On Ruscheweyh derivatives

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Abstract. Let $K_n$ be the classes of regular functions $f(z) = z + a_2z^2 + \ldots$, defined in the unit disc $E$ and satisfying

$$\Re \left( \frac{D^{n+1}f(z)}{D^nf(z)} \right) > \frac{1}{2}, \quad z \in E, \quad n = 0, 1, 2, \ldots,$$

where

$$D^nf(z) = f(z) \ast \frac{z}{(1-z)^{n+1}},$$

and $(\ast)$ is the Hadamard convolution.

(i) The author determines certain real values $\alpha$ and $\beta$ such that whenever

$$\Re \left( \frac{D^{n+1}f(z)}{D^nf(z)} - \frac{1}{2} \right)^\alpha \left( \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{1}{2} \right)^\beta > 0,$$

$z \in E, \ n = 0, 1, 2, \ldots$, then $f \in K_n$.

(ii) Let $h_n(z) = \frac{D^n f(z)}{D^n g(z)}$. The author determines the set of real values $\alpha$ and $\beta$ such that whenever $f$ and $g$ belong to $K_n$

$$\Re h_n^\alpha(z) h_{n+1}^\beta(z) > 0$$

holds for $z \in E$ and $n = 0, 1, 2, \ldots$.

1. Introduction. Let $A$ denote the class of functions $f(z)$ regular in the unit disc $E = \{z: |z| < 1\}$ and normalized by $f(0) = 0, \ f'(0) = 1$.

By $\{K_n\}$ we mean the subclasses of $A$ satisfying for every $f \in K_n$ the inequality

$$\Re \frac{(z^n f(z))^{(n+1)}}{(z^{n-1} f(z))^{(n)}} > \frac{(n+2)}{2},$$

where $n \in \mathbb{N}_0, \ \mathbb{N}_0 = 0, 1, 2, \ldots$, and $z \in E$.

* The author acknowledges partial summer support from the Faculty Research Committee at Bowling Green State University.
S. Ruscheweyh [6] introduced the classes $K_n$ and showed the basic property

$$K_{n+1} \subset K_n, \quad n \in \mathbb{N}_0.$$ 

Thus elements of $K_n$ are univalent and starlike of order $\frac{1}{2}$ ($K_0 \equiv S_{1/2}^*$).

Let

$$D^n f(z) = z(z^{n-1} f(z))^{(n)}/n!, \quad n \in \mathbb{N}_0.$$ 

We shall refer to $D^n f$ as the $n$th order Ruscheweyh derivative of $f$. Note that $D^0 f = f$, $Df(z) = zf'(z)$.

Ruscheweyh cleverly observed that

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z),$$ 

where the operation $(*)$ is the usual Hadamard product of series (i.e., if $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $f(z) = \sum_{n=0}^{\infty} b_n z^n$, then $f * g = \sum_{n=0}^{\infty} a_n b_n z^n$). This lead him to an equivalent but more practical definition for $K_n$, namely $f \in K_n$ if and only if $f \in A$ and

$$\text{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > \frac{1}{2}, \quad n \in \mathbb{N}_0,$$ 

is satisfied for $z \in E$.

Now we introduce the following classes:

**DEFINITION.** We say that $f \in S_n(a, \beta)$, $n \in \mathbb{N}_0$, if $f \in A$ and

$$P_n(f(z); a, \beta) = \left( \frac{D^{n+1} f(z)}{D^n f(z)} - \frac{1}{2} \right)^a \left( \frac{D^{n+2} f(z)}{D^{n+1} f(z)} - \frac{1}{2} \right)^\beta,$$ 

where $a, \beta$ are real numbers, then

$$\text{Re} P_n(f(z); a, \beta) > 0, \quad n \in \mathbb{N}_0, \quad z \in E.$$ 

The powers appearing in (5) are meant as principal values. For every $n \in \mathbb{N}_0$, $S_n(a, \beta)$ contains many interesting classes of univalent functions; $S_n(1, 0) = K_n$, $S_n(0, 1) = K_{n+1}$, and $S_n(a, 0)$ is contained in class of strongly starlike [1] when $|a| \geq 1$.

In section 3 we will determine a set of values of real numbers $a$ and $\beta$ for which $S_n(a, \beta) \subset K_n$, $n \in \mathbb{N}_0$. Similar problem was treated in [5].

Next, let

$$h_n(z) = \frac{D^n f(z)}{D^n g(z)}, \quad f, g \in A, \quad n \in \mathbb{N}_0.$$ 

In section 4 we will determine the set of the non-negative real numbers $a$ and $\beta$ such that

$$\text{Re} h_n^a(z) h_n^{\beta+1}(z) > 0, \quad n \in \mathbb{N}_0, \quad z \in E,$$ 

whenever $f, g \in K_n$. Special cases of this section reduces to results in [2], Theorems 1 and 2.
2. Preliminaries. We need the following results.

**Lemma 1.** Let \( w \in A \) with \( w(z) \neq 0, z \neq 0 \). If \( z_0 = r_0 e^{i\theta_0}, \ 0 < r_0 < 1 \) and \( \max_{|z| \leq r_0} |w(z)| = |w(z_0)|, \) then
\[
    z_0 w'(z_0) = mw(z_0), \quad m \geq 1.
\]

Lemma 1 may be found in [3].

**Lemma 2.** Let \( p(z) \) be regular in \( E \) with \( p(0) = 1 \) and \( \text{Re} \ p(z) > 1/2 \) in \( E \). Then
\[
    \left| p(z) - \frac{1}{1-r^2} \right| < \frac{r}{1-r^2}.
\]

Lemma 2 can be deduced from the geometrical properties of \( q(z) \), where \( q(z) = 2p(z) - 1 \).

**Lemma 3.** If \( f \in K_n, \ n \in N_0, \) then
\[
    \left| \arg \frac{D^k f(z)}{z} \right| \leq (k+1) \sin^{-1} r, \quad 0 \leq k \leq n+1.
\]

**Proof.** Since \( f \in K_n \) implies \( f \in K_j, \ 0 \leq j \leq n \), then
\[
    \frac{D^{j+1} f(z)}{D^j f(z)} = p_j(z), \quad \text{with} \ \text{Re} p_j(z) > 1/2, \ z \in E, \ 0 \leq j \leq n.
\]

Lemma 2 yields
\[
    \left| \arg \frac{D^{j+1} f(z)}{D^j f(z)} \right| \leq \sin^{-1} r, \quad 0 \leq j \leq n, \ |z| = r < 1.
\]

Moreover, \( f \in K_n \) \( \Rightarrow f \in S^* \) \( \iff \int \left( \frac{f(z)}{z} \right)^2 dz \in K \), and for \( F \in K \) we have [4] that \( |\arg F'(z)| \leq 2 \sin^{-1} r, \ |z| = r < 1 \). Consequently, \( f \in K_n \) implies
\[
    \left| \arg \frac{f(z)}{z} \right| \leq \sin^{-1} r.
\]

Applying (9) and (10) to the identity
\[
    \frac{D^k f(z)}{z} = \frac{f(z)}{z} \prod_{j=0}^{k-1} \frac{D^{j+1} f(z)}{D^j f(z)},
\]

\( 1 \leq k \leq n+1, \) we arrive at (8).

3. The classes \( S_n(a, \beta) \). Let
\[
    G_1 = \{(a, \beta)| \ (a+2\beta \leq 4k+3) \cap (a+\beta \geq 4k+1), \ \beta \geq 0, \ k \in I\},
\]
\[
    G_2 = \{(a, \beta)| \ (a+\beta \leq 4k+3) \cap (a+2\beta \geq 4k+1), \ \beta \leq 0, \ k \in I\},
\]
\[
    G_3 = \{(a, 0)| \ |a| \geq 1\} \cup \{(0, \beta)| \ |eta| \geq 1\},
\]
\[
    G = G_1 \cup G_2 \cup G_3,
\]
where \( I \) is the set of integers and \( \alpha, \beta \) being real numbers. In this section we show that for \((\alpha, \beta) \in G\) and \( f \in S_n(\alpha, \beta) \), then \( f \in K_n \). The region \( G \) is independent of \( n \).

We shall use the technique of Miller [5] to prove the following

**THEOREM 1.** \( S_n(\alpha, \beta) \subset K_n \) if \((\alpha, \beta) \in G, n \in N_o\).

**Proof.** The case where \((\alpha, \beta) \in G_o\) is trivial. Suppose \( f \in S_n(\alpha, \beta) \) and

\[
\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1}{1 - w(z)},
\]

where \( z \in E \). Then \( w(z) \) is regular in \( E \) with \( w(0) = 0, w(z) \neq \pm 1 \). To complete the proof we need to show that \( \text{Re} 1/|1 - w(z)| > 1/2 \), \( z \in E \) and \((\alpha, \beta) \in G\). To this end, it is sufficient to show \( |w(z)| < 1, z \in E \) and \((\alpha, \beta) \in G\).

Differentiating (12) and using an easy to verify identity

\[
z (D^k f(z))' = (k + 1) D^{k+1} f(z) - k D^k f(z), \quad k \in N_o,
\]

one gets

\[
z (D^{n+1} f(z))' = \frac{z (D^n f(z))'}{1 - w(z)} + \frac{zw'(z)}{(1 - w(z))^2} D^n f(z),
\]

\[
(n + 2) D^{n+2} f(z) - (n + 1) D^{n+1} f(z)
\]

\[
= \frac{(n + 1) D^{n+1} f(z) - n D^n f(z)}{1 - w(z)} + \frac{zw'(z) D^n f(z)}{(1 - w(z))^2}.
\]

Thus

\[
\frac{D^{n+2} f(z)}{D^{n+1} f(z)} = \frac{1}{n + 2} \left[ 1 + \frac{n + 1}{1 - w(z)} + \frac{zw'(z)}{1 - w(z)} \right].
\]

Substituting (12) and (13) in (5) we have

\[
P_n \{ f(z); \alpha, \beta \} = C \left( \frac{1 + w(z)}{1 - w(z)} \right)^\alpha \left( 1 + (n + 1) \frac{1 + w(z)}{1 - w(z)} + \frac{2zw'(z)}{1 - w(z)} \right)^\beta,
\]

where \( C = 2^{-\alpha - \beta} (n + 2)^{-\beta} > 0 \).

Now suppose to the contrary that there is \( z_0 \in E \) such that \( \max |w(z)| = |w(z_0)| = 1, w(z_0) \neq \pm 1 \). Then Lemma 1 shows

\[
z_0 w'(z_0)/(1 - w(z_0)) = m w(z_0)/(1 - w(z_0)).
\]

Let \( w(z_0) = e^{i\theta_0} \). Then

\[
\frac{1 + w(z_0)}{1 - w(z_0)} = \frac{\sin \theta_0}{2(1 - \cos \theta_0)} i,
\]

\[
\frac{w(z_0)}{1 - w(z_0)} = \frac{\cos \theta_0 - 1}{2(1 - \cos \theta_0)} + \frac{\sin \theta_0}{2(1 - \cos \theta_0)} i = \frac{1}{2} [-1 + \lambda i].
\]
Consequently (14) becomes

\[(15) \quad P_n(f(z_0); \alpha, \beta) = C(\lambda i)^{m/(1-m + (1 + m + n) \lambda i) \delta}
\]
\[= C |\lambda|^\delta ((1-m)^2 + (1+m+n)^2)^{\beta/2} \cos(\theta_1 + \beta \theta_2),\]

where \(\theta_1 = \arg \lambda i, \theta_2 = \arg (1-m + (1 + m + n) \lambda i)\).

Case 1. \(\lambda > 0\), \(\theta_1 = \pi/2\), and since \(1-m \leq 0\), \(\pi/2 \leq \theta_2 \leq \pi\).

(i) If \((\alpha, \beta) \in G_1\), then

\[(4k+1)\pi/2 \leq (\alpha + \beta)\pi/2 \leq \alpha \theta_1 + \beta \theta_2 \leq (\alpha + 2\beta)\pi/2 \leq (4k + 3)\pi/2.\]

Hence \(\cos(\alpha \theta_1 + \beta \theta_2) \leq 0\). This shows that \(\text{Re}(f(z_0); \alpha, \beta) \leq 0\) which contradicts \(f \in S_n(\alpha, \beta)\).

(ii) Similarly if \((\alpha, \beta) \in G_2\), then

\[(4k+1)\pi/2 \leq (\alpha + 2\beta)\pi/2 \leq \alpha \theta_1 + \beta \theta_2 \leq (\alpha + \beta)\pi/2 \leq (4k + 3)\pi/2\]

which leads to same contradiction.

Case 2. \(\lambda < 0\). Let \(\theta_3 = \arg \lambda, \theta_4 = \arg (1-m + (1 + m + n) \lambda i)\). Then

\(\theta_3 = -\theta_1 = -\pi/2, \theta_4 = -\theta_2, \cos(\alpha \theta_3 + \beta \theta_4) = \cos(\alpha \theta_1 + \beta \theta_2) \leq 0\), contradiction.

This completes the proof of Theorem 1.

Remark 1. Since \(S_n(0, 1) = K_n+1\), Theorem 1 shows the basic inclusion relationship of Ruscheweyh \(K_n+1 \subset K_n\). Also \(S_n(\alpha, \beta) \subset S_n(1, 0)\), \((\alpha, \beta) \in G\). We will generalize this latter relation in the next theorem. The set \(G\) is given by (11).

**Theorem 2.** \(S_n(\alpha, \beta) \subset S_n((\alpha - 1) t + 1, \beta t), 0 \leq t \leq 1 \) and \((\alpha, \beta) \in G\).

**Proof.** Let \(f \in S_n(\alpha, \beta)\), and

\[(16) \quad \left( \frac{D^{n+1}f(z)}{D^nf(z)} - \frac{1}{2} \right) \left( \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{1}{2} \right) = q_n(z).\]

Then \(\text{Re} q_n(z) > 0, z \in E, (\alpha, \beta) \in G\). Also by Theorem 1,

\[(17) \quad \frac{D^{n+1}f(z)}{D^nf(z)} - \frac{1}{2} = p_n(z),\]

where \(\text{Re} q_n(z) > 0\) for \(z \in E, n \in N_0\). It follows from (16) and (17) that

\[\left( \frac{D^{n+1}f(z)}{D^nf(z)} - \frac{1}{2} \right)^{(n-1)t+1} \left( \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{1}{2} \right)^{\beta t} = (p_n(z))^{1-t} |q_n(z)|^t = p(z).\]

Here \(p(0) = 1\) and

\[|\arg p(z)| \leq (1-t)|\arg p_n(z)| + t|\arg q_n(z)| \leq \pi/2,\]

which shows \(\text{Re} p(z) > 0\). This completes the proof of Theorem 2.
4. Ratios of Ruscheweyh derivatives. In [2] Burdick and Merkes obtained sharp bounds on $\alpha > 0$ and $\beta > 0$ such that

$$\text{Re} \left( \frac{f(z)}{g(z)} \right)^\alpha > 0 \quad \text{and} \quad \text{Re} \left( \frac{f'(z)}{g'(z)} \right)^\beta > 0$$

for $z \in E$ and $f, g$ varies in the classes $K$ and $S^*$ (starlike). In this section certain generalizations and extensions of these results which involve the Ruscheweyh derivatives will be obtained.

**Theorem 3.** Let

$$h_n(z) = \frac{D^n f(z)}{D^n g(z)}, \quad n \in N_0. \quad (18)$$

If $f$ and $g$ belong to $K_n$, then

$$\text{Re} \{h_n^\alpha(z) h_{n+1}^\beta(\bar{z})\} > 0, \quad (19)$$

for $z \in E$ and $\alpha \geq 0$, $\beta \geq 0$ satisfying

$$2(n+1)\alpha + 2(n+2)\beta = 1. \quad (20)$$

Here $h_n^\alpha(0) = h_{n+1}^\beta(0) = 1$. The result is sharp.

**Proof.** Using Lemma 3 and (15) we have, when $\alpha$ and $\beta$ satisfying (20),

$$|\arg h_n^\alpha(z) h_{n+1}^\beta(\bar{z})| \leq \alpha |\arg D^n f(z) - \arg D^n g(z)| +$$

$$+ \beta |\arg D^{n+1} f(z) - \arg D^{n+1} g(z)|$$

$$= \alpha \left| \arg \frac{D^n f(z)}{z} - \arg \frac{D^n g(z)}{z} \right| +$$

$$+ \beta \left| \arg \frac{D^{n+1} f(z)}{z} - \arg \frac{D^{n+1} g(z)}{z} \right|$$

$$\leq 2\alpha(n+1)\sin^{-1}r + 2\beta(n+2)\sin^{-1}r$$

$$= \sin^{-1}r < \pi/2.$$

From this inequality follows (19).

To show sharpness of results, let

$$f(z) = \frac{z}{1 - z}, \quad g(z) = \frac{z}{1 + e^{it}z},$$

$-\pi < t \leq \pi$. Using (3) we easily compute

$$D^k f(z) = \frac{z}{(1 - z)^{k+1}} * \frac{z}{1 - z} = \frac{z}{(1 - z)^{k+1}}, \quad D^k g(z) = \frac{z}{(1 + e^{it}z)^{k+1}},$$

and hence

$$h_n^\alpha(z) h_{n+1}^\beta(\bar{z}) = \left( \frac{1 + ze^{it}}{1 - z} \right)^{(n+1)\alpha + (n+2)\beta}.$$
Now since $\frac{1+ze^{it}}{1-z}$ maps the unit disc onto the half plane bounded by the line through the origin with angle of inclination $= \frac{t+\pi}{2}$, any choice of $\alpha \geq 0$, $\beta \geq 0$, satisfying $\alpha(n+1)+\beta(n+2) > \frac{1}{2}$, there exists a choice of $t$, $-\pi < t < \pi$ for which

$$\text{Re}\{h_n^\alpha(z)h_{n+1}^\beta(z)\} < 0,$$

for some $z \in E$. Thus (20) cannot be improved.

**Corollary.** If $f$ and $g$ are convex in $E$, then

$$\text{Re}\left(\frac{f(z)}{g(z)}\right)^\alpha \left(\frac{f'(z)}{g'(z)}\right)^\beta > 0$$

for $\alpha \geq 0$, $\beta \geq 0$ satisfying the relation

$$2\alpha + 4\beta = 1.$$

**Proof.** Since $K \subset \mathbb{S}_{1/2}$, the Corollary follows from Theorem 3 when $n = 0$.

**Remark 2.** For $\alpha = 0$, $\beta > 0$, and $\beta = 0$, $\alpha > 0$, our Corollary reduces to [2], Theorem 1.

Since

$$\arg \frac{f'(z)}{g'(z)} = \arg \frac{zf'(z)}{f(z)} - \arg \frac{zg'(z)}{g(z)} + \arg \frac{f(z)}{g(z)},$$

then we can easily show the following theorem.

**Theorem 4.** If $f$ and $g$ are starlike in $E$, then

$$\text{Re}\left(\frac{f(z)}{g(z)}\right)^\alpha \left(\frac{f'(z)}{g'(z)}\right)^\beta > 0$$

for $z \in E$, $\alpha \geq 0$, $\beta \geq 0$ and when

$$4\alpha + 6\beta = 1.$$

This result is sharp.

The sharpness can be established by

$$f(z) = \frac{z}{(1-z)^2}, \quad g(z) = \frac{z}{(1-e^{it}z)^2}, \quad -\pi < t \leq \pi.$$

**Remark 3.** For $\alpha = 0$, $\beta > 0$ and $\beta = 0$, $\alpha > 0$, Theorem 4 reduces to [2], Theorem 2.
References


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Reçu par la Rédaction le 21. 11. 1977