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Continuous solutions of a linear homogeneous functional equation

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Abstract. In this paper we investigate continuous solutions of equation (1) in the case where condition (iii) is satisfied. A necessary condition for continuous solutions of equation (1) is given as well as the construction of all continuous solutions in a certain particular case.

In the present paper we are concerned with continuous solutions of the functional equation

$$\varphi[f(x)] = g(x)\cdot\varphi(x), \tag{1}$$

where φ is the unknown real or complex valued function. A theory of continuous solutions of (1) has been developed in [2], where suitable references may be found.

LEMMA 1. *Let X and Y be a regular topological space and a metric space, respectively, let $G_n\colon X \to Y$, $n \in N$ (where N is the set of all positive integers), and $G\colon X \to Y$ be arbitrary functions and let $\mathcal{U}$ be a family of open subsets of X. If $\{G_n\}_{n\in N}$ is almost uniformly convergent*[1] *to G on every set in $\mathcal{U}$, then $\{G_n\}_{n\in N}$ is a.u.c. to G on $\bigcup\mathcal{U}$.*

Proof. If $\mathcal{U} = \{U_1, U_2\}$, then, in view of the regularity of X, for every compact subset $K \subset U_1 \cup U_2$ of X and for every $x \in K$ there exists an open set $U_x \subset X$ such that $x \in U_x$ and $\mathrm{Cl}\, U_x \subset U_1$ or $\mathrm{Cl}\, U_x \subset U_2$.

From the covering $\{U_x\colon x \in K\}$ of K we may choose a finite subcovering $\{V_1, \ldots, V_n\}$. Put

$$K_i = K \cap \bigcup\{\mathrm{Cl}\, V_j\colon \mathrm{Cl}\, V_j \subset U_i\}, \quad i = 1, 2.$$

It follows directly from this definition that K_i are compact, $K_i \subset U_i$, $i = 1, 2$, and $K = K_1 \cup K_2$. Hence the sequence $\{G_n\}_{n\in N}$ uniformly converges to G on K.

(1) I.e. uniformly convergent on every compact subset. In the sequel we shall abbreviate this to a.u.c.

By induction this extends to the case where $\mathcal{U} = \{U_1, \ldots, U_n\}$ is a finite family of open sets.

Now, for the general case, let us suppose that K is a compact subset of X such that $K \subset \bigcup \mathcal{U}$. Then there exists a subfamily $\{U_1, \ldots, U_n\}$ of $\mathcal{U}$ such that $K \subset \bigcup_{i=1}^{n} U_i$, and so on account of the previous considerations we get our result.

Applying this lemma we have the following

LEMMA 2. *If X and Y are a regular topological space and a metric space, respectively, and if $G_n\colon X \to Y$, $n \in N$, and $G\colon X \to Y$ are arbitrary functions, then there exists the greatest (in the sense of inclusion) open subset U of X such that G_n is a.u.c. to G in U.*

Proof. By Lemma 1, it is enough to take $U = \bigcup \mathcal{U}$, where $\mathcal{U}$ is the family of all open subsets of X on which G_n is a.u.c. to G.

Considering equation (1) in an interval $I = [\xi, a)$, where $-\infty \leqslant \xi < a \leqslant \infty$ (²), we assume the following hypotheses regarding the given functions.

(i) $f\colon I \to I$ is a continuous and strictly increasing function such that $\xi < f(x) < x$ for $x \in (\xi, a)$;

(ii) The function $g\colon I \to \mathcal{K}$, where $\mathcal{K}$ denotes the field of all real numbers or the field of all complex numbers, is continuous in I and different from zero in (ξ, a).

Moreover, we put

$$G_n = \prod_{i=0}^{n-1} g \circ f^i, \qquad n \in N, \tag{2}$$

where f^i denotes the i-th iterate of f.

We assume:

(iii) There exists a non-void open subinterval J of I such that the sequence $\{G_n\}_{n \in N}$ of functions defined by (2) is a.u.c. on J to the zero function.

Let U denote the union of all open (relatively to I) subsets of I on which the sequence $\{G_n\}_{n \in N}$ is a.u.c. to zero. By hypothesis (iii) U is non-void. Moreover, it follows from Lemma 1 that $\{G_n\}_{n \in N}$ is a.u.c. on U to zero. In the sequel the set U will play a crucial role.

THEOREM 1. *Under hypotheses* (i)–(iii) *for every continuous solution $\varphi\colon I \to \mathcal{K}$ of* (1) *we have*

$$\varphi(x) = 0 \quad \text{for } x \in I \setminus U.$$

(²) All our considerations remain true in the case where the fixed point ξ of the function f is an inner point of I or the right endpoint of I.

Proof. Let $\varphi\colon I \to \mathscr{K}$ be a continuous solution of (1). Then (see [2], Theorem 2.2) $\varphi(\xi) = 0$, and so if $\varphi(x_1) \neq 0$ for an $x_1 \in I$, then there exist positive real numbers c and δ such that $(-\delta + x_1, \delta + x_1] \subset I$ and

$$(3) \qquad |\varphi(x)| \geqslant c \quad \text{for } x \in (-\delta + x_1, \delta + x_1).$$

Moreover, because $\{f^n\}_{n \in N}$ is a.u.c. to ξ on I, we may assume that there exists an $n_0 \in N$ such that

$$(4) \qquad |\varphi[f^n(x)]| \leqslant c \cdot \varepsilon, \quad n \geqslant n_0, \; x \in (x_1 - \delta, x_1 + \delta).$$

Since

$$\varphi[f^n(x)] = G_n(x) \cdot \varphi(x), \quad n \in N, \; x \in I,$$

it follows from (3) and (4) that

$$|G_n(x)| \leqslant \varepsilon, \quad n \geqslant n_0, \; x \in (x_1 - \delta, x_1 + \delta),$$

i.e., $\{G_n\}_{n \in N}$ tends to the zero function uniformly in $(x_1 - \delta, x_1 + \delta)$ and so $x_1 \in U$. This ends the proof.

For an $x_0 \in (\xi, a)$ arbitrarily fixed, we write $I_0 = [f(x_0), x_0]$.

Under the additional hypothesis that the sequence $\{G_n\}_{n \in N}$ is bounded on $U \cap I_0$ (which is essential as it is shown in Example 2) we can give the general continuous solution of (1) (cf. remark on page 50 in [2]).

Namely, we have the following

THEOREM 2. *Suppose that hypotheses* (i)–(iii) *are fulfilled and that the sequence* $\{G_n\}_{n \in N}$ *is bounded on the set* $U \cap I_0$:

$$(5) \qquad |G_n(x)| \leqslant M, \quad n \in N, \; x \in U \cap I_0;$$

then for every continuous function $\varphi_0\colon I_0 \to \mathscr{K}$ *fulfilling the conditions*

$$\varphi_0[f(x_0)] = g(x_0) \cdot \varphi_0(x_0)$$

and

$$(6) \qquad \varphi_0(x) = 0 \quad \text{for } x \in I_0 \setminus U,$$

there exists exactly one continuous function $\varphi\colon I \to \mathscr{K}$ *which is a solution of* (1) *and such that*

$$(7) \qquad \varphi(x) = \varphi_0(x), \quad x \in I_0.$$

Proof. It follows from Theorem 2.1 in [2] that φ_0 may be uniquely extended onto $I \setminus \{\xi\}$ to a continuous solution of (1). Moreover, in view of Theorem 2.2 of [2], we may extend it further to a solution $\varphi\colon I \to \mathscr{K}$ of (1) putting $\varphi(\xi) = 0$. Hence it is enough to prove that

$$\lim_{x \to \xi} \varphi(x) = 0.$$

To this end we decompose the open (in I_0) set $U \cap I_0$ into a union of open (in I_0) and disjoint intervals

(8) $$U \cap I_0 = \bigcup_{k=1}^{a} I_k,$$

where $a \in N \cup \{\infty\}$, and we take any positive real number K such that

9) $$\max\{|\varphi_0(x)|:\ x \in I_0\} \leqslant K.$$

Fix an $\varepsilon > 0$. It follows from the continuity of φ_0 in I_0 that there exists a positive real number δ such that

(10) $$|x - \bar{x}| < \delta \quad \text{implies} \quad |\varphi_0(x) - \varphi_0(\bar{x})| < \frac{\varepsilon}{M}, \qquad x, \bar{x} \in I_0.$$

Recalling (8) we get the existence of a $k_0 \in N$ such that

(11) $$|I_k| < \delta, \qquad k > k_0,$$

where $|\cdot|$ denotes the Lebesgue measure. Moreover, there exist closed intervals J_k, $k = 1, \ldots, k_0$, fulfilling

(12) $$J_k \subset I_k, \qquad |I_k \setminus J_k| < \delta, \qquad k = 1, \ldots, k_0,$$

and such that if either x_0 or $f(x_0)$ belongs to I_k, then it belongs also to J_k. Putting

$$J_0 = \bigcup_{k=1}^{k_0} J_k,$$

we see that J_0 is a compact subset of $U \cap I_0$ and so we can find an $n_0 \in N$ such that

(13) $$|G_n(x)| < \frac{\varepsilon}{K}, \qquad n \geqslant n_0,\ x \in J_0.$$

To complete the proof let us take an arbitrary $x \in (\xi, f^{n_0}(x_0))$. Then there exist an $x^* \in I_0$ and an index $n \geqslant n_0$ such that $x = f^n(x^*)$. It follows from (1) and (2) that

(14) $$\varphi(x) = G_n(x^*) \cdot \varphi_0(x^*).$$

There are four possibilities:

(a) $$x^* \in I_0 \setminus U;$$

(b) there exists a $k > k_0$ such that $x^* \in I_k$;

(c) there exists a $k \leqslant k_0$ such that $x^* \in I_k \setminus J_k$;

(d) $$x^* \in J_0.$$

In case (a) we have $\varphi(x) = 0$ in view of (14) and (6), and in particular

$$|\varphi(x)| < \varepsilon. \tag{15}$$

In case (b) inequality (15) follows from (14), (8), (5), (11), (6) and (10).

In case (c) we have (15) by (14), (5), (12), (6) and (10), whereas recalling (14), (13) and (9) we have (15) also in case (d).

It follows from Theorem 1 that under condition (5) the construction described in Theorem 2 yields all the continuous solutions of equation (1) on I.

We conclude these considerations with two examples.

EXAMPLE 1 (cf. [1], Example 1). Take $I = [0, 1)$, $f(x) = px$, $x \in I$, $0 < p < 1$,

$$g(x) = 1 + \frac{\sin(2\cdot\pi\cdot\log_p x)}{1+\log_p x} \quad \text{for } x \in (0, 1),$$

and $g(0) = 1$. We have $f^n(x) = p^n \cdot x$ and $g[f^n(x)] = 1 + \dfrac{u}{n+v+1}$, where $v = \log_p x$, $u = \sin 2\cdot\pi\cdot v$. Hence

$$G_n(x) = \prod_{i=0}^{n-1}\left(1+\frac{u}{i+v+1}\right).$$

For this sequence we have $U = \bigcup_{k=0}^{\infty}(p^{k+1}, p^{k+\frac{1}{2}})$ and inequality (5) is fulfilled with $M = 1$.

Thus from Theorem 2 we obtain all continuous solutions of equation (1) with the functions f, g given above.

The second example shows that without (5) Theorem 2 may be false.

EXAMPLE 2. Let $I = (0, \infty]$, $f(x) = x+1$, $x \in I$ (here $\xi = +\infty$) and

$$g(x) = \begin{cases} 1 & \text{for } x = +\infty, \\ 2 & \text{for } x \in (0, 1], \\ \dfrac{1-2n}{n-1}x + 2n + \dfrac{1}{n-1} & \text{for } x \in \left(n-1, n-1+\dfrac{1}{n}\right], \\ \dfrac{n-1}{n} & \text{for } x \in \left(n-1+\dfrac{1}{n}, n-\dfrac{1}{n}\right), \\ 4x - 4n + 1 + \dfrac{3}{n} & \text{for } x \in \left[n-\dfrac{1}{n}, n-\dfrac{1}{2n}\right), \\ \dfrac{n+1}{n} & \text{for } x \in \left[n-\dfrac{1}{2n}, n\right], \text{ where } n = 2, 3, \ldots \end{cases}$$

The function g is continuous on interval I. Further, we have

$$f^n(x) = x+n, \quad x \in I, \; n \in N,$$

and

$$U = \bigcup_{k=0}^{\infty} (k, k+1),$$

whereas (5) is not fulfilled because

$$\lim_{n\to\infty} G_n(k) = \lim_{n\to\infty}\left(\frac{k+1}{k}\cdot\frac{k+2}{k+1}\cdot\ldots\cdot\frac{k+n}{k+n-1}\right) = +\infty \quad \text{for } k \in N.$$

The function $\varphi_0(x) = \frac{1}{2}-|x-\frac{3}{2}|$, defined on $I_0 = [1, f(1)] = [1, 2]$, is continuous and $\varphi_0(1) = \varphi_0(2) = 0$.

Let φ be the unique extension of φ_0 to a continuous solution of equation (1) in $(0, \infty)$. We have for the sequence $x_k = k+2-\frac{1}{2\cdot(k+2)}$,

$$\lim_{k\to\infty}\varphi(x_k) = \lim_{k\to\infty} G_k\left(2-\frac{1}{2\cdot(k+2)}\right)\cdot\varphi_0\left(2-\frac{1}{2\cdot(k+2)}\right)$$
$$= \lim_{k\to\infty}\left(\frac{k}{2}\cdot\frac{1}{2\cdot(k+2)}\right) = \frac{1}{4}.$$

On the other hand, for the sequence $y_k = k+2$, $k \in N$, we have

$$\varphi(y_k) = G_k(2)\cdot\varphi_0(2) = 0,$$

whence $\lim_{k\to\infty}\varphi(y_k) = 0$. Thus the limit $\lim_{x\to\infty}\varphi(x)$ does not exist, i.e., φ cannot be extended onto $I = (0, \infty]$ to a continuous function.

In this example only certain continuous functions φ_0 fulfilling $\varphi_0(1) = \varphi_0(2) = 0$ can be extended from $[1, 2]$ to a continuous solution φ of equation (1) in the whole I.

If condition (5) is not fulfilled, the construction of the general continuous solution of equation (1) on I is not known.

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References

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