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DIFFERENTIAL EQUATIONS WITH INFINITE NUMBER OF LIMIT CYCLES

Consider a system of differential equations

$$\dot{x}_1 = f_1, \quad \dot{x}_2 = f_2 \tag{1}$$

with f_i sufficiently regular, say of class C^2, functions defined on the (phase-)plane $\boldsymbol{R}^2$. Let $x_i(t)$, $i = 1, 2$, denote a periodic solution of (1), and K — the corresponding orbit which is a closed curve in R^2.

K is called a *limit cycle* if no closed orbit other than K lies in its sufficiently small neighbourhood. If all orbits near K wind toward K for $t \to +\infty$ (resp., $t \to -\infty$), the limit cycle K is called *stable* (resp., *unstable*). In the case where orbits lying on one side of K wind toward K for $t \to +\infty$ and those lying on the other side wind toward K for $t \to -\infty$, K is called a *semi-stable* limit cycle. For instance, the well-known Van der Pol equation

$$\dot{x} = y, \quad \dot{y} = (1-x^2)y-x$$

has a stable limit cycle which is seen in the phase diagram (see Fig. 1) obtained by modelling this equation on an analogue computer.

Recently, D'Heedene [1] presented a full analysis of the equation of the form

$$\ddot{x} + \mu \sin \dot{x} + x = 0. \tag{2}$$

It was shown that (2) has limit cycles in infinite number. Moreover, for each of them it was possible to provide a characterization of stability type.

The careful examination shows that the methods and results in [1] may be generalized to the case of the class of equations

$$\ddot{x} + \mu h(\dot{x}) + x = 0 \tag{3}$$

with h satisfying the following conditions:

$$\begin{array}{rl} \text{(i)} & h \in C^1(R), \\ \text{(ii)} & \text{periodic (with period } 2T), \\ \text{(iii)} & h(-s) = -h(s), \\ \text{(iv)} & h(s) > 0 \text{ for } 0 < s < T. \end{array} \tag{4}$$

Equation (3) is equivalent to the system

$$\dot{x} = y, \quad \dot{y} = -x - \mu h(y),$$

the analysis of which, due to the symmetry of the system, may be reduced to the half-plane $y \geqslant 0$.

Denote by M a positive constant such that $|h| \leqslant M$ and $|\dot{h}| \leqslant M$. Then the following theorem is true:

THEOREM I. *For any μ, $|\mu| < 2/M$, equation* (3) *with h satisfying* (4) *has infinitely many limit cycles displaced on the plane according to the following rule: for any $n = 1, 2, \ldots$ there is exactly one limit cycle among*

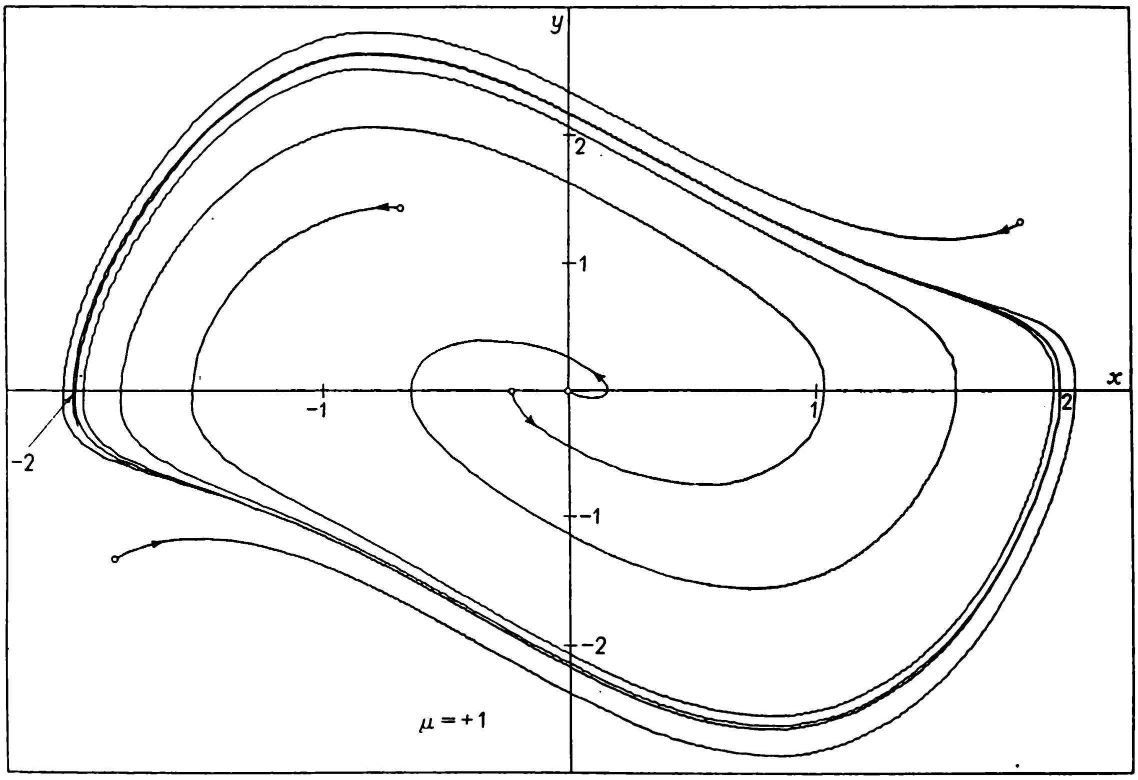


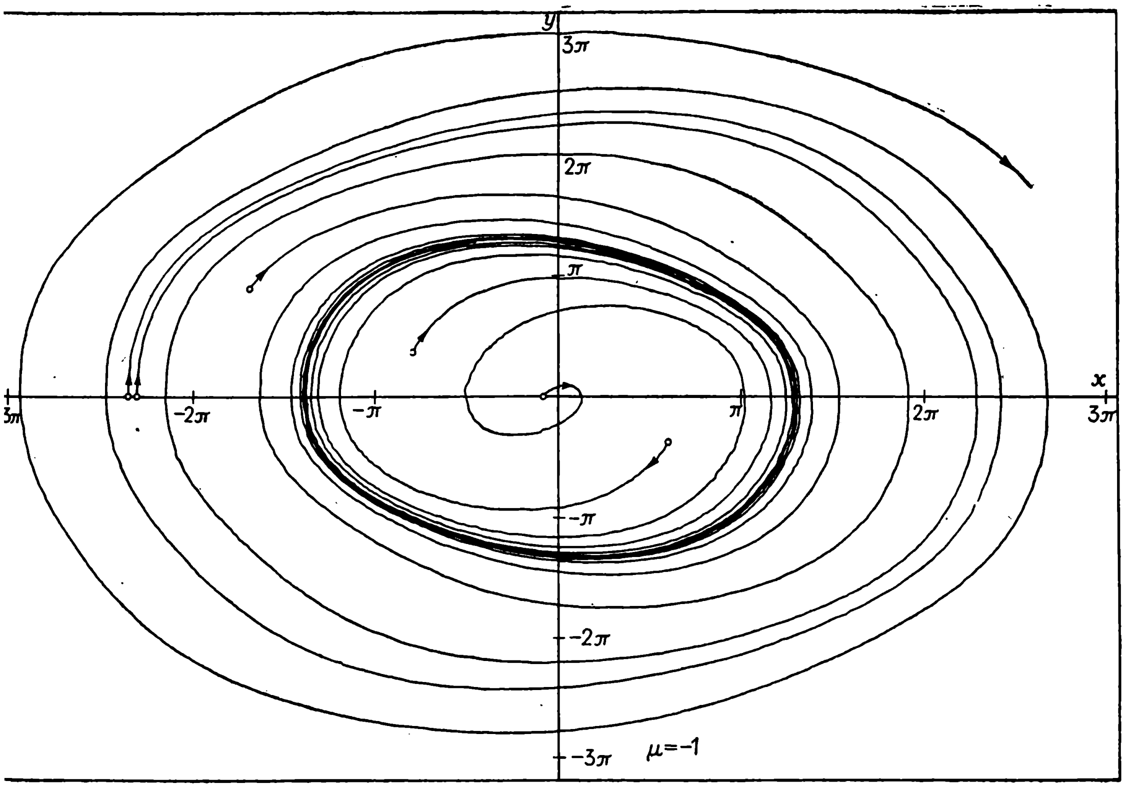
Fig. 1

all orbits passing through the points $(\mu h(y), y)$, $y \epsilon (nT, (n+1)T)$, of the plane. Moreover, for $\mu > 0$ the limit cycles which correspond to n even are stable while those corresponding to n odd are unstable. For $\mu < 0$ the opposite holds.

Condition $|\mu| < 2/M$ in Theorem I may be relaxed.

THEOREM II. *Conclusions of Theorem* I *hold true for those n and μ which satisfy the inequality*

$$n > \frac{2}{T} M|\mu| [M|\mu| + 1 + (M^2\mu^2 + 2M|\mu|)^{1/2}].$$


y
3π
2π
π
x
-3π
-2π
$-\pi$
π
2π
3π
$-\pi$
-2π
-3π
$\mu=-1$

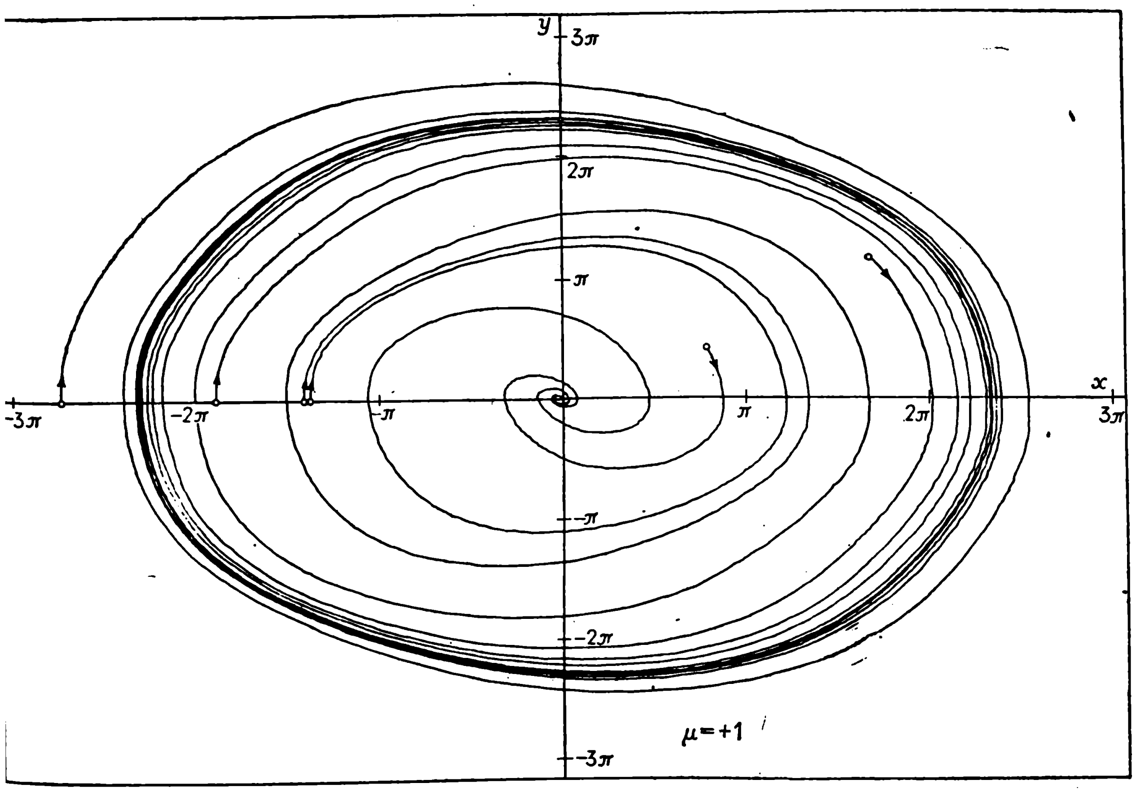

y
3π
2π
π
x
-3π
-2π
$-\pi$
π
2π
3π
$-\pi$
-2π
-3π
$\mu=+1$

Fig. 2

The proofs of Theorems I and II are parallel to those presented in [1] with minor modifications only. Therefore we omit them.

It is clear that equation (2) is a special case of (3). There we have $T = \pi$, and $M = 1$. In Fig. 2 there are given two diagrams for equation (2) with $\mu = +1$ and $\mu = -1$. The diagrams were obtained by modelling the equation on an analogue computer (like those in Fig. 1). For technical reasons the range of variable y was restricted to the interval $(0, 3\pi)$. We can observe that, in agreement with the theorems, the following holds:

(a) no limit cycle passes through the segment $x = 0$, $0 \leqslant y \leqslant \pi$;

(b) through the segment $x = 0$, $\pi \leqslant y \leqslant 2\pi$ there passes exactly one limit cycle which is unstable for $\mu = 1$ and stable for $\mu = -1$;

(c) through the segment $x = 0$, $2\pi \leqslant y \leqslant 3\pi$ there passes exactly one limit cycle which is stable for $\mu = 1$ and unstable for $\mu = -1$.

Initial points for orbits in our diagrams were arbitrarily chosen. The behaviour of orbits exhibits clearly the existence of a limit cycle: in the case of a stable (unstable) limit cycle the orbits wind toward (resp., outward) it.

There is no method as far to answer the question concerning the existence of limit cycles in a way which would be practically useful for all equations. Thus the numerical computations and the application of analogue computers seem to be the only general means for analyzing such problems.

Reference

[1] R. M. D'Heedene, *For all real μ, $\ddot{x} + \mu \sin \dot{x} + x = 0$ has an infinite number of limit cycles*, Journal of Differential Equations 5 (1969), p. 564-571.

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RÓWNANIA RÓŻNICZKOWE Z NIESKOŃCZONĄ LICZBĄ CYKLI GRANICZNYCH

STRESZCZENIE

Praca przedstawia uogólnienie wyniku D'Heedene'a, mówiącego o istnieniu nieskończenie wielu cykli granicznych dla równania (2) na ogólniejszą klasę równań

(3). W pracy nie podano dowodu tego faktu, gdyż jest on jedynie rachunkową modyfikacją oryginalnego dowodu D'Heedene'a.

Ponadto przedstawiono wykresy otrzymane przez zaprogramowanie na maszynie analogowej równania (2) rozpatrywanego przez D'Heedene'a i równania Van der Pola. Jest to w nie których przypadkach jedyna metoda pozwalająca rozstrzygnąć istnienie cyklu granicznego.