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METRIC CHARACTERIZATIONS OF BANACH SPACES

BY

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1. Introduction. It is well known that the length of the segment joining the midpoints of two sides of a triangle is less than one-half, equal to one-half, or greater than one-half the third side in hyperbolic, euclidean, and elliptic geometry, respectively. In fact, Young [8] has shown that the parallel postulate can be replaced by the above-mentioned property in order to obtain the respective geometries.

Andalafte and Blumenthal gave a complete metrization of Young's condition for euclidean geometry as follows:

THE YOUNG POSTULATE. *If p, q and r are points of a metric space M and q' and r' are the midpoints of p and q, and p and r, respectively, then* $q'r' = qr/2$.

They then showed that a complete metric space with unique lines is generated by a Banach space if and only if it satisfies the Young Postulate.

It is quite natural to ask if in a complete metric space with unique lines the segment joining the midpoints of two sides of a triangle could be a constant multiple $k \neq 1/2$ of the third side. We show in this paper that the answer is no, and surprisingly this is a direct consequence of the triangle inequality. It follows immediately that if q' and r' are chosen on the lines joining p and q, and p and r, respectively, such that $pq'/pq = pr'/pr = \lambda$ and if a constant k exists such that these conditions imply $q'r'/qr = k$, then $k = \lambda$. Finally, one is led to ask:

Does the property ($pq'/pq = pr'/pr = \lambda$ implies $q'r'/qr = \lambda$) characterize Banach spaces among the class of complete metric spaces with unique lines? (**P 823**)

Although this question remains open, we show in this paper that the property ($pq'/pq = pr'/pr = \lambda$ and $pq^*/pq = pr^*/pr = |1-\lambda|$ imply $q'r'/qr = \lambda$ and $q^*r^*/qr = |1-\lambda|$) does characterize Banach spaces among the class of complete metric spaces with unique lines. The technique will be to show that if a complete metric space with unique lines has the latter property for some λ, then it also has this property for $\lambda = 1/2$ and the afore-mentioned result of Andalatte and Blumenthal then yields the characterization.

2. The Ratio Property in M. Let M be a complete metric space with a unique line joining any pair of its distinct points. For convenience, if p, q are distinct points of M, we denote the line containing them by $L(p, q)$. The segment with end points p and q will be denoted by $S(p, q)$, and the fact that r is a point between p and q will be denoted by prq. We now assume M has the following Ratio Property:

THE RATIO PROPERTY. *There exists a positive constant k such that, for each triple p, q, r of non-collinear points of a metric space M, if q' and r' are points of $S(p, q)$ and $S(p, r)$, respectively, with $pq'/pq = pr'/pr = 1/2$, then $q'r'/qr = k$.*

LEMMA 2.1. *The number k in the Ratio Property is* 1/2.

Proof. Let p, q, r be non-collinear points of M such that $pq = pr = 1$. We define sequences of points $\{q_n\}$, $\{r_n\}$ on $L(p, q)$ and $L(p, r)$, respectively, as follows. Let q_1 and r_1 be points on $S(p, q)$ and $S(p, r)$, respectively, such that $pq_1/pq = pr_1/pr = 1/2$. If q_i and r_i $(i \geqslant 1)$ have been chosen on the respective segments $S(p, q_{i-1})$ and $S(p, r_{i-1})$, we let q_{i+1} and r_{i+1} be the points on $S(p, q_i)$ and $S(p, r_i)$, respectively, such that $pq_{i+1}/pq_i = pr_{i+1}/pr_i = 1/2$. Clearly, $\lim q_i = \lim r_i = p$.

For the indirect argument suppose $k < 1/2$. Then there exists a positive number ε such that $k = 1/2 - \varepsilon$. Now, for each i,

$$q_i r_i/pr_i = [k^i \cdot qr]/[(1/2)^i \cdot pr] = [(1/2 - \varepsilon)/(1/2)]^i \cdot qr.$$

Consequently, $\lim q_i r_i/pr_i = 0$, so there exists a positive integer n such that $q_n r_n < pr_n = pq_n$. For each i, let q_i' be the point for which $q_i' p q_i$ holds and $q_i' p = pq_i$. It follows, as above, that $q_n' r_n < pr_n = pq_n'$. Thus $q_n' r_n + q_n r_n < q_n' p + pq_n = q_n' q_n$ contrary to the triangle inequality. Therefore, $k \geqslant 1/2$.

The assumption that $k > 1/2$ implies the existence of a positive number ε which satisfies $k = 1/2 + \varepsilon$. Now, for each i,

$$q_i r_i/pr_i = [k^i \cdot qr]/[(1/2)^i \cdot pr] = [(1/2 + \varepsilon)/(1/2)]^i \cdot qr.$$

In this case, $\lim q_i r_i/pr_i = \infty$, so there exists a positive integer n for which $q_n r_n > 2pr_n$. By the definition of the sequences $\{q_n\}$ and $\{r_n\}$, $pq_n = pr_n$. Thus $q_n p + pr_n = 2pr_n < q_n r_n$ which contradicts the triangle inequality. Therefore, $k = 1/2$.

An application of the Andalafte and Blumenthal Theorem [1] gives the following

THEOREM 2.2. *A complete metric space with a unique metric line joining any pair of its distinct points is generated by a Banach space if and only if it has the Ratio Property.*

3. A generalization of the Ratio Property. In this section we introduce a more general ratio property and once again obtain a characterization of Banach spaces. Throughout this section M will denote a complete metric space with a unique line joining each pair of its distinct points. We will also assume M has the General Ratio Property, which can be stated as follows:

THE GENERAL RATIO PROPERTY. *There is some positive number λ, $\lambda < 1$, for which positive numbers $k(\lambda)$ and $\bar{k}(\lambda)$ (depending only on λ) exist such that, for each triple of non-collinear points p, q, r of a metric space M, if q', q^* and r', r^* are points on $S(p, q)$ and $S(p, r)$, respectively, such that $pq'/pq = pr'/pr = \lambda$ and $pq^*/pq = pr^*/pr = 1-\lambda$, then $q'r'/qr = k(\lambda)$ and $q^*r^*/qr = \bar{k}(\lambda)$.*

Since one of the numbers λ and $1-\lambda$ is less than or equal to $1/2$, we will assume $\lambda \geqslant 1/2$.

The proof of the following lemma is quite similar to that of Lemma 2.1 and is thus omitted.

LEMMA 3.1. *The numbers $k(\lambda)$ and $\bar{k}(\lambda)$ in the statement of the General Ratio Property satisfy the equations $k(\lambda) = \lambda$ and $\bar{k}(\lambda) = 1-\lambda$.*

THEOREM 3.2. *Let p, q, r be non-collinear points of M. If q', q^* are points on $S(p, q)$ and r', r^* are points on $S(p, r)$ for which $pq'/pq = pr'/pr = \lambda$, and $pq^*/pq = pr^*/pr = 1-\lambda$, then for each point t on $L(q, r)$ there are points t', t^* on $L(q', r')$ and $L(q^*, r^*)$, respectively, such that t', t^* lie on $S(p, t)$ and $pt'/pt = \lambda$ and $pt^*/pt = 1-\lambda$.*

Proof. Letting t' and t^* denote the points on $S(p, t)$ for which $pt'/pt = \lambda$ and $pt^*/pt = 1-\lambda$, we see, upon applying the General Ratio Property and Lemma 3.1 to the triples of non-collinear points p, q, t and p, t, r, that $q'\,t' = \lambda\cdot qt$, $q^*t^* = (1-\lambda)qt$ and $t'r' = \lambda\cdot tr$, $t^*r^* = (1-\lambda)tr$. But $q'r' = \lambda\cdot qr$ and $q^*r^* = (1-\lambda)qr$, so it follows immediately that any betweeness relation satisfied by q, r and t is also satisfied by the triples q', r' and t' and q^*, r^*, and t^*, and hence t', t^* are on the lines $L(q', r')$ and $L(q^*, r^*)$, respectively.

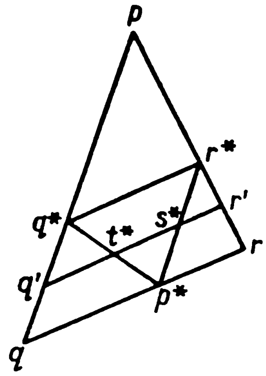
COROLLARY. *Suppose p, q, r are non-collinear points of M and suppose q', q^* are points on $S(p, q)$, r', r^* are points on $S(p, r)$ with $pq'/pq = pr'/pr = \lambda$ and $pq^*/pq = pr^*/pr = 1-\lambda$. If t, t', t^* are points on $S(q, r)$, $S(q'r')$ and $S(q^*, r^*)$, respectively, such that $qt/qr = q't'/q'r' = \lambda$ and $q^*t^*/q^*r^* = 1-\lambda$, then t' and t^* lie on $S(p, t)$.*

LEMMA 3.3. *If p, q, r are non-collinear points of M and if q', r' are points on $S(p, q)$ and $S(p, r)$, respectively, such that $pq'/pq = pr'/pr = 2\lambda-\lambda^2$, then $q'r'/qr = 2\lambda-\lambda^2$.*

Proof. Let q^*, r^*, p^* be points of the segments $S(p, q)$, $S(p, r)$ and $S(q, r)$, respectively, such that $pq^*/pq = pr^*/pr = qp^*/qr = \lambda$. It follows that $q^*q'/q^*q = r^*r'/r^*r = \lambda$. Letting t^*, s^* be points on $S(q^*, p^*)$ and

$S(r^*, p^*)$, respectively, whose distances satisfy $q^*t^*/q^*p^* = r^*s^*/r^*p^* = \lambda$, from the General Ratio Property and Lemma 3.1, we have $q't^* = \lambda \cdot qp^* = \lambda^2 \cdot qr$, $t^*s^* = (1-\lambda)q^*r^* = (\lambda-\lambda^2)qr$ and $s^*r' = \lambda \cdot p^*r = (\lambda-\lambda^2)qr$. Thus

$$q't^* + t^*s^* + s^*r' = (\lambda^2 + \lambda - \lambda^2 + \lambda - \lambda^2)qr = (2\lambda - \lambda^2)qr,$$



and it suffices to show q', t^*, s^*, r' lie on a line and in that order.

By the Corollary to Theorem 3.2, $L(r, q^*)$ and $L(r^*, p^*)$ have the point s^* in common. From Theorem 3.2, $q^*s^*/q^*r = q^*q'/q^*q = \lambda$, and so $S(q', s^*)$ contains the point t^*. In the same manner, it is seen that $S(r', t^*)$ contains the point s^*. Thus q', t^*, s^* and r' satisfy the betweeness relations $q't^*s^*$ and t^*s^*r'; so q', t^*, s^*, r' lie on the line $L(q', r')$ in that order, and the proof is complete.

LEMMA 3.4. *Let p, q, r be non-collinear points of M. Suppose q', r' are points of $S(p, q)$ and $S(p, r)$, respectively, such that $pq'/pq = pr'/pr = q'r'/qr = a$. If q^*, r^* are points on $L(p, q)$ and $L(p, r)$, respectively, such that $q'q^*q$ and $r'r^*r$ hold and $q'q^*/q'q = r'r^*/r'r = \lambda$, then $pq^*/pq = pr^*/pr = q^*r^*/qr = a + \lambda(1-a)$.*

Proof. By hypothesis,

$$pq^*/pq = (pq' + q'q^*)/pq = (a \cdot pq + \lambda \cdot q'q)/pq = (apq + \lambda(1-a)pq)/pq$$
$$= a + \lambda(1-a).$$

Similarly,

$$pr^*/pr = a + \lambda(1-a).$$

In order to prove $q^*r^*/qr = a + \lambda(1-a)$, we proceed as in the proof of Lemma 3.3. Let m^* be a point on $S(q, r)$ such that $q^*m^*/qr = \lambda$ and let t^*, s^* be points on $S(q', m^*)$ and $S(r', m^*)$, respectively, such that $q't^*/q'm^* = r's^*/r'm^* = \lambda$. Then, by the General Ratio Property and Lemma 3.1,

$$q^*t^* = \lambda \cdot qm^* = \lambda^2 \cdot qr, \tag{1}$$

$$s^*t^* = (1-\lambda)q'r' = a(1-\lambda)qr, \tag{2}$$

$$s^*r^* = \lambda \cdot rm^* = \lambda(1-\lambda)qr = (\lambda - \lambda^2)qr. \tag{3}$$

Thus $q^*t^* + s^*t^* + s^*r^* = [\lambda^2 + a(1-\lambda) + \lambda - \lambda^2]qr = [a + \lambda(1-a)]qr$. That t^*, s^* lie on $S(q^*, r^*)$ in that order follows exactly as in Lemma 3.3. We then have

$$q^*r^* = q^*t^* + t^*s^* + s^*r^* = [a + \lambda(1-a)]qr,$$

which completes the proof.

Now suppose p, q, r is any triple of non-collinear points of M. Letting q_1, r_1 denote the points of $S(p, q)$ and $S(p, r)$, respectively, such that $pq_1/pq = pr_1/pr = \lambda$, by the General Ratio Property and Lemma 3.1, $q_1r_1/qr = \lambda$. If q_2, q_3, r_2, r_3 are points on the segments $S(p, q_1)$, $S(q_1, q)$, $S(p, r_1)$ and $S(r_1, r)$, respectively, such that $pq_2/pq_1 = pr_2/pr_1 = q_1q_3/q_1q = r_1r_3/r_1r = \lambda$, by the General Ratio Property and Lemma 3.3, $q_2r/qr = pq_2/pq = pr_2/pr = \lambda^2$ and $pq_3/pq = pr_3/pr = q_3r_3/qr = 2\lambda - \lambda^2$. Letting $q_4, q_5, q_6, q_7, r_4, r_5, r_6, r_7$ denote points on the segments $S(p, q_2)$, $S(q_2, q_1)$, $S(q_1, q_3)$, $S(q_3, q)$, $S(p, r_2)$, $S(r_2, r_1)$, $S(r_1, r_3)$ and $S(r_3, r)$, respectively, such that $pq_4/pq_2 = pr_4/pr_2 = q_2q_5/q_2q_1 = r_2r_5/r_2r_1 = q_1q_6/q_1q_3 = r_1r_6/r_1r_3 = q_3q_7/q_3q = r_3r_7/r_3r = \lambda$; then from the General Ratio Property and Lemma 3.4 it follows that $pq_4/pq = pr_4/pr = q_4r_4/qr$, $pq_5/pq = pr_5/pr = q_5r_5/qr$, $pq_6/pq = pr_6/pr = q_6r_6/qr$, and $pq_7/pq = pr_7/pr = q_7r_7/qr$. Continuing, inductively, to choose points in this manner, we obtain sequences $\{q_n\}$ and $\{r_n\}$ of $S(p, q)$ and $S(p, r)$, respectively, such that the sequence $\{q_n\}$ is dense in $S(p, q)$, the sequence $\{r_n\}$ is dense in $S(p, r)$, the numbers $pq_n/pq = pr_n/pr$ are dense in the interval $[0, 1]$, and whenever $pq_n/pq = pr_n/pr = \nu$, it follows that $q_nr_n/qr = \nu$. From the continuity of the metric it follows that if $pq'/pq = pr'/pr = \nu$ $(0 \leqslant \nu \leqslant 1)$, then $q'r'/qr = \nu$. In particular, if $pq'/pq = pr'/pr = 1/2$, then $q'r'/qr = 1/2$, and we thus have the following theorem:

THEOREM 3.4. *A complete metric space with a unique metric line joining any pair of points is generated by a Banach space if and only if it satisfies the General Ratio Property.*

4. The General Ratio Property for $\lambda > 1$. We now consider the General Ratio Property as in Section 3, except we let λ be greater than 1 and we replace $1 - \lambda$ by $\lambda - 1$. We will refer to this ratio property as the General Ratio Property $(\lambda > 1)$. Again we assume M is a complete metric space with a unique metric line joining any pair of its distinct points, which has the General Ratio Property $(\lambda > 1)$.

By making slight changes in the proofs of Lemmas 2.1 and 2.2, it is easily seen that they are both valid in M. We thus have the following

THEOREM 4.1. *If p, q, r are non-collinear points of M and if q', q^* and r', r^* are points on the metric rays $R(p, q)$ and $R(p, r)$, respectively, such that $pq'/pq = pr'/pr = \lambda$ and $pq^*/pq = pr^*/pr = \lambda - 1$, then $q'r'/qr = \lambda$ and $q^*r^*/qr = \lambda - 1$.*

It suffices to show that if p, q, r are non-collinear points of M and if q', q^* are points on $S(p, q)$ and r', r^* are points on $S(p, r)$ such that $pq'/pq = pr'/pr = 1/\lambda$ and $pq^*/pq = pr^*/pr = 1 - (1/\lambda) = (\lambda - 1)/\lambda$, then $q'r'/qr = 1/\lambda$ and $q^*r^*/qr = (\lambda - 1)/\lambda$. If $pq'/pq = pr'/pr = 1/\lambda$, then $pq/pq' = pr/pr' = \lambda$, so $qr/q'r' = \lambda$ and, consequently, $q'r'/qr = 1/\lambda$. The other condition is not quite so obvious and we state it as

LEMMA 4.2. *If p, q, r are non-collinear points of M and if q', r' are points on the segments $S(p, q)$ and $S(p, r)$, respectively, such that $pq'/pq = pr'/pr = (\lambda-1)/\lambda$, then $q'r'/qr = (\lambda-1)/\lambda$.*

Proof. Choose points q^*, r^* on the respective segments $S(q', q)$ and $S(r', r)$ such that $pq'/pq^* = pr'/pr^* = 1/\lambda$. Then $pq^* = \lambda pq' = (\lambda-1)pq$ and $pr^* = \lambda pr' = (\lambda-1)pr$. Thus $q'r' = (1/\lambda)q^*r^*$ by the above-mentioned. But $pq^* = (\lambda-1)pq$ and $pr^* = (\lambda-1)pr$ implies $q^*r^* = (\lambda-1)qr$ by the General Ratio Property $(\lambda > 1)$. Thus $q'r' = (1/\lambda)q^*r^* = (1/\lambda)(\lambda-1)qr = [(\lambda-1)/\lambda]qr$.

THEOREM 4.3. *A complete metric space with a unique line joining any pair of its distinct points is a normed linear space (Banach space) if and only if it has the General Ratio Property $(\lambda > 1)$.*

Proof. By Lemma 4.2, if M has the General Ratio Property $(\lambda > 1)$, then M has the General Ratio Property of Section 3, and hence M is a normed linear space (Theorem 3.4). The converse is clear.

It is worth noting that in the Young Postulate $\lambda = 1/2$, and so $\lambda = 1-\lambda$; thus the second ratio in the General Ratio Property is really postulated in the Young Postulate. It would be interesting to know if the property "$pq^*/pq = pr^*/pr = 1-\lambda$ implies $q^*r^*/qr = 1-\lambda$" can be proved from the relation "$pq'/pq = pr'/pr = \lambda$ implies $q'r'/qr = \lambda$". If so, this would strengthen Theorem 3.4 and there would be a similar strengthening of Theorem 4.3. (**P 824**)

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