M)$ *admits a natural structure of a* C^{k+1} *vector subbundle of* $M \times G$.

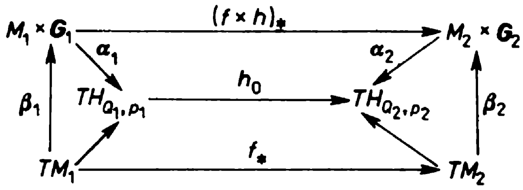
Proof. Let $p \in M$. By Proposition 4, we see that the vector bundle associated with $Q(M, p)$ by the representation Ad of the group $|K_{Q,p}|$ can be identified with $k_Q(M)$. Using Lemma 7 we see that this vector bundle structure is independent of the choice of $p \in M$ and, by Corollary 5, it is of class C^{k+1}, which completes the proof.

Now, let (M_i, G_i, Q_i), $i = 1, 2$, be lift spaces of class C^k, $k \geqslant 0$, $(f, h)\colon (M_1, G_1, Q_1) \to (M_2, G_2, Q_2)$ a lift space morphism, and $p_1 \in M_1$. By Remark 3,

(f, h) induces the quotient map h_0: $H_{Q_1,p} \to H_{Q_2,f(p)}$. By (11) and (iv) of Lemma 1, the diagram



commutes and, by (9), the diagram



commutes (where $p_2 = f(p_1)$, $\varphi_i = \varphi_{Q_i,p_i}$, $\Phi_i = \Phi_{Q_i,p_i}$ and $\alpha_i(q, X) = X\Phi_{Q_i,p_i}(q)$, $\beta_i(v) = (q, Q_i(v))$ for $q \in M_i$, $X \in G_i$, $v \in T_q M_i$ and $i = 1, 2$). By Proposition 4 and Corollary 5, the map $f \times h$ induces a C^{k+1} principal bundle homomorphism $Q_1(M_1, p) \to Q_2(M_2, f(p))$. Thus we obtain the following commutative diagram of principal bundle homomorphisms:

$$\begin{array}{ccccc} Q_1(M_1, p_1) & \to & M_1 \times G_1(M_1 \times H_{Q_1,p_1}) & \to & G_1(H_{Q_1,p_1}) \\ \downarrow & & \downarrow & & \downarrow \\ Q_2(M_2, p_2) & \to & M_2 \times G_2(M_2 \times H_{Q_2,p_2}) & \to & G_2(H_{Q_2,p_2}) \end{array}$$

Similarly, it is easy to see that the map $f \times h$ sends horizontal paths in $M_1 \times G_1$ (resp. in $Q_1(M_1, p_1)$) into horizontal paths in $M_2 \times G_2$ (resp. in $Q_2(M_2, f(p_1))$), the map $(f \times h)_*$ induces a C^{k+1} vector bundle homomorphism $k_{Q_1}(M_1) \to k_{Q_2}(M_2)$, and $(f \times h)_* T^{Q_1} \subset T^{Q_2}$.

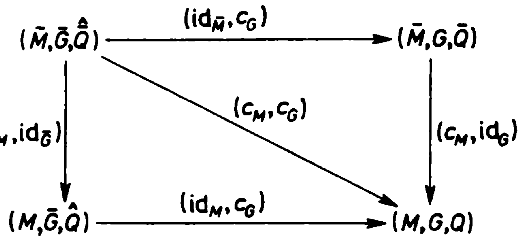
Example 6. Let (M, G, Q) be a lift space of class C^k, $k \geqslant 0$. The universal covering map c_M: $\bar{M} \to M$ induces the lift space $(\bar{M}, G, \bar{Q})$ of class C^k and the lift space morphism (c_M, id_G): $(\bar{M}, G, \bar{Q}) \to (M, G, Q)$ (clearly, $\bar{Q} = Q \circ (c_M)_*$). Let $p \in M$ and $\bar{p} \in c_M^{-1}(p)$. By Remark 1, $|K_{\bar{Q},\bar{p}}| = |K^0_{\bar{Q},\bar{p}}| = |K^0_{Q,p}|$; thus the induced principal bundle homomorphism $\bar{Q}(\bar{M}, \bar{p}) \to Q(M, p)$ (resp. the vector bundle homomorphism $k_{\bar{Q}}(\bar{M}) \to k_Q(M)$) is a covering whose group of deck transformations is $\pi_1(M)$. By Corollary 1, the quotient map $H_{\bar{Q},\bar{p}} \to H_{Q,p}$ is a covering whose group of deck transformations is $\pi_{Q,p}$ and the map $\Phi_{\bar{Q},\bar{p}}$ is the lift of $\Phi_{Q,p} \circ c_M$ with $\Phi_{\bar{Q},\bar{p}}(\bar{p}) = \varphi_{\bar{Q},\bar{p}}(e)$.

Similarly, let $\bar{G}$ be the universal covering group of G. Then the covering projection c_G: $\bar{G} \to G$ induces the lift space $(M, \bar{G}, \hat{Q})$ with the lift space morphism (id_M, c_G): $(M, \bar{G}, \hat{Q}) \to (M, G, Q)$ and the coverings

$$|K^0_{\hat{Q},p}| \to |K^0_{Q,p}|, \quad |K_{\hat{Q},p}| \to |K_{Q,p}|,$$

$$\hat{Q}(M, p) \to Q(M, p), \quad k_{\hat{Q}}(M) \to k_Q(M), \quad H_{\hat{Q},p} \to H_{Q,p}.$$

Since $\hat{Q} = \bar{Q}$, we have the commutative diagram of lift space morphisms



Example 7. In Example 2, let $m = 2$, $p = 0 \in R^n$, and

$$Q(v) = \begin{bmatrix} Q^1(v) & Q^2(v) \\ Q^3(v) & Q^4(v) \end{bmatrix}, \quad v \in R^{2n}.$$

For any piecewise C^1-path γ: $[a, b] \to R^n$, we put

$$\bar{\gamma}(E, t) = \begin{bmatrix} A(\gamma)_t^1 & A(\gamma)_t^2 \\ A(\gamma)_t^3 & A(\gamma)_t^4 \end{bmatrix}, \quad t \in [a, b].$$

(E is the identity of $GL(2, R)_0$). In this case, (5) defines a global cross-section Λ: $R^n \to Q(R^n, p) \subset R^n \times GL(2, R)_0$. Write $\Lambda^G = \mathrm{pr}_G^p \circ \Lambda$ and

$$\Lambda^G(x) = \begin{bmatrix} \Lambda^1(x) & \Lambda^2(x) \\ \Lambda^3(x) & \Lambda^4(x) \end{bmatrix}, \quad x \in R^n.$$

Now, let f: $R^n \to R$ be any continuous function not of class C^1, and g: $R^n \to R$ of class C^1 but not of class C^2. Under the above conditions, setting $Q^1(x, y) = Q^4(x, y) = f(x)$ for $x, y \in R^n$, $Q^2 = 0$ and $Q^3 = g_*$, by (3) we obtain

$$A(\gamma)_t^1 = A(\gamma)_t^4 = \exp \int_a^t f\left(\frac{d}{ds}\gamma_s\right) ds,$$

$$A(\gamma)_t^2 = 0, \quad A(\gamma)_t^3 = A(\gamma)_t^1 g(\gamma_t)$$

for any piecewise C^1-path γ: $[a, b] \to R^n$ and $t \in [a, b]$. The holonomy group at p is given by

$$|K_{Q,p}| = |K_{Q,p}^0| = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} : \alpha \in R_+ \right\},$$

and for the cross-section Λ we have

$$\Lambda^1 = \Lambda^4 = \exp f(x), \quad \Lambda^2 = 0, \quad \Lambda^3 = (\exp f(x)) g(x), \quad x \in R^n.$$

Thus

$$Q(R^n, p) = \bigcup_{x \in R^n} \{x\} \times \Lambda^G(x) |K_{Q,p}| = \bigcup_{x \in R^n} \{x\} \times \left\{ \begin{bmatrix} \alpha & 0 \\ \alpha g(x) & \alpha \end{bmatrix} : \alpha \in R_+ \right\}.$$

Therefore, the cross-section Λ is continuous but not of class C^1, and $Q(R^n, p) \subset R^n \times \mathrm{GL}(2, R)_0$ is a principal subbundle of class C^1, but not of class C^2.

In this case, $H_{Q,p} = \mathrm{SL}(2, R)$ and the map $\Phi_{Q,p}$: $R^n \to \mathrm{SL}(2, R)$ is given by

$$\Phi_{Q,p}(x) = \begin{bmatrix} 1 & 0 \\ g(x) & 1 \end{bmatrix}, \quad x \in R^n.$$

3. Flat and locally flat lift spaces. As an immediate consequence of Remark 2 and Proposition 3 we obtain

COROLLARY 7. *A lift space (M, G, Q) of class C^k, $k \geqslant 0$, is flat if and only if there exists a map Φ: $M \to G$ of class C^{k+1} such that*

$$\Phi_* = Q(\Phi \circ \pi_M).$$

Given two such maps Φ and Φ', there is a unique $g \in G$ such that $\Phi' = \Phi \cdot g$.

Using Remark 2, Corollary 7, and some facts from Example 6, we obtain the following

PROPOSITION 5. *Let (M, G, Q) be a lift space of class C^k, $k \geqslant 0$. Then the following properties are equivalent*:

(i) *(M, G, Q) is locally flat.*

(ii) *The holonomy algebra of (M, G, Q) at some point of M vanishes.*

(iii) *The induced lift space $(\bar{M}, G, \bar{Q})$ is flat.*

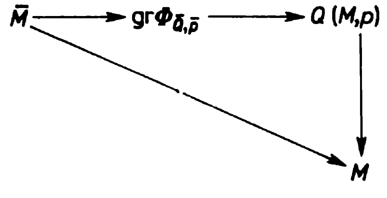
(iv) *There exists a map Φ: $\bar{M} \to G$ of class C^{k+1} such that*

$$\Phi_*(v) = Q\big((c_M)_*(v)\big)\Phi(q) \quad \textit{for } q = \bar{M},\ v \in T_q\bar{M}.$$

(v) *There exist an open cover $\{U_l\}$ of M and a family of C^{k+1}-maps Φ^l: $U_l \to G$ such that*

$$\Phi^l_*(v) = Q(v)\Phi^l(q) \quad \textit{for every } l,\ q \in U_l \textit{ and } v \in T_q M.$$

Remark 6. Let (M, G, Q) be a lift space of class C^k and let $p \in M$. If it is flat, then by Proposition 4 we have $Q(M, p) = \mathrm{gr}\,\Phi_{Q,p}$. If it is locally flat, then by Proposition 5 we obtain the commutative diagram of C^{k+1}-coverings



where $\bar{p} \in c_M^{-1}(p)$.

Remark 7. If the lift space (M, G, Q) is linear and of class C^2, then by the theorem of Ambrose and Singer (see [3], p. 89, and [4], p. 278) the holonomy algebra $k_{Q,p}$ at $p \in M$ is equal to the subspace of G, spanned by all elements of the form $\Omega_{(q,g)}(v + Q(v)g,\ w + Q(w)g)$, where $(q, g) \in Q(M, p)$,

$v, w \in T_q M$, and Ω is the curvature form of the corresponding connection $\bar{T}^Q$ (cf. Remark 4). It can be proved ([1], Proposition 3) that the holonomy algebra $k_{Q,p}$ at $p \in M$ of any lift space (M, G, Q) of class C^2 (possibly nonlinear) is equal to the subspace of G spanned by all elements of the form $g^{-1} K(X, Y)_q g$, where $(q, g) \in Q(M, p)$ and X, Y are arbitrary C^∞ vector fields on M and

$$K(X, Y) = [Q(X), Q(Y)] + XQ(Y) - YQ(X) - Q([X, Y]).$$

Example 8 (case of Abelian Lie group). Let (M, G, Q) be a lift space of class C^k, $k \geqslant 0$, with the Abelian group G. Let $p \in M$ and $\bar{p} \in c_M^{-1}(p)$. In this case $\bar{G} = G = R^n$ for $n = \dim G$, and $|K_{\bar{Q},\bar{p}}| = |K^0_{\hat{Q},\bar{p}}| = V$ is a subspace of R^n. Let $R^n = B \oplus V$ for some subspace B of R^n and let $Q = Q^B + Q^V$ be the corresponding decomposition of Q. By (2) the lift space $(\bar{M}, \bar{G}, Q^B)$ is flat, and by Corollary 7 there exists a unique C^{k+1}-map Φ: $\bar{M} \to R^n$ such that $\Phi(\bar{p}) = 0$ and $\Phi_* = Q^B$. Hence the induced covering bundle $\hat{\bar{Q}}(\bar{M}, \bar{p})$ of $Q(M, p)$ is given by

$$\hat{\bar{Q}}(\bar{M}, \bar{p}) = \bigcup_{x \in \bar{M}} \{x\} \times \{\Phi(x) + V\} \subset \bar{M} \times R^n.$$

4. Lift spaces along vector bundle morphism. Now we shall show that lift spaces may be regarded as certain generalizations of connections in principal fibre bundles of homogeneous spaces (not to be confused with homogeneous lift spaces!). Let M be a C^∞-manifold, G a connected Lie group, and F: $M \times G \to TM$ a C^∞ vector bundle morphism. A lift space (M, G, Q) of class C^k, $k \geqslant 0$, is called a C^k *lift space along* F if

$$F(q, Q(v)) = v \quad \text{for } q \in M \text{ and } v \in T_q M \cap \operatorname{im} F. \tag{12}$$

By paracompactness of M, if F is of constant rank, then there exists a linear C^∞ lift space along F.

Let $G \times M \to M$ be a transitive action of a connected Lie group G on M. For any $p \in M$, the determined principal bundle π_p: $G \ni g \mapsto gp \in M$ will be denoted by $G(M, \pi_p)$, and the isotropy group of p (i.e. the structure group of $G(M, \pi_p)$) by G_p. Define the map F_M: $M \times G \to TM$ by $F_M(q, X) = Xq$ for $q \in M$ and $X \in G$.

PROPOSITION 6. *Let (M, G, Q) be a C^k lift space along F_M, $k \geqslant 0$, and let $p \in M$. Then:*

(i) *For any piecewise C^1-path γ: $[a, b] \to M$ starting from $q \in M$, we have $\bar{\gamma}(e, t) q = \gamma_t$, $t \in [a, b]$.*

(ii) *The holonomy group $|K_{Q,p}|$ is a Lie subgroup of G_p.*

(iii) *The projection pr^E_G: $Q(M, p) \to G$ is a C^{k+1} principal bundle embedding $Q(M, p) \to G(M, \pi_p)$.*

(iv) *The holonomy algebra bundle $k_Q(M)$ is a C^{k+1} vector subbundle of* $\ker F_M$.

(v) *The map $\Phi_{Q,p}$: $M \to H_{Q,p}$ is a C^{k+1} cross-section of the bundle* pr: $H_{Q,p} \to G/G_p = M$.

Proof. (i) By (12) the paths $\bar{\gamma}(e, t)q$ and γ_t, $t \in [a, b]$, are the solutions of the following ordinary differential equation on M:

$$\dot{\xi}_t = F_M(\xi_t, Q(\dot{\xi}_t)), \ t \in [a, b], \qquad \xi_a = q.$$

Thus, by Lemma 8, they are equal.

(ii) follows from (i), the definition of the holonomy group, and Proposition 1.

(iii) and (iv) follow from (i) and (ii), and Corollaries 5 and 6.

(v) By (i) of Proposition 3, $\Phi_{Q,p}$ is of class C^{k+1}. Let $q \in M$ and let γ: $[0, 1] \to M$ be a piecewise C^1-path joining p to q. Thus, by Corollary 4 and (i) above, we have $\mathrm{pr}\,\Phi_{Q,p}(q) = \mathrm{pr}\,\Phi_{Q,p}(\gamma_1) = \bar{\gamma}(e, 1) = q$. This completes the proof.

Remark 8. Under the above conditions, if (M, G, Q) is linear, then the associated connection $\bar{T}^Q$ on $M \times G$ (cf. Remark 4) induces a C^k-connection $\mathrm{pr}_G \bar{T}^Q$ on $G(M, \pi_p)$ by the projection pr_G: $M \times G \to G$. In this case, the groups $|K_{Q,p}|$, $|K^0_{Q,p}|$, and the subbundle $\mathrm{pr}_G(Q(M, p)) \subset G(M, \pi_p)$ are its holonomy group, restricted holonomy group, and the holonomy bundle at the point $e \in G$, respectively.

Conversely, given any connection T of class C^k, $k \geqslant 0$, on the bundle $G(M, \pi_p)$, the map

(13) $\qquad Q(v) = X^v, \quad$ where $X^v g \in T_g$ for $g \in G$ and $v \in T_{gp}M$,

defines a linear C^k lift space (M, G, Q) along F_M such that $\mathrm{pr}_G \bar{T}^Q = T$.

Let the Lie group G_i act on M_i transitively and let (M_i, G_i, Q_i) be a C^k lift space along F_{M_i}, $k \geqslant 0$, for $i = 1, 2$. Let (f, h): $(M_1, G_1, Q_1) \to (M_2, G_2, Q_2)$ be a lift space morphism. Then for any $p \in M_1$ the pair $(h, h|_{G_1})_p$ is a principal bundle homomorphism $G_1(M_1, \pi_p) \to G_2(M_2, \pi_{f(p)})$ for which the induced map $M_1 \to M_2$ is equal to f (hence f is analytic). In fact, by (i) of Proposition 6 and (iv) of Lemma 1, we obtain

$$f(gp) = f(\gamma_1) = \overline{f \circ \gamma}(e, 1) f(p) = h(\bar{\gamma}(e, 1)) f(p) = h(g) f(p)$$

for any $g \in G$ and a piecewise C^1-path γ: $[0, 1] \to M$ joining p to gp.

If (M_i, G_i, Q_i), $i = 1, 2$, are linear, then we obtain the commutative diagram of connection homomorphisms

$$\begin{array}{ccc} \bar{T}^{Q_1} & \xrightarrow{(f \times h)_*} & \bar{T}^{Q_2} \\ \downarrow{\scriptstyle \mathrm{pr}_{G_1}} & & \downarrow{\scriptstyle \mathrm{pr}_{G_2}} \\ \mathrm{pr}_{G_1} \bar{T}^{Q_1} & \xrightarrow{h_*} & \mathrm{pr}_{G_2} \bar{T}^{Q_2} \end{array}$$

where $\mathrm{pr}_{G_i} \bar{T}^{Q_i}$ is the induced connection on $G_i(M_i, \pi_{p_i})$ for $p_1 = p$ and $p_2 = f(p)$.

REFERENCES

[1] A. Całka, *The curvature of lift spaces* (to appear).
[2] C. Ehresmann, *Les connexions infinitésimales dans un espace fibre différentiable*, p. 29-55 in: Colloque de topologie, Bruxelles 1950.
[3] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol I, New York 1963.
[4] R. Sulanke und P. Wintgen, *Differentialgeometrie und Faserbündel*, Berlin 1972.
[5] H. Yamabe, *On an arcwise connected subgroup of a Lie group*, Osaka Mathematical Journal 2 (1950), p. 13-14.

MATHEMATICAL INSTITUTE
WROCŁAW UNIVERSITY
WROCŁAW

Reçu par la Rédaction le 14. 11. 1980;
en version modifiée le 9. 10. 1981