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On a characterization of the Euclidean sphere *

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The aim of this paper is to prove a theorem of S. Gołąb. Proved as early as 1942, it has not been published so far, because the original proof of the author was too complicated.

To formulate this theorem it is necessary to introduce a certain notion, namely the notion of a B-straight curve and a B-plane curve lying on a surface embedded in a three-dimensional Euclidean space. This notion had been explained (in a more general manner than it is necessary for our considerations) in the paper of S. Gołąb *Généralisations des équations de Bonnet-Kowalewski dans l'espace à un nombre arbitraire de dimensions*, Ann. de la Soc. Pol. de Math. 22 (1949), pp. 97-156, 128-138, and also in the paper of K. Tryuk *On B-curvatures of curves on surfaces of the Euclidean space*, Ann. Pol. Math. 2 (1955), p. 15.

The above-mentioned theorem may be formulated in a simple way:

If a surface S is such that each of its geodesics is a B-plane curve and is not a B-straight one, then S is a part of the sphere.

DEFINITION 1. A surface S of regularity class C^n $(n \geqslant 1)$ determined by vector equation
$$\bar{v} = \bar{v}(u_1, u_2)$$
or by parametric equations
$$x_i = \varphi_i(u_1, u_2) \qquad (i = 1, 2, 3)$$
is called a *regular* one if for every point the rank of the matrix
$$\left\| \frac{\partial x_i}{\partial u_\lambda} \right\| \qquad (i = 1, 2, 3;\ \lambda = 1, 2)$$
is equal to two.

DEFINITION 2. I. A curve C lying on a surface S of class C^1 is called *B-straight* if the plane tangent to S along C remains constantly perpendicular to a fixed direction. II. A curve C lying on a surface S of class C^1 is called *B-plane* if the plane tangent to S along C is constantly parallel

* The author wishes to express his thanks to Miss K. Tryuk for calling his attention to this problem.

to a fixed direction (i.e. the tangent plane circumscribes a cylindrical surface).

THEOREM. *If every geodesic of a regular surface S of class C^3 is a B-plane curve and is not B-straight, then the surface S is a part of the sphere.*

First we shall prove five lemmas.

LEMMA 1. *If in a neighbourhood of point P on a regular surface S of class C^3 the Gaussian curvature K is identically equal to zero, then in this neighbourhood of the point P there exist geodesics which are simultaneously B-straight.*

Proof. The above assumptions (1) imply that through every point of the neighbourhood in question passes one segment of a straight line lying on S (it is a segment of a generatrix of a developable surface). This segment is a geodesic. Since the plane tangent to S along it is fixed, this geodesic is a B-straight line.

LEMMA 2. *If the set of ombilics of the surface S of class C^2 is dense, then every point of S is an ombilic.*

Proof. Since the function

$$R(u_1, u_2) = \operatorname{rank} \|g_{\lambda\mu}, h_{\lambda\mu}\| \qquad (\lambda, \mu = 1, 2)$$

where $g_{\lambda\mu}, h_{\lambda\mu}$ signify the coefficients of the first and second differential form of the surface respectively, is low semicontinuous, the equation $R(u_1, u_2) = 0$, true on the dense set, implies the identity $R(u_1, u_2) \equiv 0$.

LEMMA 3. *If S' denotes the spherical mapping of the surface S (of the class C^3) and if a point P of S is elliptic ($K > 0$) or hyperbolic ($K < 0$) of S, then the mapping of a B-plane curve C lying in the neighbourhood of P is a great-circle arc C' lying in a neighbourhood of point P' on S' and, conversely, every great-circle arc C' in a neighbourhood of the point P' is assigned to a B-plane curve C lying in a neighbourhood of P on S.*

Proof. Note that the spherical mapping of the neighbourhood of an elliptic ($K > 0$) or hyperbolic ($K < 0$) point P is a one-to-one mapping (2). Hence in the neighbourhoods of points P and P', to every curve C lying on S corresponds one curve C' on S' and vice versa, to every curve C' corresponds one curve C.

1. Let C be a B-plane curve of S lying in the neighbourhood of the point P. From the fact of the curve C being a B-plane curve it is seen that the unit normal vector to the surface S along the curve C remains

(1) A. Hoborski, *Geometria różniczkowa, cz. II. Teoria powierzchni i zarys teorii tensorów*, Kraków 1928, p. 122. The proof of the theorem stating that a surface of class C^2, having the curvature of Gauss identically null, is developable, leaves much to be desired with respect to accuracy.

(2) D. Hilbert and S. Cohn-Vossen, *Geometria poglądowa*, Warszawa 1956, p. 183.

parallel to a fixed plane π. When the vector undergos a parallel displacement so that its one end is identical with the centre O' of the spherical surface S', then its other end traces a curve C' on surface S' which is the mapping of curve C. Since the above-mentioned vector is constantly parallel to the plane π and has the fixed origin O', then it must turn in a plane π_1 which is parallel to the plane π and passes throught the point O'. The curve C' is a great-circle arc as a part of the intersection of the unit sphere and plane π_1.

2. If C' is a great-circle arc of the spherical surface S' in the neighbourhood of point P', then the normal vector to the surface S along curve C whose image is C' is constantly parallel to the plane of arc C'. Curve C is therefore B-plane in the neighbourhood of point P whose image is point P'.

LEMMA 4. *If every geodesic C of a regular surface S of class C^3 is a B-plane curve and if it is not a B-straight curve, then the set of elliptic or hyperbolic points ($K \neq 0$) of S is a dense set.*

Proof. Suppose that the set in question is not dense. Thus there is a point P and a such neighbourhood of P that the curvature K of S in this neighbourhood of P is identically equal to zero. From lemma 1 we conclude that in the neighbourhood of the point P there exist geodesics which are B-straight. This conclusion contradicts the assumption, whence our set is a dense one.

LEMMA 5. *If every geodesic of a regular surface S of class C^3 is a B-plane curve and if a point P is elliptic or hyperbolic ($K \neq 0$), then:* a) *the spherical mapping of any neighbourhood of the point P is necessarily a geodesic mapping,* b) *the Christoffel symbols of the second kind satisfy the following four conditions:*

$$(1)\qquad \begin{aligned} &1.\ \left\{ {2 \atop 11} \right\}' = \left\{ {2 \atop 11} \right\}, \qquad 3.\ 2\left\{ {1 \atop 12} \right\}' - \left\{ {2 \atop 22} \right\}' = 2\left\{ {1 \atop 12} \right\} - \left\{ {2 \atop 22} \right\},\\ &2.\ \left\{ {1 \atop 22} \right\}' = \left\{ {1 \atop 22} \right\}, \qquad 4.\ 2\left\{ {2 \atop 12} \right\}' - \left\{ {1 \atop 11} \right\}' = 2\left\{ {2 \atop 12} \right\} - \left\{ {1 \atop 11} \right\}. \end{aligned}$$

Proof. a) 1. Since the geodesic C is B-plane, by lemma 3 its image C' on S' is a great-circle arc, whence it is geodesic on S' (the set of geodesics on a sphere is identical with the set of great-circles). 2. If C' is geodesic from a neighbourhood of the point P', then it is a great-circle arc of S', and by virtue of lemma 3 it is an image of a certain B-plane curve C lying in a neighbourhood of the point P. Then it will be shown that C is a geodesic on the surface S. Suppose that it is not so. Let us take any point P_1 lying on curve C in a neighbourhood of point P and let us draw throught P_1 the geodesic C_1 tangent to C at point P_1. The geodesic C_1 is by assumption B-plane. Its image on S' is a great-circle arc C_1' of the sphere. The arc C_1' is tangent to C' at the point P_1'. Since arcs C' and C_1'

lie in a plane determined by centre O' and the common tangent at point P_1', then they must be identical. This corollary implies that two different curves C and C_1 from the neighbourhood of P (elliptic or hyperbolic) correspond to the same curve $C' = C_1'$ in the neighbourhood of P' on S'. This conclusion contradicts the one-to-one spherical mapping in the neighbourhood of an elliptic or hyperbolic point P. So $C = C_1$ and the geodesic C' on S' corresponds to the geodesic C on S. Thus the spherical mapping of the neighbourhood of an elliptic or hyperbolic point P on S is simultaneously a geodesic one.

b) The second part of our theorem results from the first one, and from the theorem which says that equations (1) form a necessary and sufficient condition for the existence of geodesic correspondence among two surfaces S and S' ([3]).

Now we shall return to the proof of our principal theorem. Suppose for an indirect proof that the point P lying on S is not an ombilic. Thus (lemma 2) there exists a neighbourhood of P on S such that none of its points is ombilical. Hence at every point of the neighbourhood in question two principal directions are determined and in the whole neighbourhood there exists an orthogonal system of curvature lines. We consider it as a system of curvilinear coordinates. Next let us make a spherical mapping of the surface S. Suppose that the Gauss curvature K of S does not vanish at the point P. Thus there exists a neighbourhood of the point P such that K is not equal to zero at any point of this neighbourhood.

Between the coefficients g_{ik}, h_{ik} $(i, k = 1, 2)$ of the first and second differential form of S respectively, and the coefficients g_{ik}' $(i, k = 1, 2)$ of the first differential form of S' we have the following well-known relations ([4])

$$g_{ik}' = 2Hh_{ik} - Kg_{ik} \qquad (i, k = 1, 2) \tag{2}$$

where H and K denote the mean and the Gauss curvature of S. We have

$$H = \frac{1}{2}\,\frac{g_{22}h_{11} - 2g_{12}h_{12} + g_{11}h_{22}}{g_{11}g_{22} - g_{12}^2}, \qquad K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}.$$

For the above-mentioned system of curvilinear coordinates the coefficients g_{12} and h_{12} both vanish identically and relations (2) have a simpler form:

$$\begin{aligned} g_{11}' &= 2Hh_{11} - Kg_{11} = \frac{h_{11}^2}{g_{11}}, \\ g_{12}' &= 2Hh_{12} - Kg_{12} = 0, \\ g_{22}' &= 2Hh_{22} - Kg_{22} = \frac{h_{22}^2}{g_{22}}, \end{aligned} \tag{3}$$

([3]) L. Bianchi, *Lezioni di geometria differenziale I*, p., 314, 1922.

([4]) M. Biernacki, *Geometria różniczkowa, II*, Warszawa 1954, p. 87.

where

$$H = \frac{g_{22}h_{11} + g_{11}h_{22}}{2g_{11}g_{22}}, \qquad K = \frac{h_{11}h_{22}}{g_{11}g_{22}}.$$

In order to make use of conditions (1) of lemma 5, let us calculate the Christoffel symbols of the second kind for S and S'. From the general formula

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = \frac{1}{2} g^{kl}\{\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}\}$$

where

$$g = g_{11}g_{22} - g_{12}^2, \qquad g^{11} = \frac{g_{22}}{g}, \qquad g^{12} = -\frac{g_{12}}{g}, \qquad g^{22} = \frac{g_{11}}{g};$$

taking into consideration the equality $g_{12} = 0$, we get for the surface S the following values

$$\begin{aligned}
&\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{\partial_1 g_{11}}{2g_{11}}, &\qquad &\begin{Bmatrix} 2 \\ 11 \end{Bmatrix} = -\frac{\partial_2 g_{11}}{2g_{22}}, \\
&\begin{Bmatrix} 1 \\ 12 \end{Bmatrix} = \frac{\partial_2 g_{11}}{2g_{11}}, &\qquad &\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial_1 g_{22}}{2g_{22}}, \\
&\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -\frac{\partial_1 g_{22}}{2g_{11}}, &\qquad &\begin{Bmatrix} 2 \\ 22 \end{Bmatrix} = \frac{\partial_2 g_{22}}{2g_{22}}.
\end{aligned} \tag{4}$$

From (3) we analogously obtain for S':

$$\begin{aligned}
&\begin{Bmatrix} 1 \\ 11 \end{Bmatrix}' = \frac{\partial_1 g'_{11}}{2g'_{11}} = \frac{\partial_1\left(\dfrac{h_{11}^2}{g_{11}}\right)}{2\cdot\dfrac{h_{11}^2}{g_{11}}} = \frac{2g_{11}\partial_1 h_{11} - h_{11}\partial_1 g_{11}}{2g_{11}h_{11}}, \\
&\begin{Bmatrix} 1 \\ 12 \end{Bmatrix}' = \frac{\partial_2 g'_{11}}{2g'_{11}} = \frac{2g_{11}\partial_2 h_{11} - h_{11}\partial_2 g_{11}}{2g_{11}h_{11}}, \\
&\begin{Bmatrix} 1 \\ 22 \end{Bmatrix}' = -\frac{\partial_1 g'_{22}}{2g'_{11}} = \frac{h_{22}g_{11}(h_{22}\partial_2 g_{11} - 2g_{22}\partial_1 h_{22})}{2g_{22}^2 h_{11}^2}, \\
&\begin{Bmatrix} 2 \\ 11 \end{Bmatrix}' = -\frac{\partial_2 g'_{11}}{2g'_{22}} = \frac{h_{11}g_{22}(h_{11}\partial_1 g_{22} - 2g_{11}\partial_2 h_{11})}{2g_{11}^2 h_{22}^2}, \\
&\begin{Bmatrix} 2 \\ 12 \end{Bmatrix}' = \frac{\partial_1 g'_{22}}{2g'_{22}} = \frac{2g_{22}\partial_1 h_{22} - h_{22}\partial_1 g_{22}}{2h_{22}g_{22}}, \\
&\begin{Bmatrix} 2 \\ 22 \end{Bmatrix}' = \frac{\partial_2 g'_{22}}{2g'_{22}} = \frac{2g_{22}\partial_2 h_{22} - h_{22}\partial_2 g_{22}}{2h_{22}g_{22}}.
\end{aligned} \tag{5}$$

Remark. From assumption $K \neq 0$ and from the last formula of conditions (1) it follows that $h_{11}\cdot h_{22} \neq 0$. Thus the right-hand sides of formulas (5) are meaningful.

Substituting (4) and (5) into (1) we get four partial differential equations. Adding to them two Mainardi-Codazzi equations, we get the following system of six partial differential equations:

$$
(6)\qquad
\begin{aligned}
&1.\quad \frac{h_{11}g_{22}(h_{11}\partial_2 g_{11}-2g_{11}\partial_2 h_{11})}{2g_{11}^2h_{22}^2}=-\frac{\partial_2 g_{11}}{2g_{22}},\\
&2.\quad \frac{h_{22}g_{11}(h_{22}\partial_1 g_{22}-2g_{22}\partial_1 h_{22})}{2g_{22}^2h_{11}^2}=-\frac{\partial_1 g_{22}}{2g_{11}},\\
&3.\quad \frac{2g_{11}\partial_2 h_{11}-h_{11}\partial_2 g_{11}}{g_{11}h_{11}}-\frac{2g_{22}\partial_2 h_{22}-h_{22}\partial_2 g_{22}}{2g_{22}h_{22}}=\frac{\partial_2 g_{11}}{g_{11}}-\frac{\partial_2 g_{22}}{2g_{22}},\\
&4.\quad \frac{2g_{22}\partial_1 h_{22}-h_{22}\partial_1 g_{22}}{g_{22}h_{22}}-\frac{2g_{11}\partial_1 h_{11}-h_{11}\partial_1 g_{11}}{2g_{11}h_{11}}=\frac{\partial_1 g_{22}}{g_{22}}-\frac{\partial_1 g_{11}}{2g_{11}},\\
&5.\quad 2g_{11}g_{22}\partial_2 h_{11}=(g_{11}h_{22}+g_{22}h_{11})\partial_2 g_{11},\\
&6.\quad 2g_{11}g_{22}\partial_1 h_{22}=(g_{11}h_{22}+g_{22}h_{11})\partial_1 g_{22}.
\end{aligned}
$$

Now we show that system (6) leads to a contradiction of the assumption: P is not an ombilical point. Writing equations 1, 2, 5 and 6 in a slightly different form, we obtain

$$
(6')\qquad
\begin{aligned}
&1'.\quad \partial_2 h_{11}=\partial_2 g_{11}\frac{g_{11}^2h_{22}^2+g_{22}^2h_{11}^2}{2g_{11}h_{11}g_{22}^2},\\
&2'.\quad \partial_1 h_{22}=\partial_1 g_{22}\frac{g_{11}^2h_{22}^2+g_{22}^2h_{11}^2}{2g_{22}h_{22}g_{11}^2},\\
&5'.\quad \partial_2 h_{11}=\partial_2 g_{11}\frac{g_{11}h_{22}+g_{22}h_{11}}{2g_{11}g_{22}},\\
&6'.\quad \partial_1 h_{22}=\partial_1 g_{22}\frac{g_{11}h_{22}+g_{22}h_{11}}{2g_{11}g_{22}}.
\end{aligned}
$$

Note further that the partial derivatives $\partial_2 g_{11}$ and $\partial_1 g_{22}$ cannot be equal to zero simultaneously and identically in a whole neighbourhood of P. Indeed, for $\partial_1 g_{11}\equiv 0$, $\partial_1 g_{22}\equiv 0$ we find by Frobenius' formula, expressing the curvature K

$$K=-\frac{1}{2\sqrt{g_{11}g_{22}}}\left[\partial_2\left(\frac{\partial_2 g_{11}}{\sqrt{g_{11}g_{22}}}\right)+\partial_1\left(\frac{\partial_1 g_{22}}{\sqrt{g_{11}g_{22}}}\right)\right],$$

that K vanishes in the whole neighbourhood of P. This conclusion contradicts the hypothesis made at the begining of the proof. Thus in the neighbourhood of P there exist a point P_1 at which we have

$$\partial_2 g_{11}\neq 0 \quad \text{or} \quad \partial_1 g_{22}\neq 0\,.$$

Since the point P_1 satisfies the same conditions as point P, systems of equations (6) or (6′) are satisfied in a neighbourhood of point P_1. For instance let $\partial_2 g_{11} \neq 0$ at the point P_1. Comparing the right-hand sides of equations 1′ and 5′, we obtain after simple computations

$$\frac{h_{11}}{g_{11}} = \frac{h_{22}}{g_{22}}. \tag{7}$$

Condition (7) expresses, however, that the point P_1 is an ombilic. This statement contradicts the statement that in the neighbourhood in question of point P ombilical points do not exist. The above contradiction proves that in spite of our assumption the point P is ombilical. Thus it is proved that every point P of S at which the curvature K is not equal to zero is an ombilic. From the last conclusion and the assumption that S and S' are of class C^3 it follows that the surface S is a part of the sphere. Hence the proof of our theorem is finished.

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