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A NOTE ON THE LITTLEWOOD-PALEY SQUARE FUNCTION INEQUALITY

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1. For f 2π-periodic in H^p, $1 < p < \infty$, $0 \le n_0 < n_1 \ldots$ a lacunary sequence of integers, $n_{j+1} \ge bn_j$, $b > 1$, $j = 1, 2, \ldots$, we define

$$\gamma(x) \equiv \gamma(x; f) = \Big\{\sum_{j=0}^{\infty} |S_{n_j}(x) - S_{n_{j-1}}(x)|^2\Big\}^{1/2}, \tag{1}$$

where S_n, $n = 0, 1, \ldots$, denotes the nth partial sum of the Fourier series of f and $S_{n_{-1}} \equiv 0$. A celebrated theorem of Littlewood and Paley asserts that

$$A_p\|f\|_p \le \|\gamma\|_p \le B_p\|f\|_p, \tag{2}$$

where the positive constants A_p, B_p depend, for a fixed b, on p only. The above definition makes sense for functions in L^p and (2) remains valid (with, in general, different constants). For functions in L^p we may, as in [B2], adopt a slightly different definition of γ:

$$(\gamma(x))^2 = \sum_{j=0}^{\infty}\Big\{\Big|\sum_{k=n_{j-1}}^{n_j-1} \hat{f}(k)e^{ikx}\Big|^2 + \Big|\sum_{k=-n_j}^{-n_{j+1}+1} \hat{f}(k)e^{ikx}\Big|\Big\}. \tag{1*}$$

Again (2) is valid with possibly different values of the constants. *In what follows, when we refer to L^p functions, γ will be understood as in* (1*).

Recently J. Bourgain [B2] determined the asymptotic behavior of B_p by showing that

$$B_p = O(p), \quad p \to \infty, \quad B_p = O((p-1)^{-3/2}), \quad p \to 1^+. \tag{3}$$

In an earlier paper [B1] Bourgain showed that A_p can be taken independent of p if $p < 2$ (in particular the left inequality of (2) holds for $p = 1$). In [P1] it is proved that the "asymmetry" in the behavior of B_p (see (3)) disappears if we restrict ourselves to H^p functions.

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In this note we shall prove that

$$A_p^{-1} = O(p \log p)\,,\ p \to \infty \quad \text{(reminder: } \gamma \text{ is given by } (1^*)). \tag{4}$$

A natural conjecture here appears to be $A_p^{-1} = O(p^{1/2})$, and it is conceivable that some probabilistic arguments (e.g. those in [B1]) can prove it. Anyway, the example of a lacunary f shows that the estimate $O(p^{1/2})$, if true, cannot be improved. The "raison d'être", or rather the excuse, for the present incomplete result is the simplicity of its proof, which will be given in the next section. In the last section we offer some additional comments.

Finally, we should mention that important contributions to the Littlewood–Paley theory, where (2) belongs, are due to A. Zygmund. Indeed, the work of Marcinkiewicz and Zygmund substantially improved and complemented that of Littlewood and Paley, and the pre-war state of affairs can be found in the second volume of Zygmund's book [Z]. E. Stein in his address to the 1981 University of Chicago conference honoring Zygmund gave an excellent presentation of the post-war progress, to which one should add that in 1983 Rubio de Francia [R] extended the right-hand side inequality (2), for $p > 2$, to all sequences (not necessarily lacunary).

2. In view of definition (1^*) it is enough to consider functions in H^p. Standard arguments then show that we may even assume that f is a polynomial.

Let now l be an integer such that $b^l > p$ and write

$$f_m = \sum_j \Delta_{lj+m}\,, \quad m = 0, 1, \ldots, l-1\,, \quad \text{where } \Delta_i = S_{n_i} - S_{n_{i-1}}\,.$$

Then

$$\|f\|_p \le \sum_{m=0}^{l-1} \|f_m\|_p$$

and l can be taken of the order of $\log p$ (e.g. $[\log p/\log b] + 1$). Thus it suffices to prove that

$$\|f_m\|_p \le Cp\|\gamma\|_p\,. \tag{5}$$

Here and in what follows C will denote a positive constant, not necessarily the same at each occurrence, which depends on b only.

To simplify our notation we assume $f = f_m$, for some $m = 0, 1, \ldots,$ and write S_j for S_{n_j}. Since we have assumed that f is a (Taylor) polynomial, f equals S_N for some positive integer N.

After these reductions we proceed to the proof of (4).

First case: $p = 2k$, $k = 1, 2, \ldots$ We start with the formula

$$\begin{aligned} f^k &= S_N^k = (\Delta_N + S_{N-1})^k \\ &= \Delta_N^k + \binom{k}{1} \Delta_N^{k-1} S_{N-1} + \ldots + \binom{k}{m} \Delta_N^{k-m} S_{N-1}^m + \ldots \\ &\quad + \binom{k}{k-1} \Delta_N S_{N-1}^{k-1} + S_{N-1}^k . \end{aligned}$$

On writing the analogous formulae for $S_{N-1}^k, S_{N-2}^k, \ldots, S_0^k$ and adding the resulting equalities we obtain

$$\begin{aligned} (6) \qquad f^k &= \sum_j \Delta_j^k + \binom{k}{1} \sum_j \Delta_j^{k-1} S_{j-1} + \ldots \\ &\quad + \binom{k}{m} \sum_j \Delta_j^{k-m} S_{j-1}^m + \ldots + \binom{k}{k-1} \sum_j \Delta_j S_{j-1}^{k-1} , \end{aligned}$$

where the summations extend from 0 to N and $S_j \equiv 0$ if $j < 0$. Now (6) implies

$$(7) \qquad \|f^k\|_2 = \|f\|_{2k}^k \le \sum_{m=0}^{k-1} \binom{k}{m} \|F_m\|_2 ,$$

where $F_m = \sum_j \Delta_j^{k-m} S_{j-1}^m$.

In order to estimate the norms $\|F_m\|_2$ we shall use two simple facts:

a) The spectrum of $\Delta_j^{k-m} S_{j-1}^m$ is contained in the interval $[n_{j-1}, kn_j]$. It follows, because of the structure of the spectrum of f $(= f_m)$, that the spectra of $\Delta_j^{k-m} S_{j-1}$ and $\Delta_i^{k-m} S_{i-1}$ intersect only in the case where $|j - i| \le 1$.

b) Because of the large gaps in the spectrum of f, the partial sums S_j coincide with the de la Vallée Poussin means of f, and hence they are bounded by CM_f, where M_f denotes the Hardy–Littlewood maximal function of f.

On using these two facts we have

$$\begin{aligned} \|F_m\|_2^2 &\le C \sum \int |\Delta_j|^{2k-2m} M_f^{2m} = C \int \gamma^{2k-2m} M_f^{2m} \\ &\le C \|\gamma\|_{2k}^{2k-2m} \|M_f\|_{2k}^{2m} = C \|\gamma\|_{2k}^{2k-2m} \|f\|_{2k}^{2m} . \end{aligned}$$

In the last inequality we have used the Hardy–Littlewood maximal function inequality with an absolute constant. This is possible since $2m$ lies in the interval $[2, 2k]$ (see [Z], Ch. I, Thm. 13.15). We conclude that

$$\|F_m\|_2 \leq C\|\gamma\|_{2k}^{k-m}\|f\|_{2k}^m, \quad m = 0, 1, \ldots \tag{8}$$

Collecting our results and writing $t = \|f\|_{2k}/\|\gamma\|_{2k}$, we obtain

$$t^k \leq C \sum_{m=0}^{k-1} \binom{k}{m} t^m = C\{(1+t)^k - t^k\}, \quad \text{or}$$

$$t \leq \frac{C^{1/k}}{(1+C)^{1/k} - C^{1/k}}.$$

Since $C^{1/k} \leq C$ and $(1+C)^{1/k} - C^{1/k} \geq 1/(k(1+C))$, the desired estimate $t \leq Ck$ follows.

Second case. We present only the idea of the proof, since it is essentially the same as the one used in [P]. Let $2k$ be the largest even integer less than p. The F. Riesz factorization for H^p shows that there is a holomorphic h such that $|h| = |f|^{p/(2k)-1}$. The result of the first case applied to the function $g = fh$ implies

$$\|f\|_p^{p/(2k)} = \|g\|_{2k} \leq Cp\log p\|\gamma_g\|_{2k}. \tag{9}$$

Taking into account the structure of the spectrum of f (after the reductions made before the proof of the first case), we obtain an estimate of the form

$$\gamma_g \leq 2(M_f\delta(h) + M_h\delta(f)), \tag{10}$$

where δ denotes "smoothed" version of γ (see [P]), from which we obtain

$$\|\gamma_g\|_{2k} \leq C\|\gamma_f\|_p. \tag{11}$$

Combining (9) and (11) we obtain the desired estimate.

3. Remark. It is well known that the constant in the classical L^p-L^2 inequality for lacunary series behaves like $p^{1/2}$ as $p \to \infty$. The usual proof is based on the analogous property for Rademacher functions, while if we try a direct algebraic approach, as the one we use in this article, we obtain a $p\log p$ behavior for the constant. This was the reason, as we said in Section 1, that probabilistic arguments may lead to better results.

Let us mention finally that the estimate $O(p^{1/2})$ for A_p, if true, coupled with Bourgain's estimate $O(p)$ for B_p, gives an almost immediate proof of the $O((p-1)^{-3/2})$ result of Bourgain. Indeed, in this case the norm in $L^{p'}$, $1/p' + 1/p = 1$, of the multiplier in $L^{p'}$ (which is equal to the norm of the same multiplier in L^p) given by $m(n) = +1$ or -1 and constant in each interval $[n_j, n_{j+1}]$, will be $O(p^{1/2})O(p)$ as $p \to \infty$, i.e. $O((p'-1)^{-3/2})$ as $p' \to 1^+$. The corresponding estimate for B_p is now an almost immediate corollary.

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