COLLOQUIUM MATHEMATICUM

VOL. XLV 1981 FASC. 2

ON CONTINUA HAVING THREE TYPES OF OPEN SUBSETS

BY

WITOLD BULA (KATOWICE)

We proved [1] that every compact metric space having finitely many types of open subsets is totally disconnected. This implies that every metric continuum has infinitely many types of open subsets and leads to the problem of existence of non-metric continua having only finitely many types of open subsets. In this paper we consider non-metric continua having exactly three types of (non-empty) open subsets. We showed in [1] that such continua are perfectly normal and here we prove that they are indecomposable and homogeneous.

Let X be a *continuum* (i.e., a compact connected Hausdorff space) and let F be its closed subset, $\emptyset \neq F \neq X$. By X/F we mean the space which can be obtained from X by shrinking F to a point which will be denoted by $[F]$. The expression $Q_F: X \to X/F$ denotes the quotient map. The phrase $X-F = U \cup V$ *is a separation* (and all phrases like that) means, besides the equality of sets, that U and V are open in X, non-empty and disjoint. For instance, a point p is a *cut point* of X (p cuts X between q and r) if there exists a separation $X-\{p\} = U \cup V$ (such that $q \in U$ and $r \in V$).

A partially ordered set $(E, <)$ is said to be a *pseudotree* (*tree*) if the set $\{q \in E: q < p\}$ is linearly ordered (well ordered) for each $p \in E$.

Let X be a continuum such that the set R of its cut points is non-empty. For a given $p \in R$ we define a relation $<$ on R by assuming $q < r$ if $q \neq r$ and either $p = q$ or $p \neq q$ and q cuts X between p and r.

The following two lemmas are obvious:

LEMMA 1. *The relation $<$ is a partial order on R. In addition, $(R, <)$ is a pseudotree.*

LEMMA 2. *Let $L \subset R$ be a chain. Then there exists a family of separations $X-\{r\} = U_r \cup V_r$ for $r \in L$ such that*

(*) *$p \in U_r$ for each $r \in L-\{p\}$ and if $r, s \in L$, $r < s$, then $U_r \cup \{r\} \subset U_s$.*

Let Y be a non-degenerate continuum. Then Y contains at least three topologically distinct open subsets. One of these subsets is Y itself. Among non-compact subsets there are connected and disconnected ones; the existence of connected ones follows from the existence of a non-cut point of Y; the union of two disjoint open subsets of Y is an example of a disconnected open subset of Y.

Let X be a continuum having only three types of open subsets. The following lemma follows from the observation as above:

LEMMA 3. (a) *Every (non-empty) connected open subset of X is equal to X or is homeomorphic to the complement of a non-cut point of X. Thus every two connected open proper subsets of X are homeomorphic.*

(b) *Every two disconnected open subsets of X are homeomorphic.*

Let us recall the following two lemmas from [1]:

LEMMA 4. *If U is a connected open subset of X, then U is dense in X.*

LEMMA 5. *The continuum X is perfectly normal.*

COROLLARY 1. *The continuum X has the Souslin property.*

This corollary follows from Lemma 5 because X is compact.

LEMMA 6. *If F is a closed subset of X, $\emptyset \neq F \neq X$, and $X-F$ is connected, then X/F is homeomorphic to X.*

Proof. The set $X-F$ is homeomorphic to the complement of a non-cut point q of X. Hence X/F, being homeomorphic to the Alexandroff compactification of $X-F$, therefore also of $X-\{q\}$, is homeomorphic to X.

LEMMA 7. *The continuum X has no cut point.*

Proof. Assume, to the contrary, that the set R of all cut points of X is non-empty.

(1) *If F is a closed subset of X, $\emptyset \neq F \neq X$, then X/F is homeomorphic to X.*

Indeed, $X-F$ is homeomorphic to the complement of a point in X.

(2) *The set R is dense in X.*

First we show that R has at least two points.

Let $p \in R$. Let $X-\{p\} = U \cup V$ be a separation and let $F \subset V$ be a closed set having a non-empty interior. By Lemma 4 the set $X-F$ is disconnected, so $[F]$ is a cut point of the continuum X/F which is, by (1), homeomorphic to X. Thus X/F has at least two cut points, namely $[F]$ and $Q_F(p)$.

To prove the density of R, assume that U is an open subset of X, $\emptyset \neq U \neq X$. Let $F = X-U$. The space X/F is homeomorphic to X, so it contains a cut point q, $q \neq [F]$. One can see that $Q_F^{-1}(q) \in U$ is a cut point of X.

Now, let $p \in R$ and let $<$ be the partial order on R defined as above.

Let $L \subset R$ be a maximal chain. And let a family of separations $X - \{q\} = U_q \cup V_q$ for $q \in L$ satisfy condition (*) from Lemma 2.

(3) *The chain L has no greatest element.*

Assume that $s \in L$ is the greatest element of L. The set $L \cap V_s$ is empty because $U_r \cup \{r\} \subset U_s \subset X - V_s$ for $r \in L - \{s\}$. But, in view of (2), there is a point $t \in V_s \cap R$. One can see that $L \cup \{t\}$ is a chain and $t \notin L$. This contradicts the maximality of L.

Let $A_L = \bigcap \{V_q \cup \{q\}: q \in L\}$.

(4) *The point $[A_L]$ is a non-cut point of the continuum X/A_L and $[A_L]$ is a point of local connectedness of X/A_L.*

Indeed, L has no greatest element, so

$$X - A_L = \bigcup \{U_q \cup \{q\}: q \in L\}.$$

Hence $X - A_L$ is connected. But $X/A_L - [A_L]$ is homeomorphic to $X - A_L$, so it is connected as well. The point $[A_L]$ is a point of local connectedness of X/A_L because A_L is the intersection of a decreasing family of closed connected sets and is contained in interiors of these sets.

(5) *The continuum X is locally connected at points of the set $X - R$ and is not locally connected at points of the set R.*

This holds because if $q \in X - R$, then $X - \{q\}$ is homeomorphic to the set $X - A_L$, so q is a point of local connectedness of X. On the other hand, by Lemma 4, the continuum X cannot be locally connected. Since for every two points $r, s \in R$ the sets $X - \{r\}$ and $X - \{s\}$ are homeomorphic, the continuum X is not locally connected at any point of R.

(6) *The set R is uncountable.*

Indeed, R is the set of all points at which X is not locally connected, so, by the Frolík theorem [2], the set R contains a non-degenerate continuum of convergence.

(7) *The chain L is well ordered by $<$.*

Let $\emptyset \neq A \subset L$. Assume that A has no least element. Then $p \notin A$ and $\bigcup \{V_q \cup \{q\}: q \in A\} = \bigcup \{V_q: q \in A\}$. Hence the open set $U = \bigcup \{V_q: q \in A\}$ is connected and is not dense because $U \cap U_p = \emptyset$, which contradicts the conclusion of Lemma 4.

(8) *Every antichain in R is at most countable.*

Let B be an antichain containing at least two points. Then $p \notin B$. For each $q \in B$ let $X - \{q\} = W_q \cup Z_q$ be a separation such that $p \in W_q$. We show that for distinct points $q, r \in B$ the sets Z_q and Z_r are disjoint. Let $q, r \in B$, $q \neq r$. The point q does not cut X between p and r, so $p, r \in W_q$; analogously, $p, q \in W_r$. Thus $r \notin Z_q \cup \{q\}$ and $q \in W_r$, whence $Z_q \cup \{q\} \subset W_r \subset X - Z_r$. By Corollary 1, the space X has the Souslin property, so B is at most countable.

(9) *There exists a maximal chain $M \subset R$ which is isomorphic to the ordinal $\gamma \geqslant \omega+1$.*

Indeed, R is an uncountable tree every antichain of which is at most countable.

To complete the proof, let $M = \{q_\alpha: \alpha < \gamma\}$, where $\gamma \geqslant \omega+1$, be a chain in R. Let a family of separations $X - \{q_\alpha\} = U_\alpha \cup V_\alpha$ for $\alpha < \gamma$ satisfy condition (*) from Lemma 2. Consider the set $W = \bigcup\{U_n \cup \{q_n\}: n < \omega\}$. The set W is connected and, for $q_n \in U_{n+1}$, open. But $W \subset U_\omega \subset X - V_\omega$, so W is not dense. This contradicts the conclusion of Lemma 4.

COROLLARY 2. *The continuum X is homogeneous.*

Proof. Indeed, no point cuts X, and so, by Lemma 3 (a), for every two points $p, q \in X$ the sets $X - \{p\}$ and $X - \{q\}$ are homeomorphic.

COROLLARY 3. *The continuum X has no point of local connectedness.*

Proof. Indeed, by Lemma 4, the continuum X has a point at which it is not locally connected. Hence, by Corollary 2, it is not locally connected at any point.

LEMMA 8. *If U is an open disconnected subset of X, then every component of U has an empty interior.*

Proof. Assume that the set U has a component with a non-empty interior. Hence every open non-empty subset of X has such a component and we can define by induction a sequence

$$X \supset W_1 \supset K_1 \supset \operatorname{int} K_1 \supset \operatorname{cl} W_2 \supset W_2 \supset K_2 \supset \operatorname{int} K_2 \supset \ldots,$$

where W_1 is an arbitrary open non-dense subset of X, W_n is an open set, and K_n is its component with a non-empty interior.

Let $F_0 = \bigcap\{K_n: n < \omega\}$. The set F_0 is a continuum, $F_0 \neq X$. The set $X - F_0$ cannot be connected. If it were, then, by Lemma 6, the continuum X/F_0 would be homeomorphic to X, and X/F_0 would be locally connected at the point $[F_0]$, which contradicts the conclusion of Corollary 3.

The choice of a non-dense open set W_1 was arbitrary, so in every open subset of X we can find a continuum F such that $X - F$ is disconnected.

Let $X - F_0 = U_0 \cup V_0$ be a separation. Let $F_1 \subset V_0$ be a continuum such that $X - F_1$ is disconnected. Choose a separation $X - F_1 = U_1 \cup V_1$ such that $U_0 \cup F_0 \subset U_1$. By induction we can find a sequence of disjoint continua $F_1, F_2, \ldots$ such that $X - F_n$ is disconnected, and a sequence of separations $X - F_n = U_n \cup V_n$ such that $U_n \cup F_n \subset U_{n+1}$. Let $F = \bigcap\{V_n \cup F_n: n < \omega\}$. The set F is a continuum. For $F_n \subset U_{n+1}$ we have $F_n \cap F = \emptyset$ and $F = \bigcap\{V_n: n < \omega\}$. Thus the set $X - F$ is connected. The argumentation similar to the previous one leads to a contradiction.

LEMMA 9. *The continuum X is indecomposable.*

Proof. Let $K \subset X$ be a continuum, $\emptyset \neq K \neq X$. Let $p \in X - K$. Choose open sets U and V such that $K \subset U$, $p \in V$, and $U \cap V = \emptyset$. By

Lemma 5 the set U is disconnected. The continuum K is contained in the component of U, so, by Lemma 8, it has an empty interior.

The lemmas give the announced theorem.

THEOREM. *If a Hausdorff continuum has only three topologically distinct non-empty open subsets, then it is non-metrizable, perfectly normal, indecomposable, and homogeneous.*

The answer in the affirmative to the following problem would imply that Hausdorff continua having three types of open subsets did not exist.

PROBLEM (**P 1235**). Let X be a Hausdorff continuum having the Souslin property or, in particular, being perfectly normal. Let K be a continuum of convergence of X. Is it true that all save a countable number of points of K are non-local separating points of X?

REFERENCES

[1] W. Bula, *On compact Hausdorff spaces having finitely many types of open subsets*, Colloquium Mathematicum 41 (1979), p. 211-214.

[2] Z. Frolík, *Concerning topological convergence of sets*, Czechoslovak Mathematical Journal 10 (1960), p. 160-180.

Reçu par la Rédaction le 22. 8. 1978;
en version modifiée le 30. 3. 1979