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A HOMOMORPHISM THEOREM FOR PARTIAL ALGEBRAS

BY

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To Heinrich Behnke
on his 70th birthday, October 9, 1968

1. The essence of a homomorphism theorem seems to be the intrinsic characterization of — roughly speaking — "all homomorphic images". This is obvious in the case of algebras $(A, f) = (A, (f_i)_{i \in I})$ of *type* $\Delta = (K_i)_{i \in I}$, i.e. of sets A together with *fundamental operations* $f_i \colon A^{K_i} \to A$ (where the *arities* K_i as well as the *index-domain* I may be arbitrary finite or infinite sets). A *homomorphism*

$$\varphi \colon (A, f) \to (B, g)$$

into an algebra $(B, g) = (B, (g_i)_{i \in I})$ of the same type Δ, $g_i \colon B^{K_i} \to B$, is a mapping $\varphi \colon A \to B$ such that

$$\varphi(f_i(\mathfrak{a})) = g_i(\varphi \circ \mathfrak{a}) \tag{1}$$

for each index $i \in I$, each sequence $\mathfrak{a} \in A^{K_i}$, i.e., $\mathfrak{a} \colon K_i \to A$. A *congruence relation* of an algebra (A, f) is an equivalence relation, R, in A, compatible with the algebraic structure f, i.e., for each $i \in I$, $\mathfrak{a}, \mathfrak{a}' \in A^{K_i}$,

(2) if $(\mathfrak{a}_k, \mathfrak{a}'_k) \in R$, for each $k \in K_i$, then $(f_i(\mathfrak{a}), f_i(\mathfrak{a}')) \in R$ (in other words, R is a subalgebra of the cartesian product $(A, f) \times (A, f)$).

One then has the following three *basic statements*:

I. Each homomorphism $\varphi \colon (A, f) \to (B, g)$ induces in (A, f) a congruence relation $R = R_\varphi$.

II. Each congruence relation R in (A, f) is induced by a surjective homomorphism $\varphi \colon (A, f) \to (B, g)$.

(Take, e.g., the *natural homomorphism* $\pi = \pi_R$ onto the *factor algebra* $(A, f)/R = (A/R, f/R)$, where the *factor structure* f/R is obtained by the usual definition by representatives.)

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III. For a surjective homomorphism $\varphi: (A, f) \to (B, g)$ and an arbitrary homomorphism $\psi: (A, f) \to (C, h)$, there is a — necessarily unique — homomorphism $\omega: (B, g) \to (C, h)$ such that $\psi = \omega \circ \varphi$, if and only if $R_\varphi \subset R_\psi$.

The usual *standard factorization* is a rather special case of statement III: Given an arbitrary homomorphism $\psi: (A, f) \to (C, h)$, one takes as φ the natural homomorphism $\pi: (A, f) \to (A, f)/R$, where $R = R_\psi$. Then $\psi = \omega \circ \pi$; furthermore, since $R_\pi = R_\psi$, the homomorphism ω is injective, hence an isomorphism onto the subalgebra $\operatorname{im} \psi = \operatorname{im} \pi \subset (C, h)$. This is what is usually called the *Homomorphism Theorem.* One might well ask the question if the natural homomorphism $\pi: (A, f) \to (A, f)/R$ actually does or does not yield any information that could not be obtained by an arbitrary (surjective) homomorphism $\psi: (A, f) \to (C, h)$ with "kernel" R — no matter what the nature of the elements of C. If there is no additional information obtainable from $\pi = \pi_R$, why this traditional enthusiasm for the standard factorization? Why pass, from a given surjective homomorphism ψ, to its isomorphic copy π_R?

For a precise formulation of the intrinsic characterization of "all homomorphic images", we are considering a certain algebra (A, f), and we introduce, into the class of all surjective homomorphisms $\varphi: (A, f) \to (B, g)$, $\psi: (A, f) \to (C, h), \ldots$, a quasi-ordering $\succ$:

(3) $\varphi \succ \psi$ if and only if there is a — necessarily unique — homomorphism $\omega: (B, g) \to (C, h)$ such that $\psi = \omega \circ \varphi$.

(If so, we may call φ a *right factor* of ψ, or say that φ *dominates* ψ.) For this reflexive and transitive relation, an obvious equivalence theorem of Bernstein type holds:

(4) $\varphi \succ \psi \succ \varphi$ if and only if there is an isomorphism $\omega: (B, g) \to (C, h)$ such that $\psi = \omega \circ \varphi$.

(We may call φ and ψ *isomorphic* and write $\varphi \cong \psi$.) Then we can restate III (restricted to surjective homomorphisms):

(5) $\varphi \succ \psi$ if and only if $R_\varphi \subset R_\psi$.

In particular,

(6) $\varphi \cong \psi$ if and only if $R_\varphi = R_\psi$.

Hence, together with the basic statements I and II, we have a one-one correspondence between all congruence relations of (A, f) and all classes of isomorphic surjective homomorphisms from (A, f) onto other algebras of the same type. By (5), this one-one correspondence can be considered as an order anti-isomorphism.

2. So far, we have summarized the well-known situation in the theory of *complete*, or *full algebras*, where all fundamental operations, f_i, are everywhere defined, $\operatorname{dom} f_i = A^{K_i}$. The situation becomes much more complicated if we pass to the case of *partial algebras*, (A, f), where dom $f_i \subset A^{K_i}$ (even dom $f_i = \emptyset$ is admitted). Here, one even has to reformulate the definition of homomorphisms; instead of equation (1), one has the homomorphism implication:

(7) if $f_i(\mathfrak{a}) = a$, then $g_i(\varphi \circ \mathfrak{a}) = \varphi(a)$,

for each index $i \in I$, each sequence $\mathfrak{a}: K_i \to A$ and each element $a \in A$. Also, condition (2) for a congruence relation must be restated more carefully:

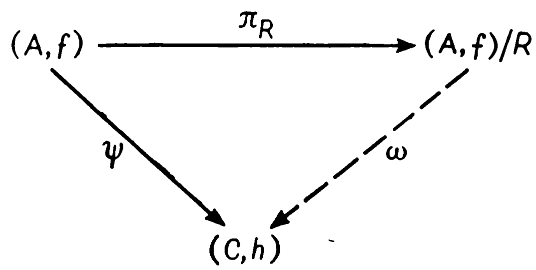
(8) If $(\mathfrak{a}_k, \mathfrak{a}'_k) \in R$, for each $k \in K_i$, and if $f_i(\mathfrak{a}) = a$, $f_i(\mathfrak{a}') = a'$, then $(a, a') \in R$,

for each $i \in I$, $\mathfrak{a}, \mathfrak{a}': K_i \to A$, $a, a' \in A$. Then our basic statements I and II still hold, but III is no longer true. In fact, if we still stick to the idea that the "kernel" of a homomorphism should be nothing but its induced congruence relation, then we must introduce a stronger notion of homomorphism.

Homomorphism φ: $(A, f) \to (B, g)$ is called *strong* provided that, in addition to (7):

(9) If $g_i(\mathfrak{b}) = b$, then $\mathfrak{b} = \varphi \circ \mathfrak{a}$, $b = \varphi(a)$, $f_i(\mathfrak{a}) = a$, for some sequence $\mathfrak{a}: K_i \to A$, some element $a \in A$,

and this should again hold for each index $i \in I$, and each sequence $\mathfrak{b}$ of type K_i in im φ (rather than in B — in order to escape an obvious, highly undesirable relativity), and each element $b \in$ im φ. Now, as shown, e.g., in Pierce's book [11], the three basic statements of section 1 remain true if we admit partial algebras, but replace "surjective homomorphism", whenever it occurs, by "strong surjective homomorphism" ("epimorphism", in the nomenclature of Pierce). In particular, each congruence relation R in (A, f) is induced by a strong surjective homomorphism (statement II), e.g., the natural homomorphism $\pi = \pi_R$ onto the factor-algebra $(A, f)/R$. But now, statement III also becomes true. This is an immediate consequence of the fact that, if a surjective homomorphism $\varphi: (A, f) \to (B, g)$ is strong, then g is the *final structure* for the mapping $\varphi: (A, f) \to B$ in the sense of Bourbaki [2] (cf. also [15]). That is, given an arbitrary partial algebra (C, h) and an arbitrary mapping $\omega: B \to C$, then $\omega \circ \varphi: (A, f) \to$ $\to (C, h)$ is a homomorphism if and only if $\omega: (B, g) \to (C, h)$ is. In particular, g is the *poorest* (*finest*, *weakest*) *algebraic structure* on B such that $\varphi: (A, f) \to (B, g)$ is an homomorphism. Concerning the standard factorization aspect of statement III, note that ω in the commutative diagram



where $R = R_\psi$, is no longer an isomorphism. Even if we assume ψ to be surjective, and hence ω bijective, ω is an isomorphism only if ψ is strong. The situation is exactly as in topology: For the surjective case, the final structure is what is to-day called the "identification topology"; the mapping then has been called "strongly continuous" by Alexandroff-Hopf. Furthermore, a bijective continuous mapping is not a homeomorphism in general.

3. Unfortunately, many homomorphisms of partial algebras — as many continuous mappings in topology — are not strong. In particular, statement III as discussed in the previous section, does not lead to a classification of arbitrary (instead of strong) surjective homomorphisms, in any case not in terms of an intrinsic description of their isomorphism classes.

As observed by B. Banaschewski, there is an easy method of intrinsically characterizing those classes of isomorphic surjective homomorphisms by the following modified definition of "kernels": Given the homomorphism $\psi: (A, f) \to (C, h)$ as in the diagram above, add to the kernel $R = R_\psi$, as hitherto considered, the structure $\omega^{-1}(h)$ on A/R that is the unique result of transporting (lifting) the structure h against the injection ω from C onto A/R. One can also define $\omega^{-1}(h)$ as the only structure on A/R that makes $\omega: (A/R, \omega^{-1}(h)) \to (C, h)$ a strong homomorphism, hence an isomorphism in case $\psi: A \to C$ is surjective, hence $\omega: A/R \to C$ bijective. (In more systematic terms, $\omega^{-1}(h)$ is the *initial structure* (Bourbaki [2]) for the mapping $\omega: A/R \to (C, h)$.) In general, $\omega^{-1}(h)$ is different, in fact *richer* (*stronger*) than the factor structure f/R, equality taking place if and only if $\psi: (A, f) \to (C, h)$ is strong.

There are, of course, other examples of classifications of homomorphisms of certain kinds by means of two or more invariants, not only the most natural one, the induced equivalence (congruence) relation. In fact, Hoehnke [8], interpreting Lyndon's paper [10], has hinted upon the following two-data kernel notion for homomorphisms $\psi: (A, f) \to (C, h)$ between relational systems (models): One adds to the equivalence relation R_ψ the initial structure $\psi^{-1}(h)$, obtained by lifting the relational structure h against ψ. In other words, one continues Banaschewski's procedure by additionally lifting $\omega^{-1}(h)$ against natural projection π_R, thus obtaining a relational structure no longer in A/R, but in A itself. (Here, R is a congruence relation — in the sense, Lyndon [10] apparently has in mind — of the initial structure $\psi^{-1}(h) = \pi_R^{-1}(\omega^{-1}(h))$, which is an enrichment of the original relational structure f: in general, R is not a congruence relation of f itself.)

Both Banaschewski's and Hoehnke's proposals enable us to analogize the three basic statements of section 1, hence to obtain classifications of surjective homomorphisms for partial algebras or relational systems respectively.

4. The main subject of this paper is a new notion of kernel for homomorphisms of partial algebras, which will enable us to classify not only surjective homomorphisms, as in section 3, but arbitrary epimorphisms in the category of all partial algebras of type Δ; strongness, important in section 2, plays no further role.

Let us first note that homomorphism $\varphi: (A, f) \to (B, g)$ is an *epimorphism* in the category sense, i.e. *right cancellable*, if and only if im φ generates (B, g),

$$\overline{\operatorname{im}\varphi} = B$$

($\overline{}$ the symbol for the generated subalgebra). We may call such a homomorphism *almost surjective*. In fact, if $\overline{\operatorname{im}\varphi} = B$ and $\omega\circ\varphi = \omega'\circ\varphi$, where $\omega, \omega': (B, g) \to (C, h)$, then $\omega = \omega'$, since the set of all $b \in B$ such that $\omega(b) = \omega'(b)$ is a subalgebra of (B, g). In order to prove the converse, consider the subalgebra $B_0 := \overline{\operatorname{im}\varphi}$, and take the free union of two copies of the partial algebra (B, g) with amalgamation B_0. To be more precise: Start from a bijection $\delta: B - B_0 \to \operatorname{im}\delta$ such that $B \cap \operatorname{im}\delta = \emptyset$. Let $C := B \cup \operatorname{im}\delta$ and $\omega := \operatorname{id}_{B_0} \cup \delta: B \to C$. Make sure that there is an algebraic structure h on C such that ω as well as $\operatorname{id}_B: B \to C$ become homomorphisms from (B, g) into (C, h). In order to do that, we have to show, cf. [15], that the equivalence relations induced by ω and id_B are congruence relations of (B, g), but this is trivial, since ω and id_B are injective. Furthermore, one has to show that

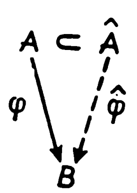
if $\omega\circ\mathfrak{b} = \mathfrak{b}'$ and $g_i(\mathfrak{b}) = b$ and $g_i(\mathfrak{b}') = b'$, then $\omega(b) = b'$,

for all indices $i \in I$, all sequences $\mathfrak{b}, \mathfrak{b}': K_i \to B$, and all elements $b, b' \in B$. This follows from the construction: From $\omega\circ\mathfrak{b} = \mathfrak{b}'$ we obtain that both sequences are equal and in B_0. Hence the elements b and b' are equal, and in B_0, since B_0 is a subalgebra, implying that $\omega(b) = b = b'$. Having thus introduced h such that $\omega, \operatorname{id}_B: (B, g) \to (C, h)$ are homomorphisms, observe that $\omega\circ\varphi = \operatorname{id}_B\circ\varphi$ since ω and id_B coincide on $B_0 \subset \operatorname{im}\varphi$ by construction. Since φ is an epimorphism by hypothesis, $\omega = \operatorname{id}_B$, which gives $\overline{\operatorname{im}\varphi} = B_0 = B$.

Before we describe our new kernel notion, let us illustrate that the class of almost surjective homomorphisms is in fact much more comprehensive than the subclass of surjective homomorphisms. E.g., among the epimorphisms are all (identical) *embeddings* $\mathrm{id}_A: (A, f) \to (B, g)$, be they strong or not, i.e., all *strong* or *weak extensions* of partial algebra (A, f), provided that they are *minimal extensions*, i.e., generated by the subset A. Important minimal extensions are the *minimal completions* as studied in [4], i.e., the strong, complete, minimal extensions. As another special kind of (weak) minimal extensions, take $B = A$, and consider all algebraic structures, g, on A richer than the given structure f, i.e., for which $\mathrm{id}_A: (A, f) \to (A, g)$ is a homomorphism. Hence the comparison of algebraic structures on the same set as introduced in [14] also falls, as a special case, under the pattern of (almost) surjective homomorphisms.

5. Banaschewski's proposal as discussed in section 3 involves the congruence relation $R = R_\varphi$ and an enrichment of the factor structure f/R. The method for relational systems as mentioned in section 3 is somewhat dual to Banaschewski's proposal: Instead of first factorizing modulo R, then enriching the factor structure f/R, it first enriches relational structure f itself, and factorizes afterwards. The method we are going to present is similar; the only — but essential — difference is that the enrichment of the relational structure f as described in section 3 is no longer an algebraic structure: The enriched operations would become multi-valued. So, instead of remaining within the set A (with its given algebraic structure f), we will "enrich" by splitting under the various values of those multi-valued operations, i.e., by passing to a suitable extension of (A, f). In fact, the latter, together with a suitable congruence relation in it, will represent our new kernel suitable for the classification of all almost surjective homomorphisms.

In order to describe this extension, we have to recollect some facts about the *universal* or *free completion* as studied in [4]. With each partial algebra A (we will frequently omit mentioning the algebraic structure f) is associated its free completion $\hat{A}$, the latter being a strong, complete extension of A such that each homomorphism φ from A into an arbitrary complete algebra B can be uniquely extended to a homomorphism $\hat{\varphi}$ from $\hat{A}$ into B:



So the extension of A to $\hat{A}$ is the solution of a universal problem. In particular, $\hat{A}$ is unique up to unique isomorphism "over A", i.e. leaving all elements of A fixed. Besides this *categorical description* of $\hat{A}$, there is the following *internal characterization* (*axiomatization*): A strong complete extension (B, g) of (A, f) is (a model of) $\hat{A}$ if and only if the following three *Axioms of Free Completion* hold:

FC1. If $g_i(\mathfrak{b}) \in A$, then $\mathfrak{b}$ is in A, i.e. $\operatorname{im} \mathfrak{b} \subset A$;

FC2. If $g_i(\mathfrak{b}) = g_j(\mathfrak{c}) \notin A$, then $i = j$ and $\mathfrak{b} = \mathfrak{c}$;

FC3. $\bar{A} = B$.

Since (A, f) is a *strong relative algebra* of (B, g), i.e., the extension (B, g) is strong, the conclusion of FC1 can be strengthened to $g_i(\mathfrak{b}) = f_i(\mathfrak{b})$. Hence, if partial algebra (A, f) is *discrete*, i.e., dom $f_i = \emptyset$ for each $i \in I$, FC1 simply runs: $g_i(\mathfrak{b}) \notin A$. This makes it obvious that the above axioms are nothing but rather *Generalized Peano Axioms*. In particular, FC3 is the generalization of the Axiom of Complete Induction and may be called the *Axiom of Algebraic Induction*, since it gives rise to an obvious inductive method of proving statements for all elements $b \in B$ (as first, to the author's knowledge, used systematically by Löwig [9]). FC2 states the injectivity of the operations g_i, when suitably restricted, and the pairwise disjointness of their images. Finally, FC1 may be interpreted in terms of quasi-order. In an arbitrary partial algebra (B, g), we call the binary relation G which holds between any term $\mathfrak{b}_k (k \in K_i)$ of any sequence $\mathfrak{b}$ and any element $g_i(\mathfrak{b})$ the *graph of the partial algebra* (B, g) (as suggested by Diener [5]). The quasi-order generated by the graph G may be called the *natural quasi-order*. FC1 now states that, if $a \in A$, and $(b, a) \in G$, then $b \in A$. This can be strengthened: If $a \in A$ and $b < a$ ($<$ the symbol for the natural quasi-order), then $b \in A$. In an arbitrary quasi-ordered set $(B, <)$, one may call such a subset A an *initial segment*. FC1 then can be formulated briefly: A is an initial segment of partial algebra (B, g). In [4], (B, g) was then called a *normal extension*.

6. In the sequel, the *intermediate initial segments* of free completion $\hat{A}$ will play an important role, "intermediate" meaning: lying between A (the least intermediate initial extension) and $\hat{A}$ (the greatest one). These intermediate extensions can be characterized without reference to A, exactly by the axioms FC1, FC2, and FC3. This is an easy consequence of the discussion of the transitivities of these three extension properties. Let us list them here, the proofs being either trivial or straightforward. Speaking of a chain of strong extensions,

$$(A, f) \subset (B, g) \subset (C, h),$$

we have:

1. If A is FC1 in B, and B is FC1 in C, then A is FC1 in C;
2. If A is FC2 in B, and B is FC1 and FC2 in C, then A is FC2 in C;
3. If A is FC3 in B, and B is FC3 in C, then A is FC3 in C;

4. If A is FC1 in C, then A is FC1 in B;
5. If A is FC2 in C, then A is FC2 in B;
6. If A is FC3 in C, and B is FC1 in C, then A is FC3 in B;
7. If A is FC1 and FC2 in C and FC3 in B, then B is FC1 in C;
8. If A is FC2 in C, then B is FC2 in C;
9. If A is FC3 in C, then B is FC3 in C.

In particular, if A is FC1, FC2, and FC3 in C, then

$$A \text{ is FC3 in } B \text{ iff } B \text{ is FC1 in } C.$$

I.e., A generates B iff B is an initial segment of C. By specializing once $C = \hat{B}$, twice $C = \hat{A}$, we obtain

THEOREM 1. *The following statements on the (strong) extension B of A are equivalent*:

(i) A *is* FC1, FC2, *and* FC3 *in* B;

(ii) $\hat{B} = \hat{A}$ (*i.e.*, $\hat{B}$ *is a model of* $\hat{A}$);

(iii) B *is an initial segment of* (*a suitable model of*) $\hat{A}$;

(iv) A *generates* B, *and* B *is* (*strongly*) *embeddable into* (*a model of*) $\hat{A}$.

In fact, for (i) ⇒ (ii), specify $C = \hat{B}$. (ii) ⇒ (iii) is trivial. For (iii) ⇒ (i) as well as (iii) ⇔ (iv), take $C = \hat{A}$.

Assume now at least FC1 between A and B, and let $\varphi: A \to C$ be an arbitrary homomorphism. An *initial extension* of φ is a homomorphic extension, ψ, from an intermediate initial segment of B (considered as a relative algebra of B), still into C. With this notion, we come to a useful extension of the universality property of A. This idea, at least in a special case, is due to W. Hutter, so we may refer to it as:

THEOREM 2 (Hutter's Theorem). *Let (B, g) be a normal, minimal extension of (A, f). Then each homomorphism $\varphi: (A, f) \to (C, h)$ has a maximal initial extension, $\tilde{\varphi}$.*

Define $\tilde{\varphi}$ to be the union of all initial extensions ψ of φ; since φ itself is a ψ, $\varphi \subset \tilde{\varphi}$. If ψ' and ψ'' are two such extensions, dom ψ' and dom ψ'', and hence their intersection dom $\psi' \cap$ dom ψ'', are intermediate initial segments. Hence A generates dom $\psi' \cap$ dom ψ'' (rule 6 given above), so ψ' and ψ'', being equal on A, coincide on this intersection, showing that $\tilde{\varphi}$ is single-valued, i.e., a function. Its domain, being the union of intermediate initial segments, is again an intermediate initial segment. Furthermore, $\tilde{\varphi}$ is a homomorphism: Take $\mathfrak{b}: K_i \to \operatorname{dom} \tilde{\varphi}$ and $g_i(\mathfrak{b}) = b \in \operatorname{dom} \tilde{\varphi}$. Then $b \in \operatorname{dom} \psi$, for some initial extension ψ, and so $\operatorname{im} \mathfrak{b} \subset \operatorname{dom} \psi$, showing that

$$h_i(\tilde{\varphi} \circ \mathfrak{b}) = h_i(\psi \circ \mathfrak{b}) = \psi(b) = \tilde{\varphi}(b).$$

Note that the maximal initial extension $\tilde{\varphi}$ depends on C. If we extend $C, \tilde{\varphi}$ may become larger. In this connection, we have the useful

THEOREM 3. *Let B be a normal, minimal extension of A, D a normal extension of C, and $\varphi: A \to C$ a homomorphism. Let φ_C be the maximal initial extension of $\varphi: A \to C$, φ_D that of $\varphi: A \to D$. Then*

$$\varphi_C = \varphi_D | \varphi_D^{-1}(C). \tag{10}$$

For clearly $\varphi_C \subset \varphi_D$, hence $\varphi_C \subset \varphi_D | \varphi_D^{-1}(C)$. On the other hand, since C is an initial segment of D, $\varphi_D^{-1}(C)$ is an initial segment of B. Hence $\varphi_D | \varphi_D^{-1}(C)$ is an initial extension of $\varphi: A \to C$, showing that this restriction of φ_D really is φ_C.

We are mainly concerned with the special case $B = \hat{A}$. If, in this case, C is also complete, then the universality property of $\hat{A}$ states that $\tilde{\varphi}$ actually is $\hat{\varphi}$, defined in section 5. In other words, dom $\tilde{\varphi} = \hat{A}$, for another independent proof of that cf. the end of the next section. If C is not complete, take $D = \hat{C}$. Then, as a special case of Theorem 3,

$$\tilde{\varphi} = \hat{\varphi} | \hat{\varphi}^{-1}(C), \tag{11}$$

where $\hat{\varphi}$ is the homomorphic extension of φ from $\hat{A}$ into $\hat{C}$.

7. A homomorphism $\varphi: (A, f) \to (B, g)$ may be called *closed* provided that, in addition to (7),

$$\text{if } g_i(\varphi \circ \mathfrak{a}) = b, \text{ then } \varphi(f_i(\mathfrak{a})) = b, \tag{12}$$

for each index $i \in I$, each sequence $\mathfrak{a}: K_i \to A$, and each element $b \in B$. An important property of closed homomorphisms, heavily taken advantage of by Baumann and Pfanzagl [1] (they introduced them as *double homomorphisms*), is their preservation of closed subsets (subalgebras) and closure: Under such a homomorphism, the image of a closed subset of (A, f) is a closed subset of (B, g), hence

$$\varphi(\overline{M}) = \overline{\varphi(M)} \tag{13}$$

for any subset $M \subset A$. This is not true for arbitrary homomorphism. It is by this property that closed homomorphisms can be considered as the analogues of the closed continuous mappings of topology (as strong homomorphisms were the analogues of strongly continuous mappings) (¹). As in topology (cf., e.g., Bourbaki [3]), a homomorphism is closed if and only if it is strong, and if, in addition, its image im φ is a closed subset of (B, g), and if, finally, its congruence relation $R = R_\varphi$ is closed also.

(¹) Grätzer [7] calls our closed homomorphisms *strong*, our strong homomorphisms *full*.

Here, the congruence relation R is *closed* (²) provided that, in addition to (8),

(14) if $\mathfrak{a} \in \operatorname{dom} f_i$ and $(\mathfrak{a}_k, \mathfrak{a}'_k) \in R$, for each $k \in K_i$, then $\mathfrak{a}' \in \operatorname{dom} f_i$, for all indices $i \in I$, all sequences $\mathfrak{a}, \mathfrak{a}' : K_i \to A$.

Closed homomorphisms and closed congruence relations play an outstanding role for our kernels. The link is the following characterization of our maximal initial extension $\tilde{\varphi}$ of an arbitrary homomorphism $\varphi: A \to C$ with respect to free completion $\hat{A}$ as considered at the end of section 6. The idea of this characterization may be traced back, at least in a special case, to R. Kerkhoff, so we may well call it:

THEOREM 4 (Kerkhoff's Theorem). *The maximal extension (with respect to $\hat{A}$) of homomorphism $\varphi: (A, f) \to (C, h)$ is the only closed initial extension.*

We first prove that $\tilde{\varphi}$ is closed, i.e.

$$\text{if } h_i(\varphi \circ \mathfrak{b}) = c, \text{ then } \tilde{\varphi}(g_i(\mathfrak{b})) = c,$$

where $\mathfrak{b}$ is an arbitrary sequence of type K_i in $\operatorname{dom} \tilde{\varphi} \subset \hat{A}$, g_i the fundamental operation in $\hat{A}$, and $c \in C$. By the completeness of $\hat{A}$, $b := g_i(\mathfrak{b})$ exists. Assume $b \notin \operatorname{dom} \tilde{\varphi}$. $\operatorname{dom} \tilde{\varphi} \cup \{b\}$ still is an initial segment — FC1 — of $\hat{A}$, since $\operatorname{dom} \tilde{\varphi}$ has properties FC1 and FC2 with respect to $\hat{A}$ and FC3 with respect to C, i.e., generates, $\operatorname{dom} \tilde{\varphi} \cup \{b\}$. We define

$$\psi(x) := \begin{cases} \tilde{\varphi}(x) & \text{if } x \in \operatorname{dom} \tilde{\varphi}, \\ c & \text{if } x = b. \end{cases}$$

ψ is a homomorphism. For let $g_j(\mathfrak{x}) = x$, where $\mathfrak{x}$ is a sequence of type K_j in $\operatorname{dom} \tilde{\varphi} \cup \{b\}$, $x \in \operatorname{dom} \tilde{\varphi} \cup \{b\}$. If $x \in \operatorname{dom} \tilde{\varphi}$, $\mathfrak{x}$ is also in $\operatorname{dom} \tilde{\varphi}$, since $\operatorname{dom} \tilde{\varphi}$ is an initial segment of $\hat{A}$. Hence

$$h_j(\psi \circ \mathfrak{x}) = h_j(\tilde{\varphi} \circ \mathfrak{x}) = \tilde{\varphi}(x) = \psi(x).$$

If $x = b$, since $b \notin A$, by FC2 we have $j = i$ and $\mathfrak{x} = \mathfrak{b}$, whence

$$h_j(\psi \circ \mathfrak{x}) = h_i(\tilde{\varphi} \circ \mathfrak{b}) = c = \psi(b) = \psi(x).$$

After all, ψ is an initial extension of φ, contradicting the maximality of $\tilde{\varphi}$. Hence $b \in \operatorname{dom} \tilde{\varphi}$, which gives

$$\tilde{\varphi}(g_i(\mathfrak{b})) = \tilde{\varphi}(b) = h_i(\tilde{\varphi} \circ \mathfrak{b}) = c.$$

It remains to show that any closed initial extension ψ of φ is actually $\tilde{\varphi}$. In fact, $\psi \subset \tilde{\varphi}$ by the maximality of $\tilde{\varphi}$. In order to establish equality,

(²) Grätzer [7]: strong congruence relation.

we show by algebraic induction on $b \in \hat{A}$:

$$(*) \qquad \text{if } b \in \operatorname{dom}\tilde{\varphi}, \text{ then } b \in \operatorname{dom}\psi.$$

This is trivial for $b \in A$. Then we prove (*) for $b = g_i(\mathfrak{b})$, under the inductive hypothesis that (*) holds for all terms of the sequence $\mathfrak{b}$. Since $b \in \operatorname{dom}\tilde{\varphi}$ and $\operatorname{dom}\tilde{\varphi}$ is an initial segment, these terms are in fact in $\operatorname{dom}\tilde{\varphi}$, hence in $\operatorname{dom}\psi$ by hypothesis. Hence

$$h_i(\psi \circ \mathfrak{b}) = h_i(\tilde{\varphi} \circ \mathfrak{b}) = \tilde{\varphi}(b).$$

But ψ is closed, so $\psi(g_i(\mathfrak{b})) = \tilde{\varphi}(b)$, and so $b = g_i(\mathfrak{b}) \in \operatorname{dom}\psi$, completing the proof.

The properties of $\hat{A}$ we have really used in this proof were completeness, FC1, FC2, and FC3. That an extension B of A with these properties is indeed the free completion (as pointed out in section 5), can be derived from Theorem 2 and 4 as an immediate consequence. To this end, consider a homomorphism $\varphi: A \to C$, where C is complete, and its maximal initial extension (with respect to B), $\tilde{\varphi}$. By Theorem 4, $\tilde{\varphi}: \operatorname{dom}\tilde{\varphi} \ (\subset B) \to C$ is closed, hence, as C is complete, $\operatorname{dom}\tilde{\varphi}$ is also complete, thus a subalgebra of B. Since $\operatorname{dom}\tilde{\varphi}$ generates B, it equals B. (For general information about this fairly general form of the so-called *Recursion Theorem* — various proofs, special cases, etc., cf. Diener [6].)

8. We are now ready to introduce our new kernel. As before, we are concerned with some partial algebra A and its homomorphisms $\varphi: A \to B$. As in Theorem 4, their associated maximal initial extensions $\tilde{\varphi}$ will be understood with respect to the free completion $\hat{A}$. Recall that the domain of $\tilde{\varphi}$ is an (intermediate) initial segment of $\hat{A}$, hence the partial algebra $\operatorname{dom}\tilde{\varphi}$ (a relative algebra of $\hat{A}$) is generated by the subset A,

$$(15) \qquad \bar{A} = \operatorname{dom}\tilde{\varphi}.$$

By Theorem 4, $\tilde{\varphi}: \operatorname{dom}\tilde{\varphi} \to B$ is a closed homomorphism; hence by (13) and (15), we obtain

$$(16) \qquad \overline{\operatorname{im}\varphi} = \operatorname{im}\tilde{\varphi}.$$

In particular, $\varphi: A \to B$ is almost surjective if and only if $\tilde{\varphi}: \operatorname{dom}\tilde{\varphi} \to B$ is surjective.

Now, the congruence relation induced by $\tilde{\varphi}$ is called the *kernel* of the original homomorphism $\varphi: A \to B$, or rather of the couple (φ, B):

$$(17) \qquad \ker(\varphi, B) = R_{\tilde{\varphi}}.$$

Remember that $\tilde{\varphi}$ depended on partial algebra B, and the same still holds for $R_{\tilde{\varphi}}$. (The kernels suggested in section 3 also depended on B.)

Note that the congruence relation induced by φ itself is just the restriction of our kernel:

$$R_{\varphi} = R_{\tilde{\varphi}} \cap (A \times A). \tag{18}$$

Because of the closedness of $\tilde{\varphi}$, $R_{\tilde{\varphi}}$ is a closed congruence relation in its field $\operatorname{dom} \tilde{\varphi}$. Hence we have the following counterpart of the basic statement I:

THEOREM 5. *The kernel of an arbitrary homomorphism $\varphi: A \to B$ is a closed congruence relation in an intermediate initial segment of $\hat{A}$.*

Observe that the complete case, as considered in section 1, is a special case of our present notion: If A is complete, then $\hat{A} = A$, and $\tilde{\varphi} = \varphi$, hence $R_{\tilde{\varphi}} = R_{\varphi}$.

We also have the following extension of the basic statement II:

THEOREM 6. *Each closed congruence relation R in an intermediate initial segment D of $\hat{A}$ is the kernel of some almost surjective homomorphism $\varphi: A \to B$.*

For there is a strong surjective homomorphism $\Phi: D \to B$ such that $R_{\Phi} = R$. Since R is closed, Φ is. Let φ be the restriction of Φ to A. Then Φ is a closed initial extension of φ, whence $\Phi = \tilde{\varphi}$ by Theorem 4. Since Φ is surjective, φ is almost surjective.

Combining Theorems 5 and 6, the closed initial congruence relations R (closed congruence relations in intermediate initial segments) are precisely the kernels of almost surjective homomorphisms, i.e. epimorphisms.

Finally we have the following extension of the basic statement III:

THEOREM 7. *Let $\varphi: A \to B$ be an almost surjective, and $\psi: A \to C$ an arbitrary homomorphism. Then there is a — necessarily unique — homomorphism $\omega: B \to C$ such that $\psi = \omega \circ \varphi$, if and only if $R_{\tilde{\varphi}} \subset R_{\tilde{\psi}}$.*

In fact, if $\psi = \omega \circ \varphi$, where $\omega: B \to C$, then $\omega \circ \tilde{\varphi}: \operatorname{dom} \tilde{\varphi} \to C$ is an initial extension of $\psi: A \to C$, whence $\omega \circ \tilde{\varphi} \subset \tilde{\psi}$, so that $R_{\tilde{\varphi}} \subset R_{\omega \circ \tilde{\varphi}} \subset R_{\tilde{\psi}}$. Conversely, if $R_{\tilde{\varphi}} \subset R_{\tilde{\psi}}$, then $\operatorname{dom} \tilde{\varphi} \subset \operatorname{dom} \tilde{\psi}$ and

$$R_{\tilde{\varphi}} \subset R_{\tilde{\psi}} \cap (\operatorname{dom} \tilde{\varphi} \times \operatorname{dom} \tilde{\varphi}) = R_{\tilde{\psi} | \operatorname{dom} \tilde{\varphi}}.$$

Since $\tilde{\varphi}: \operatorname{dom} \tilde{\varphi} \to B$ is strong and surjective, there is (cf. section 2) a homomorphism $\omega: B \to C$ such that

$$\omega \circ \tilde{\varphi} = \tilde{\psi} | \operatorname{dom} \tilde{\varphi}, \tag{19}$$

whence

$$\omega \circ \varphi = \omega \circ \varphi | A = \tilde{\psi} | A = \psi.$$

Let us observe that

(20) $$\operatorname{im}\omega = \tilde{\psi}(\operatorname{dom}\tilde{\varphi}),$$

which follows from (19).

9. Restricting now ourselves to almost surjective homomorphisms $\varphi: A \twoheadrightarrow B$, $\psi: A \twoheadrightarrow C$, we define, in analogy to (3):

(21) $(\varphi, B) \succ (\psi, C)$ if and only if there is a — necessarily unique — $\omega: B \to C$ such that $\psi = \omega \circ \varphi$.

Again, $\succ$ is a quasi-ordering in the class of all almost surjective homomorphisms, for which again a Bernstein type equivalence theorem holds:

(22) $(\varphi, B) \succ (\psi, C) \succ (\varphi, B)$ if and only if $(\varphi, B) \cong (\psi, C)$,

with the obvious definition of $\cong$. We then have

(23) $(\varphi, B) \succ (\psi, C)$ if and only if $\ker(\varphi, B) \subset \ker(\psi, C)$.

In particular,

(24) $(\varphi, B) = (\psi, C)$ if and only if $\ker(\varphi, B) = \ker(\psi, C)$.

Hence there is a one-one correspondence between the kernels and the classes of isomorphic almost surjective homomorphisms; again, this one-one correspondence might be interpreted as an order anti-isomorphism.

Concerning the ordering of the kernels, we have the basic fact:

THEOREM 8. *The kernels, i.e., the closed congruence relations in intermediate initial segments of $\hat{A}$, constitute a closure system on $\hat{A} \times \hat{A}$.*

I.e., $\hat{A} \times \hat{A}$ itself is such a kernel (since $\hat{A}$ is complete), and the intersection of an arbitrary non-empty family of kernels R_t $(t \in T)$ is again a kernel.

As a corollary, the kernels constitute a complete lattice, with $\hat{A} \times \hat{A}$ as the greatest, id_A as the least element, and intersection as greatest lower bound (infimum). We call this lattice the *kernel lattice of partial algebra* A, denoted by $\operatorname{Ker}(A)$.

If A is complete, $\operatorname{Ker}(A)$ simply is the congruence lattice of A. In general, $\operatorname{Ker}(A)$ is much larger; the larger, the less complete the partial algebra A is.

In order to make that more precise, one can establish a natural order embedding of the *congruence lattice*, $\operatorname{Con}(A)$, into our kernel lattice $\operatorname{Ker}(A)$. Let us first observe that, as another corollary of Theorem 8, any subset $M \subset \hat{A} \times \hat{A}$ generates a kernel, $\tilde{M}$, which is the intersection, hence the least, of all kernels containing M. Now, if M is an arbitrary congruence relation, R, in A, $\tilde{M} = \tilde{R}$ assumes a very concrete meaning

THEOREM 9. *Let R be a congruence relation in A. Then the kernel generated by R is the kernel of any strong surjective homomorphism $\varphi: A \to B$ that induces R.*

In particular,

$$\tilde{R} = \ker(\pi_R, A/R). \tag{25}$$

For $R_\varphi = R$ implies $R \subset R_{\tilde{\varphi}} = \ker(\varphi, B)$, hence $\tilde{R} \subset \ker(\varphi, B)$ by the minimality of $\tilde{R}$. In order to prove the converse inclusion, take a homomorphism $\psi: A \to C$ such that $R_{\tilde{\psi}} = \ker(\psi, C) = \tilde{R}$. Then

$$R_\varphi = R \subset \tilde{R} \cap (A \times A) = R_\psi$$

by (18). Since $\varphi: A \to B$ was assumed strong and surjective, there is (cf. section 2) a homomorphism $\omega: B \to C$ such that $\psi = \omega \circ \varphi$, which gives

$$\ker(\varphi, B) \subset \ker(\psi, C) = \tilde{R}.$$

As a consequence of (25) and (18) (as already used in the proof), we have

$$R = \tilde{R} \cap (A \times A). \tag{26}$$

This makes it evident that the restriction of the closure operator $\sim$ to congruence relations R in A is an order embedding of $\mathrm{Con}(A)$ into $\mathrm{Ker}(A)$:

(27) $R \subset S$ if and only if $\tilde{R} \subset \tilde{S}$

for any two congruence relations R, S in A. Note that $R = \tilde{R}$ if and only if the congruence relation R is closed.

10. The reasonable thing to do now is to begin a dictionary translating properties of almost surjective homomorphisms $\varphi: A \to B$ into properties of their kernels $R = R_{\tilde{\varphi}}$. Easy examples of this kind:

(28) φ is closed if and only if the field of R equals A;

(29) B is complete if and only if the field of R equals $\hat{A}$.

For φ is closed if and only if $\varphi = \tilde{\varphi}$, i.e., if and only if $\operatorname{dom}\varphi = \operatorname{dom}\tilde{\varphi}$. On the other hand, if B is complete, $\operatorname{dom}\tilde{\varphi} = \hat{A}$. This is the universality property of $\hat{A}$ (cf. the end of section 7). Conversely, if $\operatorname{dom}\tilde{\varphi} = \hat{A}$, B, the homomorphic image (under surjective homomorphism $\tilde{\varphi}$) of the complete algebra $\hat{A}$, must be complete also.

Further easy examples of translation:

(30) φ is injective iff $R \cap (A \times A) = \mathrm{id}_A$, i.e., iff A intersects each class modulo R in at most one point;

(31) φ is surjective (onto B) iff A intersects each class modulo R in at least one point.

Hence φ is bijective iff A is a complete representative system modulo R.

Note that each injective, almost surjective homomorphism $\varphi: A \to B$ is isomorphic with the inclusion homomorphism $\mathrm{id}_A: A \to C$ into a suitable extension C, not necessarily strong, but also generated by the image, i.e., by A itself. (This is due to a more or less obvious set-theoretical substitution procedure, which, as has been shown by Zermelo, can be carried out in a neat "constructive" way. First use of this procedure within algebra seems to have been made by van der Waerden; this has been generalized by Doerge, Pickert, and possibly others.) In short, the kernels R with the property described in (30) are in one-one correspondence with the weak minimal extensions of A, i.e., with their isomorphism classes. For such weak extensions B, C, one may abbreviate

$$B \succ C \text{ instead of } (\mathrm{id}_A, B) \succ (\mathrm{id}_A, C).$$

This means, that there is a homomorphism $\omega: B \to C$ over A, i.e., leaving all elements of A fixed. E.g., $A \succ B$ for each weak extension B. In fact, $(\mathrm{id}_A, A) \succ (\varphi, B)$ for each almost surjective homomorphism $\varphi: A \to B$.

The weak extensions B for which inclusion $\mathrm{id}_A: A \to B$ is a bijection are in fact the enrichments (A, g) of the given algebraic structure f of A. Hence the enrichments of f are in one-one correspondence with those kernels R for which A is a complete representative system.

Speaking of weak extensions, one will quite naturally ask for the strong ones among them.

THEOREM 10. *The homomorphism $\varphi: A \to B$ is strong and injective (i.e., an isomorphism onto* $\mathrm{im}\,\varphi$*) if and only if, for its kernel R, the equation*

$$R \cap (A \times \mathrm{D}A) = \mathrm{id}_A \tag{32}$$

holds.

Here, $\mathrm{D}A$ is understood with respect to $\mathrm{dom}\,\tilde{\varphi} = \bar{A}$, $\mathrm{D}A$ being the subset obtained from A by adding to A all results of single applications of the fundamental operations to sequences in A:

$$\mathrm{D}A := A \cup \bigcup_{i \in I} \mathrm{dom}\,\tilde{\varphi} \cap \hat{f}_i(A^{K_i}),$$

where $\hat{f}_i$ is a fundamental operation of the free completion $\hat{A}$. $\mathrm{D}A - A$, or sometimes the whole of $\mathrm{D}A$, is called the *first Baire class* over A (with respect to partial algebra $\mathrm{dom}\,\tilde{\varphi}$); remember that the transfinite iteration of operator D finally leads to the full closure — (cf. Słomiński [12], also [13]).

Proof. Let $a \in A$, $x \in \mathrm{D}A$, and $(a, x) \in R$, i.e. $\tilde{\varphi}(a) = \tilde{\varphi}(x)$. If $x \in A$, $\varphi(a) = \varphi(x)$, hence $a = x$ if φ is injective. If $x = \hat{f}_i(\mathfrak{a})$, $\mathfrak{a}: K_i \to A$, then

$$\varphi(a) = \tilde{\varphi}(a) = \tilde{\varphi}(x) = g_i(\tilde{\varphi} \circ \mathfrak{a}) = g_i(\varphi \circ \mathfrak{a}),$$

where g_i is a fundamental operation of B. Hence $a = f_i(\mathfrak{a})$ if φ is strong and injective, hence $a = x$, proving (32).

Conversely, if (32) holds, φ is injective by (30). Let $\varphi(a) = g_i(\tilde{\varphi} \circ \mathfrak{a})$, $a \in A$, $\mathfrak{a}: K_i \to A$. Since $\tilde{\varphi}$ is closed, $\tilde{\varphi}(a) = \varphi(a) = \tilde{\varphi}(\hat{f}_i(\mathfrak{a}))$, in particular, $\hat{f}_i(\mathfrak{a}) \in \operatorname{dom} \tilde{\varphi}$, hence $f_i(\mathfrak{a}) \in \mathrm{D}A$, whence $a = \hat{f}_i(\mathfrak{a}) = f_i(\mathfrak{a})$ by (32). This shows that φ is strong.

Hence the strong minimal extensions of A (i.e., their isomorphism classes) are in one-one correspondence with the kernels R that fulfill (32). In particular, combining this with (29), the (strong) minimal completions (their isomorphism classes) are in one-one-correspondence with the congruence relations R of $\hat{A}$ that fulfill (32); this has been proved (in essentially the same way) in [4], Theorem 8.

Again, one might strengthen strongness and injectivity by adding normality, where $\varphi: A \to B$ may be called *normal* if and only if $\operatorname{im} \varphi$ is normal in, i.e., is an initial segment of, B.

THEOREM 11. *The homomorphism $\varphi: A \to B$ is strong, injective, and normal if and only if, for its kernel R, the equation*

$$R \cap (A \times \text{ field } R) = \mathrm{id}_A \tag{33}$$

holds.

The proof is an almost literal copy of the proof given in [4], Theorem 9, for the special case $\varphi = \mathrm{id}_A: A \to B$, B complete.

Hence the strong, normal, minimal extensions of A (i.e., their isomorphism classes) are in one-one correspondence with the kernels R fulfilling (33). Again, restricting that to completions, one gets the mentioned result ([4], Theorem 9).

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