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A CLASSIFICATION OF SECONDARY COHOMOLOGY OPERATIONS

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1. Cohomology operations and, more generally, natural transformations between functors representable as homotopy class functors $[\ ,A]$ and $[\ ,B]$ are classified (see, for example, 3.3 of [2]) by their correspondence with $[A, B]$, the set of homotopy classes of maps from A to B. We give here a classification of secondary cohomology (in the sense of Chapter 1, Section 3, of [1]) and homotopy operations by defining an equivalence relation on homotopy classes of maps $\psi\colon P_f \to C$, where $P_f \to A$ is induced from the path fibration $PB \to B$ by a map $f\colon A \to B$.

$$\begin{tikzcd} & P_f \arrow[r, "\psi"] \arrow[d, "p"] & C \\ X \arrow[ur, "\xi"] \arrow[r, "u"'] & A \arrow[r, "f"'] & B \end{tikzcd}$$

Let $\langle\psi\rangle$ be the equivalence class of ψ and Ψ the secondary operation determined by ψ.

THEOREM. *There is a bijection, given by $\langle\psi\rangle \to \Psi$, between equivalence classes of maps and secondary operations defined with respect to the triple $(A, B; C)$.*

This result extends to higher-order operations and dualizes in the usual way.

2. We consider based spaces, maps and homotopies. Let $p = p_f$: $P_f \to A$ be the fibre space induced by a map $f\colon A \to B$. Thus

$$P_f = \{(a, l) \in A \times PB\colon f(a) = l(1)\}.$$

For each class $u\colon X \to A$, denote by $[X, P_f\colon u]$ the set of homotopy classes in $[X, P_f]$ of liftings of u; thus

$$[X, P_f\colon u] = \{\xi \in [X, P_f]\colon p_*(\xi) = u\}.$$

Given a class $\psi\colon P_f \to C$, define a secondary operation Ψ by

$$\Psi(u) = \psi_*[X, P_f\colon u].$$

The operation Ψ is defined on the kernel of $f_*\colon [X, A] \to [X, B]$ and as values takes subsets of $[X, C]$.

Let $f, g\colon A \to B$ be maps and let $\psi_1\colon P_f \to C$ and $\psi_2\colon P_g \to C$ be homotopy classes.

Definition. The classes ψ_1 and ψ_2 are *equivalent* if there are maps $V_1\colon P_g \to P_f$ and $V_2\colon P_f \to P_g$ over A satisfying $\psi_1 \circ \{V_1\} = \psi_2$ and $\psi_2 \circ \{V_2\} = \psi_1$.

Denote the equivalence class of $\psi\colon P_f \to C$ by $\langle\psi\rangle$.

3. Secondary operations are partially ordered by inclusion, that is $\Psi_1 \leqslant \Psi_2$ if and only if $\Psi_1(u) \subseteq \Psi_2(u)$ for all u. We now show how this order is determined by a relation between the defining classes $\psi_1\colon P_f \to C$ and $\psi_2\colon P_g \to C$. Let $f, g\colon A \to B$ be maps, and $f_*, g_*\colon [X, A] \to [X, B]$ the induced functions.

LEMMA 1. *There is a map $P_f \to P_g$ over A if and only if* $\ker f_* \subseteq \ker g_*$ *for all X.*

Proof. (i) Suppose $\ker f_* \subseteq \ker g_*$. Then $g_*(\{p_f\}) = f_*(\{p_f\}) = 0$, and hence $p_f\colon P_f \to A$ can be lifted to a map $P_f \to P_g$.

(ii) Suppose that $V\colon P_f \to P_g$ is over A and that the composite

$$X \xrightarrow{u} A \xrightarrow{f} B$$

is trivial. Then there is a lifting $U\colon X \to P_f$ of u and $VU\colon X \to P_g$ is a lifting of u; thus

$$X \xrightarrow{u} A \xrightarrow{g} B$$

is trivial.

Let the classes $\psi_1\colon P_f \to C$ and $\psi_2\colon P_g \to C$ determine secondary operations Ψ_1 and Ψ_2.

LEMMA 2. *The operations Ψ_1 and Ψ_2 satisfy $\Psi_1 \leqslant \Psi_2$ if and only if there is a map $V\colon P_f \to P_g$ over A such that $\psi_2 \circ \{V\} = \psi_1$.*

Proof. (i) Suppose $\Psi_1 \leqslant \Psi_2$. Then, by Lemma 1, there is a map $P_f \to P_g$ over A. This means that the composite

$$P_f \to A \xrightarrow{g} B$$

is trivial. Now $\Psi_1\{p_f\} \subseteq \Psi_2\{p_f\}$, that is

$$\psi_{1*}[P_f, P_f\colon \{p_f\}] \subseteq \psi_{2*}[P_f, P_g\colon \{p_f\}].$$

Thus there is a map $V\colon P_f \to P_g$ over A such that $\psi_{1*}(1) = \psi_{2*}\{V\}$, that is $V^*(\psi_2) = \psi_1$.

(ii) Let $V\colon P_f \to P_g$ be a map over A which satisfies $\psi_2 \circ \{V\} = \psi_1$. Then $V_*[X, P_f\colon u] \subset [X, P_g\colon u]$, and hence

$$\Psi_1(u) = \psi_{1*}[X, P_f\colon u] = \psi_{2*} V_*[X, P_f\colon u] \subset \psi_{2*}[X, P_g\colon u] = \Psi_2(u).$$

Since $\Psi_1 = \Psi_2$ if and only if both $\Psi_1 \leqslant \Psi_2$ and $\Psi_2 \leqslant \Psi_1$, the Theorem is now immediate.

REFERENCES

[1] F. P. Peterson and N. Stein, *Secondary cohomology operations: two formulas*, American Journal of Mathematics 81 (1959), p. 281-305.

[2] N. E. Steenrod, *Cohomology operations*, Symposium Internacional de Topologia Algebraica (Mexico), UNESCO (1958), p. 165-185.

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