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GRAPHS MINIMAL WITH RESPECT TO CONTRACTIONS IN SOME SUBFAMILIES OF MAXIMAL PLANAR GRAPHS

Abstract. Let $\mathcal{M}_5$ denote the family of all maximal planar graphs which have no vertices of degree less than 5 and no 5-cycles with at least two vertices in each of its domains. The purpose of this paper is to obtain a description of the family of all minimal graphs in $\mathcal{M}_5$. This permits us to characterize the family $\mathcal{M}_5$ in terms of contractions of vertices of degree 5. We prove also that there is only one minimal graph in the family $\mathcal{M}_5 \setminus \mathcal{R}$, where $\mathcal{R}$ is the family of all minimal graphs in $\mathcal{M}_5$.

1. Preliminaries. A *graph* is understood here as a maximal planar graph without loops or multiple edges. $V(G)$ denotes a set of vertices of a graph G, $\deg a$ denotes the degree of vertex a, and a adj b means that a and b are adjacent. A *cycle* α is defined as a closed simple path, and $|\alpha|$ denotes the length of α. We denote by $C(G)$ the family of all cycles in a graph G which split the plane into domains each of which contains at least two vertices of G. Note that, since every maximal planar graph G has a unique family of faces, we may talk about domains of G without referring to any embedding of G. Let $\mathcal{M}$ denote the family of all maximal planar graphs with all vertices of degree at least 5, and $\mathcal{M}_k \subset \mathcal{M}$ be the family of all graphs which have no cycles of length k in $C(G)$. The family $\mathcal{M}_5$ has been considered by Birkhoff in his separation theorem [1], and in [2] which has played an important role in many investigations on the four-color problem. If a adj b, then $G_{a,b}$ denotes the graph which is obtained from G by contracting the edge ab. Let $\mathcal{H} \subset \mathcal{M}$. A graph $G \in \mathcal{H}$ is said to be *minimal* in the family $\mathcal{H}$ if for all vertices $a, b \in V(G)$ such that a adj b and $\deg a = 5$ we have $G_{a,b} \notin \mathcal{H}$.

Let G be a maximal plane graph and let $\alpha = (a_1, a_2, \ldots, a_k)$ be a cycle in G. In what follows, we assume that any cyclic permutation

$$(a_i, a_{i+1}, \ldots, a_k, a_1, \ldots, a_{i-1})$$

of vertices of α is the same cycle α; however, $(a_k, a_{k-1}, \ldots, a_1)$ and each of its cyclic permutations generate another cycle denoted by $\tilde{\alpha}$. Int α denotes a bounded (unbounded) domain corresponding to an anti-clockwise (clockwise) orientated cycle α. $\deg_\alpha(a)$ is the number of vertices adjacent to a and belonging to Int $\alpha \cup \alpha$. We denote by $V(\text{Int } \alpha)$ the set of vertices of a given graph belonging to Int α. The *neighbourhood* $N(a)$ of a vertex a is the cycle whose vertices are adjacent to a and whose orientation is such that $a \in \text{Int } N(a)$. It is not difficult to see that for every $n \geqslant 5$ there is exactly one graph in $\mathcal{M}$, denoted by $G(n)$, which has two vertices a and b of degree n, and

$$N(a) \cap N(b) = \emptyset, \qquad V(G) = \{a, b\} \cup N(a) \cup N(b),$$

and every vertex in $N(a) \cup N(b)$ is of degree 5. Moreover, one can easily show that there is exactly one graph in $\mathcal{M}$, denoted by A, with 12 vertices of degree 5 and 3 vertices of degree 6 (see Fig. 1). The family of all $G(n)$'s, $n \geqslant 5$, is denoted by $\mathcal{S}$. We denote a j-cycle α by w_j ($j = 3, 4, 5$) if the restriction of G to Int $\alpha \cup \alpha$ is isomorphic to

(a) $G(5)$ for $j = 3$;
(b) $G(5) \setminus e$ for $j = 4$, where e is an edge of $G(5)$;
(c) $G(5) \setminus x$ for $j = 5$, where x is a vertex of $G(5)$.

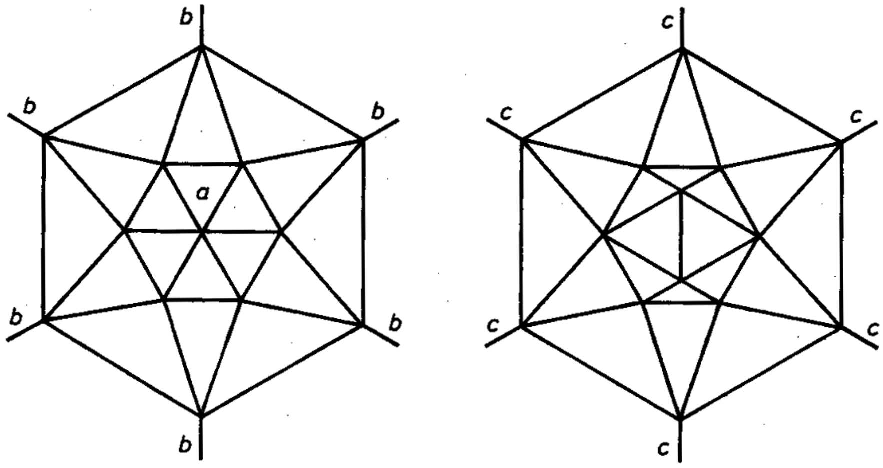

Fig. 1. Graphs $G(6)$ and A

The purpose of this paper is to obtain a description of the family of all minimal graphs in $\mathcal{M}_5$ (Theorem 1). This allows us to characterize $\mathcal{M}_5$ in terms of contractions of vertices of degree 5. Namely, every graph in $\mathcal{M}_5$ can be obtained from the set $\mathcal{S}$ by the operations opposite to contractions of vertices of degree 5. Theorem 2 follows from Theorem 1 and states that there is only one minimal graph in the family $\mathcal{M}_5 \setminus \mathcal{S}$. The proofs of those theorems in Section 4 follow from eight lemmas which are considered in Sections 2 and 3.

2. Some properties of the cycles of $C(G)$. The following lemma is very simple, therefore we can omit its proof.

LEMMA 1. *Let* $\alpha_1, \alpha_2, \alpha_3$ *be three cycles in a graph* G *such that*

$$\alpha_1 = (a_1, a_2, \ldots, a_n), \quad \alpha_2 = (a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_s),$$
$$\alpha_3 = (a_p, a_{p+1}, \ldots, a_n, a_1, b_s, b_{s-1}, \ldots, b_1).$$

If $b_j \in V(\mathrm{Int}\,\alpha_1)$ *for* $j = 1, \ldots, s$, *then*

$$\mathrm{Int}\,\alpha_2 \cup \mathrm{Int}\,\alpha_3 \subset \mathrm{Int}\,\alpha_1, \quad \mathrm{Int}\,\alpha_2 \cap \mathrm{Int}\,\alpha_3 = \emptyset,$$
$$V(\mathrm{Int}\,\alpha_2) \cup V(\mathrm{Int}\,\alpha_3) \cup \{b_j\colon j = 1, \ldots, s\} = V(\mathrm{Int}\,\alpha_1),$$
$$\deg b_j = \deg_{\alpha_2}(b_j) + \deg_{\alpha_3}(b_j) - 2 \quad \text{for } j = 1, \ldots, s.$$

LEMMA 2. $\mathcal{M}_6 \subset \mathcal{M}_5 \subset \mathcal{M}_4 \subset \mathcal{M}_3$.

Proof. It is sufficient to prove the following implication for $j = 3, 4, 5$: If there exists a cycle $\alpha \in C(G)$, $|\alpha| = j$, then there exists a cycle $\beta \in C(G)$, $|\beta| = j+1$. Let $\alpha = (a_1, a_2, \ldots, a_5)$. Since $V(\mathrm{Int}\,\alpha) \neq \emptyset$, there exists $b \in V(\mathrm{Int}\,\alpha)$ adjacent to two consecutive vertices of the cycle α. Therefore, we can assume that there exists a cycle $\delta = (a_1, a_2, b)$ such that

(i_1) $V(\mathrm{Int}\,\delta) = \emptyset$.

Let us consider a cycle $\beta = (a_2, a_3, a_4, a_5, a_1, b)$. By Lemma 1 and ($i_1$) we have

(i_2) $\mathrm{Int}\,\beta \subset \mathrm{Int}\,\alpha$,

(i_3) $V(\mathrm{Int}\,\beta) \cup \{b\} = V(\mathrm{Int}\,\alpha)$,

(i_4) $\deg b = \deg_\beta(b)$.

Let us note that $|V(\mathrm{Int}\,\tilde{\beta})| \geqslant 2$ since (i_2) implies

$$\mathrm{Int}\,\tilde{\alpha} \subset \mathrm{Int}\,\tilde{\beta},$$

and hence

$$V(\mathrm{Int}\,\tilde{\alpha}) \subset V(\mathrm{Int}\,\tilde{\beta}).$$

By (i_4), $\deg_\beta(b) \geqslant 5$. Hence at least one of the following cases holds:

(a) $|V(\mathrm{Int}\,\beta)| \geqslant 2$.

(b) There exist vertices $a_k, a_l \in \beta$ $(3 \leqslant k < l \leqslant 5)$ such that $a_k \operatorname{adj} b$ and $a_l \operatorname{adj} b$.

It is enough to show that the case (b) leads to a contradiction. Let

$$\gamma_1 = (a_2, \ldots, a_k, b), \quad \gamma_2 = (a_k, \ldots, a_l, b), \quad \gamma_3 = (a_l, \ldots, a_1, b)$$

be three cycles. Then $3 \leqslant k < l \leqslant 5$ implies $|\gamma_j| \leqslant 4$ for $j = 1, 2, 3$. Since $G \in \mathcal{M}$, we obtain

$$\sum_{j=1}^{3} |V(\mathrm{Int}\,\gamma_j)| = 0.$$

Lemma 1 also implies that

$$V(\mathrm{Int}\,\gamma_i) \cap V(\mathrm{Int}\,\gamma_j) = \emptyset \quad \text{for } i \neq j$$

and

$$\bigcup_{j=1}^{3} V(\text{Int } \gamma_j) = V(\text{Int } \beta).$$

So, by (i_3) we arrive at a contradiction

$$0 = |V(\text{Int } \beta)| = |V(\text{Int } \alpha)| - 1 \geqslant 1.$$

The proof in cases $|\alpha| = 3$ and 4 is analogous.

LEMMA 3. *Let $G \in \mathcal{M}_5$. If a and b are consecutive vertices of a cycle $\alpha \in C(G)$, $|\alpha| = 6$, and $V(\text{Int } \alpha)$ has only one vertex in $N(a) \cup N(b)$, then all vertices in $V(\text{Int } \alpha)$ are of degree 5.*

Proof. Let $\alpha = (a_1, a_2, a_3, a_4, a, b)$. Because a and b are consecutive vertices of α, there exists a vertex

$$c \in V(\text{Int } \alpha) \cap N(a) \cap N(b).$$

Since c is the only vertex belonging to $V(\text{Int } \alpha) \cap (N(a) \cup N(b))$, we have c adj a_1 and c adj a_4. Therefore, there exists a cycle $\beta = (a_1, a_2, a_3, a_4, c)$. Because $c \in V(\text{Int } \alpha)$, we get by Lemma 1:

(i_1) $\text{Int } \beta \subset \text{Int } \alpha$;

(i_2) $V(\text{Int } \alpha) = V(\text{Int } \beta) \cup \{c\}$;

(i_3) $\deg c = \deg_\beta(c) + 2$.

Let us note that (i_1) implies $\text{Int } \tilde{\alpha} \subset \text{Int } \tilde{\beta}$, and therefore

$$2 \leqslant |V(\text{Int } \tilde{\alpha})| \leqslant |V(\text{Int } \tilde{\beta})|.$$

Since $|\tilde{\beta}| = 5$, we get $|V(\text{Int } \beta)| \leqslant 1$. Hence, by ($i_2$), we have

$$2 \leqslant |V(\text{Int } \alpha)| \leqslant |V(\text{Int } \beta)| + 1 \leqslant 2.$$

This means that $|V(\text{Int } \beta)| = 1$, and we have

(i_4) there is a vertex d such that $V(\text{Int } \beta) = \{d\}$ and $\deg d = 5$.

Finally, it is enough to check that $\deg c = 5$ by (i_3) and (i_4), and (i_2), (i_4) imply

$$V(\text{Int } \alpha) = \{c, d\} \quad \text{and} \quad \deg d = 5.$$

LEMMA 4. *Let $G \in \mathcal{M}_5$. If a, b, c are consecutive vertices of a cycle $\alpha \in C(G)$, $|\alpha| = 6$, and $\deg a = \deg b = \deg c = 5$, then all vertices in $V(\text{Int } \alpha)$ or in $V(\text{Int } \tilde{\alpha})$ are of degree 5.*

Proof. Let $\alpha = (a_1, a_2, a_3, a, b, c)$. Without loss of generality we may assume that

$$N(b) = (c, d_1, d_2, a, d_3).$$

Hence $d_1, d_2 \in V(\text{Int } \alpha)$. The following cases are possible:

(a) a_3 adj d_2 and a_1 adj d_1,

(b) a_3 adj d_3 or a_1 adj d_3.

In case (b), the set $V(\text{Int }\tilde{\alpha})$ has only one vertex in $N(a) \cup N(b)$ or has only one vertex in $N(b) \cup N(c)$. Hence, by Lemma 3, all vertices in Int $\tilde{\alpha}$ are of degree 5. Whereas in case (a) there is a cycle $\beta = (a_1, a_2, a_3, d_2, d_1)$. Since $d_1, d_2 \in V(\text{Int }\alpha)$, Lemma 1 yields

(i_1) Int $\beta \subset$ Int α;

(i_2) $V(\text{Int }\alpha) = V(\text{Int }\beta) \cup \{d_1, d_2\}$;

(i_3) $\deg d_1 = \deg_\beta(d_1) + 2$;

(i_4) $\deg d_2 = \deg_\beta(d_2) + 2$.

It is clear that

$$|V(\text{Int }\tilde{\beta})| \geqslant |V(\text{Int }\tilde{\alpha})| \geqslant 2$$

by (i_1). Hence $|V(\text{Int }\beta)| \leqslant 1$ by $G \in \mathscr{M}_5$. We will show now that

(i_5) $\deg_\beta(d_1) = \deg_\beta(d_2) = 3$.

Let us note that, by (i_3) and (i_4),

$$\deg_\beta(d_1) \geqslant 3 \quad \text{and} \quad \deg_\beta(d_2) \geqslant 3.$$

If $V(\text{Int }\beta) = \emptyset$, then ($i_5$) follows since d_2 and d_1 are consecutive vertices in β. If $|V(\text{Int }\beta)| = 1$, then ($i_5$) is obvious since $|\beta| = 5$. By (i_3), (i_4) and (i_5) we get $\deg d_1 = \deg d_2 = 5$. Using (i_2), we note that if $V(\text{Int }\beta) = \emptyset$, then $V(\text{Int }\alpha) = \{d_1, d_2\}$; moreover, if $|V(\text{Int }\beta)| = 1$, then

$$V(\text{Int}\,\alpha) = \{d_1, d_2, e\}, \quad \text{where } \{e\} = V(\text{Int }\beta) \text{ and } \deg e = 5.$$

LEMMA 5. *Let* $G \in \mathscr{M}_5$ *and for* $a, b \in V(G)$ *let* $\deg a = \deg b = 5$ *and* a adj b. *If there exist cycles* $\alpha, \beta \in C(G)$, $|\alpha| = |\beta| = 6$, *such that*

$$\alpha \cap \beta \cap N(a) = \{b\} \quad \text{and} \quad \alpha \cap \beta \cap N(b) = \{a\},$$

then all vertices in $V(\text{Int }\alpha)$ *or in* $V(\text{Int }\tilde{\alpha})$ *are of degree* 5.

Proof. Let

$$\alpha = (a_1, a_2, a_3, a_4, a, b), \quad \beta = (b_1, b_2, b_3, b_4, a, b)$$

and

$$N(a) \cap N(b) = \{c, d\}.$$

Since $\alpha \cap \beta \cap N(b) = \{a\}$, without loss of generality we may assume that $N(b) = (a, d, a_1, b_1, c)$. Hence we have

(1) $b_1 \in V(\text{Int }\alpha)$ and $a_1 \in V(\text{Int }\tilde{\beta})$,

(2) $c \in V(\text{Int }\alpha)$ and $d \in V(\text{Int }\tilde{\alpha})$.

Since $\alpha \cap \beta \cap N(a) = \{b\}$, the cycle $N(a)$ is either (b, c, b_4, a_4, d) or (b, c, a_4, b_4, d). In the former case $b_4 \in V(\text{Int }\alpha)$ and, by (1), (2) and Lemma 3, all the vertices in $V(\text{Int }\tilde{\alpha})$ are of degree 5. Otherwise, we get

(3) $b_4 \in V(\text{Int }\tilde{\alpha})$ and $a_4 \in V(\text{Int }\beta)$.

By (k, l) we will understand that $b_j \in V(\text{Int }\alpha)$ for $1 \leqslant j \leqslant k-1$ and $b_k = a_l$. Because of (1) and (3) we should consider k and l such that $2 \leqslant k \leqslant l \leqslant 3$.

(2,2) Since $b_2 = a_2$, there is a cycle

$$\alpha_1 = (a_2, a_3, a_4, c, b_1).$$

Since b_1, $c \in V(\text{Int } \alpha)$, Lemma 1 gives

(i_1) $\text{Int } \alpha_1 \subset \text{Int } \alpha$;

(i_2) $V(\text{Int } \alpha) = V(\text{Int } \alpha_1) \cup \{b_1, c\}$;

(i_3) $\deg b_1 = \deg_{\alpha_1}(b_1)+2$;

(i_4) $\deg c = \deg_{\alpha_1}(c)+2$.

By (i_1) we have $V(\text{Int } \tilde{\alpha}) \subset V(\text{Int } \tilde{\alpha}_1)$. Since $2 \leqslant |V(\text{Int } \tilde{\alpha})|$ and $|\tilde{\alpha}_1| = 5$, we obtain $|V(\text{Int } \alpha_1)| \leqslant 1$. Because c and b_1 are consecutive vertices in the cycle α_1, $|\alpha_1| = 5$, $|V(\text{Int } \alpha_1)| \leqslant 1$ analogously as in the proof of (i_5) in Lemma 4, we can demonstrate first that

$$\deg_{\alpha_1}(b_1) = \deg_{\alpha_1}(c) = 3,$$

and then that all vertices in $V(\text{Int } \alpha)$ are of degree 5.

(3,2) Since $b_3 = a_2$, there are cycles

$$\alpha_1 = (a_4, a_3, a_2, b_4) \quad \text{and} \quad \alpha_2 = (a_2, a_1, d, b_4).$$

Because d, $b_4 \in V(\text{Int } \tilde{\alpha})$, Lemma 1 yields

(j_1) $\text{Int } \alpha_1 \cup \text{Int } \alpha_2 \subset \text{Int } \tilde{\alpha}$;

(j_2) $V(\text{Int } \tilde{\alpha}) = V(\text{Int } \alpha_1) \cup V(\text{Int } \alpha_2) \cup \{b_4, d\}$;

(j_3) $\deg d = \deg_{\alpha_2}(d)+2$;

(j_4) $\deg b_4 = \deg_{\alpha_1}(b_4)+\deg_{\alpha_2}(b_4)$.

By (j_1), $\text{Int } \tilde{\alpha}_1 \cap \text{Int } \tilde{\alpha}_2 \supset \text{Int } \alpha$. Hence

$$2 \leqslant |V(\text{Int } \alpha)| \leqslant |V(\text{Int } \tilde{\alpha}_1)| \quad \text{and} \quad 2 \leqslant |V(\text{Int } \tilde{\alpha}_2)|.$$

Since $|\tilde{\alpha}_1| = |\tilde{\alpha}_2| = 4$, Lemma 2 implies

(j_5) $V(\text{Int } \alpha_1) = V(\text{Int } \alpha_2) = \emptyset$.

By (j_3) and (j_5) we get

(j_6) $\deg_{\alpha_2}(d) = 3$, $\deg_{\alpha_1}(b_4) \leqslant 3$, $\deg_{\alpha_2}(b_4) = 2$.

By (j_2) and (j_5), we have $V(\text{Int } \tilde{\alpha}) = \{b_4, d\}$. Conditions ($\mathrm{j}_3$), ($\mathrm{j}_4$) and ($\mathrm{j}_6$) give $\deg d = 5 = \deg b_4$.

The cases (3,3) and (2,3) can be reduced, by the reflexion of G in the plane, to the cases (2,2) and (3,2), respectively.

LEMMA 6. *Let $G \in \mathcal{M}_5$ and $\deg a = 5$ for $a \in V(G)$. If there exist cycles $\alpha, \beta \in C(G)$, $|\alpha| = |\beta| = 6$, such that $a \in \alpha \cap \beta$ and $\alpha \cap \beta \cap N(a) = \emptyset$, then there exists a cycle $\gamma \in \{\alpha, \tilde{\alpha}, \beta, \tilde{\beta}\}$ such that all vertices in* Int γ *are of degree* 5.

Proof. Let

$$\alpha = (a_1, a_2, \ldots, a_5, a) \quad \text{and} \quad \beta = (b_1, b_2, \ldots, b_5, a).$$

Since $N(a) \cap \alpha \cap \beta = \emptyset$, a_1, a_5, b_1, b_5 are different vertices of $N(a)$. Without loss of generality we may assume that $N(a) = (a_1, b_1, a_5, b_5, c)$. Therefore, we get

(1) $b_1 \in V(\text{Int }\alpha)$, $b_5 \in V(\text{Int }\tilde{\alpha})$, $a_1 \in V(\text{Int }\tilde{\beta})$, $a_5 \in V(\text{Int }\beta)$.
(2) $c \in V(\text{Int }\tilde{\alpha}) \cap V(\text{Int }\tilde{\beta})$.

By (k, l) we will understand that $b_j \in V(\text{Int }\alpha)$ for $1 \leqslant j \leqslant k-1$ and $b_k = a_l$. By (1), we should consider k and l such that $2 \leqslant k \leqslant l \leqslant 4$.

(2,2) Because $b_2 = a_2$, there exist cycles

$$\alpha_1 = (a_2, a_3, a_4, a_5, b_1) \quad \text{and} \quad \alpha_2 = (a_1, a_2, b_1).$$

Since $b_1 \in V(\text{Int }\alpha)$, by Lemma 1 we get
(i_1) $\text{Int }\alpha_1 \cup \text{Int }\alpha_2 \subset \text{Int }\alpha$;
(i_2) $V(\text{Int }\alpha) = V(\text{Int }\alpha_1) \cup V(\text{Int }\alpha_2) \cup \{b_1\}$;
(i_3) $\deg b_1 = \deg_{\alpha_1}(b_1) + \deg_{\alpha_2}(b_1)$.

By (i_1) we have $\text{Int }\tilde{\alpha} \subset \text{Int }\tilde{\alpha}_1 \cap \text{Int }\tilde{\alpha}_2$, and hence

$$2 \leqslant |V(\text{Int }\tilde{\alpha}_1)| \quad \text{and} \quad 2 \leqslant |V(\text{Int }\tilde{\alpha}_2)|.$$

Therefore, by Lemma 2 and since $|\tilde{\alpha}_1| = 5$ and $|\tilde{\alpha}_2| = 3$, we get
(i_4) $|V(\text{Int }\alpha_1)| \leqslant 1$ and $|V(\text{Int }\alpha_2)| = 0$.

By (i_2) and (i_4) we have

$$2 \leqslant |V(\text{Int }\alpha)| \leqslant |V(\text{Int }\alpha_1)| + |V(\text{Int }\alpha_2)| + 1 \leqslant 2.$$

This means that $|V(\text{Int }\alpha_1)| = 1$ and there is a vertex d such that
(i_5) $V(\text{Int }\alpha_1) = \{d\}$ and $\deg d = 5$.

Since $|\alpha_1| = 5$ and $|\alpha_2| = 3$, we obtain

$$\deg_{\alpha_1}(b_1) = 3 \quad \text{and} \quad \deg_{\alpha_2}(b_1) = 2,$$

so, by (i_3), $\deg b_1 = 5$. Finally, it is enough to note that the equality

$$V(\text{Int }\alpha) = V(\text{Int }\alpha_1) \cup \{b_1\} = \{b_1, d\}$$

follows from (i_2), (i_4) and (i_5).

(3,2) Since $b_3 = a_2$, there exist cycles

$$\beta_1 = (b_3, b_2, b_1, a_1) \quad \text{and} \quad \beta_2 = (a, b_5, b_4, b_3, a_1).$$

Because $a_1 \in V(\text{Int }\tilde{\beta})$, by Lemma 1 we have
(j_1) $\text{Int }\beta_1 \subset \text{Int }\tilde{\beta}$, $\text{Int }\beta_2 \subset \text{Int }\tilde{\beta}$;
(j_2) $V(\text{Int }\tilde{\beta}) = V(\text{Int }\beta_1) \cup V(\text{Int }\beta_2) \cup \{a_1\}$;
(j_3) $\deg a_1 = \deg_{\beta_1}(a_1) + \deg_{\beta_2}(a_1) - 1$.

By the inequality $|V(\text{Int }\beta)| \geqslant 2$, ($j_1$) and Lemma 2 we get
(j_4) $|V(\text{Int }\beta_1)| = 0$ and $|V(\text{Int }\beta_2)| \leqslant 1$.

Since $c \in V(\text{Int }\tilde{\beta})$, by ($j_2$) and ($j_4$) we have
(j_5) $V(\text{Int }\beta_2) = \{c\}$ and $\deg c = 5$.

By (j_4), (j_5) and since $|\beta_1| = 4$, $|\beta_2| = 5$ we get

$$\deg_{\beta_1}(a_1) \leqslant 3 \quad \text{and} \quad \deg_{\beta_2}(a_1) = 3.$$

From this and (j_3) we conclude that

$$5 \leqslant \deg a_1 = \deg_{\beta_1}(a_1) + \deg_{\beta_2}(a_1) - 1 \leqslant 5.$$

Thus it is enough to note that

$$V(\text{Int } \tilde{\beta}) = V(\text{Int } \beta_2) \cup \{a_1\} = \{c, a_1\},$$

which follows by (j_2) and (j_5).

(3,3) Because $b_3 = a_3$, there exist cycles

$$\alpha_1 = (a_1, a_2, a_3, b_2, b_1) \quad \text{and} \quad \alpha_2 = (a_3, a_4, a_5, b_1, b_2).$$

Since $b_1, b_2 \in V(\text{Int } \alpha)$ and by Lemma 1 we get

(k_1) $\text{Int } \alpha_1 \cup \text{Int } \alpha_2 \subset \text{Int } \alpha$;

(k_2) $V(\text{Int } \alpha) = V(\text{Int } \alpha_1) \cup V(\text{Int } \alpha_2) \cup \{b_1, b_2\}$;

(k_3) $\deg b_1 = \deg_{\alpha_1}(b_1) + \deg_{\alpha_2}(b_1)$;

(k_4) $\deg b_2 = \deg_{\alpha_1}(b_2) + \deg_{\alpha_2}(b_2) - 2$.

The inequality $|V(\text{Int } \tilde{\alpha})| \geqslant 2$ together with (k_1) gives

(k_5) $|V(\text{Int } \alpha_1)| \leqslant 1$, $|V(\text{Int } \alpha_2)| \leqslant 1$.

Let us note that we cannot have $|V(\text{Int } \alpha_1)| = |V(\text{Int } \alpha_2)| = 1$ because in this case $\deg_{\alpha_1}(b_2) = \deg_{\alpha_2}(b_2) = 3$, which leads to a contradiction with (k_4). Namely, we then have

$$5 \leqslant \deg b_2 = \deg_{\alpha_1}(b_2) + \deg_{\alpha_2}(b_2) - 2 = 4.$$

Similarly, in all other cases of condition (k_5) we obtain

$$|V(\text{Int } \alpha_1)| = 0 = |V(\text{Int } \alpha_2)|.$$

Hence

$$V(\text{Int } \alpha) = \{b_1, b_2\} \quad \text{and} \quad \deg b_1 = \deg b_2 = 5$$

and

$$|V(\text{Int } \alpha_1)| = 1 \quad \text{or} \quad |V(\text{Int } \alpha_2)| = 1.$$

Therefore there exists a vertex d such that

$$V(\text{Int } \alpha) = \{b_1, b_2, d\} \quad \text{and} \quad \deg b_1 = \deg b_2 = \deg d = 5.$$

We will show now that cases (2,3) and (4,2) cannot occur.

(2,3) Since $b_2 = a_3$, there exist cycles

$$\alpha_1 = (a_1, a_2, a_3, b_1) \quad \text{and} \quad \alpha_2 = (a_3, a_4, a_5, b_1).$$

Since $b_1 \in V(\text{Int } \alpha)$ and by Lemma 1 we get

(l_1) $\text{Int } \alpha_1 \subset \text{Int } \alpha$, $\text{Int } \alpha_2 \subset \text{Int } \alpha$;

(l_2) $V(\text{Int } \alpha) = V(\text{Int } \alpha_1) \cup V(\text{Int } \alpha_2) \cup \{b_1\}$.

Since $|V(\text{Int } \tilde{\alpha})| \geqslant 2$ and $|\alpha_1| = |\alpha_2| = 4$, by Lemma 2 we get
(l_3) $|V(\text{Int } \alpha_1)| = |V(\text{Int } \alpha_2)| = 0$.
Conditions (l_2) and (l_3) lead to a contradiction, since $|V(\text{Int } \alpha)| \geqslant 2$.
(4,2) Since $b_4 = a_2$, there exist cycles

$$\beta_1 = (a, b_5, b_4, a_1) \quad \text{and} \quad \beta_2 = (b_4, b_3, b_2, b_1, a, a_1).$$

By Lemma 1 and the fact that $a_1 \in V(\text{Int } \tilde{\beta})$ we have
(m_1) $\text{Int } \beta_1 \subset \text{Int } \tilde{\beta}$;
(m_2) $V(\text{Int } \tilde{\beta}) = V(\text{Int } \beta_1) \cup V(\text{Int } \beta_2) \cup \{a_1\}$.
Since $b_j \in V(\text{Int } \alpha)$ for $1 \leqslant j \leqslant 3$ and also by Lemma 1, we get
(m_3) $\text{Int } \beta_2 \subset \text{Int } \alpha$.
Let us note that, by (m_1), $\text{Int } \beta \subset \text{Int } \tilde{\beta}_1$, and hence

$$2 \leqslant |V(\text{Int } \beta)| \leqslant |V(\text{Int } \tilde{\beta}_1)|.$$

Therefore, by Lemma 2 we have
(m_4) $|V(\text{Int } \beta_1)| = 0$.
Since $c \in V(\text{Int } \tilde{\alpha}) \cap V(\text{Int } \tilde{\beta})$, by ($m_2$) and ($m_3$) we get $c \in V(\text{Int } \beta_1)$. Thus we arrive at a contradiction with (m_4).

The proof in case (2,4) is analogous to that of (2,2). The remaining cases, i.e. (3,4), (4,3) and (4,4), can be reduced, by the reflexion of G in the plane, to cases (2,3), (3,2) and (2,2), respectively.

3. The local characterization of the family $\mathscr{S}$.

LEMMA 7. *The following conditions are equivalent for $G \in \mathscr{M}$:*

(i) *The neighbourhood of every vertex of degree 5 has at least four vertices of degree 5.*

(ii) *There exists a set $F \neq \emptyset$, $F \subset V(G)$, such that*

(a) $\deg a = 5$ *for* $a \in F$,

(b) *the neighbourhood of every vertex in F contains at least four vertices of F.*

(iii) $G \in \mathscr{S}$.

Proof. (ii) $\Rightarrow$ (iii). Assume that there exists $a_1 \in F$ whose neighbourhood $N(a_1)$ has a vertex $a \notin F$. Let $\deg a = n \geqslant 5$ and

$$N(a) = (a_1, a_2, \ldots, a_n), \quad \text{and} \quad N(a_1) = (a, a_n, b_1, b_2, a_2).$$

Since $a_1 \in F$ and $a \notin F$, vertices a_n, b_1, b_2, a_2 belong to F. Let $2 \leqslant k \leqslant n-1$. If $a_k \in F$, then

$$a_{k-1}, b_k, b_{k+1}, a_{k+1} \in F, \quad \text{where } N(a_k) = (a_{k-1}, b_k, b_{k+1}, a_{k+1}, a).$$

Therefore, $a_i, b_i \in F$ and $\deg a_i = \deg b_i = 5$ for $i = 1, 2, \ldots, n$. Since a_n adj b_1 and $\deg a_n = 5$, we have b_1 adj b_n. Put $\beta = (b_1, b_2, \ldots, b_n)$. Since $\deg_\beta(b_i) = 4$ for $i = 1, 2, \ldots, n$, there exists $b \in V(G)$ such that $N(b) = \tilde{\beta}$ and $\deg b = n$. This means that G has vertices a and b satisfying the conditions of the definition of $G(n)$. If the neighbourhood of every vertex from F has all vertices in F, then, analogously, we can show that $G = G(5)$.

(i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are obvious.

LEMMA 8. *The following conditions are equivalent for* $G \in \mathcal{M}$:

(i) *The neighbourhood of every vertex of degree 5 has three consecutive vertices of degree 5 and there exists a vertex of degree 5 whose neighbourhood does not have four vertices of degree 5.*

(ii) *For every vertex a of degree 5 there exists a cycle α such that*

(a) $\alpha = w_j$ *for some* $j = 3, 4, 5$;

(b) *all vertices of α are of degree* $\geqslant 6$;

(c) $a \in V(\mathrm{Int}\ \alpha)$.

Proof. (i) $\Rightarrow$ (ii). Let F be the set of all vertices of degree 5 for which there does not exist a cycle α satisfying the conditions (a), (b), and (c). Directly from the definition of the set F we get the implication

(1) a adj b and $\deg a = \deg b = 5$ imply $a \in F$ iff $b \in F$.

Now, we will show that

(2) If a is a vertex of degree 5 and $N(a)$ does not have four vertices of degree 5, then $a \notin F$.

Let $N(a) = (a_1, a_2, \ldots, a_5)$ and $\deg a_i = 5$ for $i = 1, 2, 3$ and $\deg a_j \geqslant 6$ for $j = 4, 5$. Let $N(a_2) = (a_1, c_1, c_2, a_3, a)$. Since $N(a_1)$ and $N(a_3)$ have three consecutive vertices of degree 5, $\deg c_1 = \deg c_2 = 5$. Therefore, $N(a_2)$ has all vertices of degree 5. Hence there exists a cycle β such that

$$a \in N(a_2) \subset V(\mathrm{Int}\ \beta) \quad \text{and} \quad \beta = w_5.$$

It is not difficult to note that if β does not have property (b), then β has two consecutive vertices of degree 5. Therefore, there exists a cycle γ such that $\mathrm{Int}\ \beta \subset \mathrm{Int}\ \gamma$ and $\gamma = w_4$. If γ does not have property (b), then there exists a cycle δ such that $\mathrm{Int}\ \gamma \subset \mathrm{Int}\ \delta$, $\delta = w_3$, and all vertices of δ are of degree $\geqslant 6$. Hence $a \notin F$. By (1) and (2), we see that the neighbourhood of every vertex of F has at least four vertices in F. By Lemma 7 the set F is empty.

(ii) $\Rightarrow$ (i) is obvious.

COROLLARY. *If G satisfies condition* (i) *of Lemma* 8, *then* $G \notin \mathcal{M}_5$.

4. Description of the family of all minimal graphs in $\mathcal{M}_5$.

THEOREM 1. *The following conditions are equivalent for* $\mathcal{M}_5$:

(i) *G is minimal in the family* $\mathcal{M}_5$.

(ii) *The neighbourhood of every vertex of degree 5 has three consecutive vertices of degree 5.*

(iii) *The neighbourhood of every vertex of degree 5 has four vertices of degree 5.*

(iv) $G \in \mathcal{S}$.

Proof. (i) $\Rightarrow$ (ii). Let us suppose that there is a vertex a, $\deg a = 5$, whose neighbourhood has two non-consecutive vertices of degree $\geqslant 6$. We will show that G is not minimal in $\mathcal{M}_5$. Let

$$N(a) = (a_1, a_2, \ldots, a_5).$$

Without loss of generality we may assume that one of the following cases occurs:

(1) $\deg a_i \geqslant 6$ for $i = 1, 2, \ldots, 5$;

(2) $\deg a_1 = 5$ and $\deg a_i \geqslant 6$ for $i = 2, \ldots, 5$;

(3) $\deg a_j = 5$ for $j = 1, 3$ and $\deg a_i \geqslant 6$ for $i = 2, 4, 5$.

(4) $\deg a_j = 5$ for $j = 1, 2$ and $\deg a_i \geqslant 6$ for $i = 3, 5$.

(1) If $G_{a,a_1} \notin \mathcal{M}_5$, $G_{a,a_2} \notin \mathcal{M}_5$ and $G_{a,a_3} \notin \mathcal{M}_5$, then there exist cycles $\alpha, \beta \in C(G)$, $|\alpha| = |\beta| = 6$, such that $\alpha \cap \beta \cap N(a) = \emptyset$. But, by Lemma 6, this is impossible.

(2) If $G_{a,a_3} \notin \mathcal{M}_5$ and $G_{a,a_4} \notin \mathcal{M}_5$, then there exist cycles $\alpha, \beta \in C(G)$, $|\alpha| = |\beta| = 6$, and $a, a_3 \in \alpha$, $a, a_4 \in \beta$. If $\alpha \cap \beta \cap N(a) = \emptyset$, then, by Lemma 6, we arrive at a contradiction. Let

$$N(a_1) = (a, a_5, b_1, b_2, a_2).$$

If $\alpha \cap \beta \cap N(a) \neq \emptyset$, then $a_1 \in \alpha \cap \beta$ and one of the following subcases is possible:

(2a) $b_1 \in \alpha$ and $b_2 \in \beta$;

(2b) $b_1 \notin \alpha$ or $b_2 \notin \beta$.

In (2a) we have $a_3, a, a_1, b_1 \in \alpha$ and $a_4, a, a_1, b_2 \in \beta$, which, by Lemma 5, leads to a contradiction. In (2b) we have $a_3, a, a_1, b_2 \in \alpha$ or $a_4, a, a_1, b_1 \in \beta$, which, by Lemma 3, leads to a contradiction.

(3) If $G_{a,a_1} \notin \mathcal{M}_5$ and $G_{a,a_3} \notin \mathcal{M}_5$, then there exist cycles $\alpha, \beta \in C(G)$, $|\alpha| = |\beta| = 6$, such that $a, a_1 \in \alpha$ and $a, a_3 \in \beta$. If $\alpha \cap \beta \cap N(a) = \emptyset$, then we get a contradiction by Lemma 6. If $\alpha \cap \beta \cap N(a) \neq \emptyset$, then $a_1, a, a_3 \in \alpha$ or $a_1, a, a_3 \in \beta$, and we get a contradiction by Lemma 4.

(4) Let

$$N(a_1) = (a, a_5, c_1, c_2, a_2) \quad \text{and} \quad N(a_2) = (a, a_1, c_2, c_3, a_3).$$

Two subcases are possible:

(4a) $\deg c_2 = 5$;

(4b) $\deg c_2 \geqslant 6$.

In (4a), if $G_{a,a_4} \notin \mathcal{M}_5$, then there is a cycle $\alpha \in C(G)$, $|\alpha| = 6$, such that $a, a_4 \in \alpha$ and one of the vertices c_1, c_2, c_3 belongs to α. If $c_2 \in \alpha$, then we get a contradiction by Lemma 4. If $c_1 \in \alpha$ or $c_3 \in \alpha$, then we get a contradiction by Lemma 3. In case (4b), if $G_{a,a_4} \notin \mathcal{M}_5$ and $G_{a_1,c_1} \notin \mathcal{M}_5$, then there exist cycles $\alpha, \beta \in C(G)$, $|\alpha| = |\beta| = 6$, such that $a_4, a \in \alpha$ and $c_1, a_1 \in \beta$. If one of the vertices c_1, c_3 belongs to α or one of the vertices c_3, a_4 belongs to β, then we arrive at a contradiction by Lemma 3. Otherwise, we get $c_2 \in \alpha$ and $a_3 \in \beta$. But then we can assume that $a_4, a, a_1, c_2 \in \alpha$ and $a_3, a, a_1, c_1 \in \beta$ and we arrive at a contradiction by Lemma 5.

(iii) $\Rightarrow$ (iv) follows from the Corollary to Lemma 8.

(iv) $\Rightarrow$ (i) is obvious, since $\mathscr{S} \subset \mathscr{M}_5$.

THEOREM 2. *The following conditions are equivalent:*

(i) *G is minimal in the family $\mathscr{M}_5 \setminus \mathscr{S}$.*

(ii) $G = A$ (see Fig. 1).

Proof. (i) $\Rightarrow$ (ii). Since G is minimal in the family $\mathscr{M}_5 \setminus \mathscr{S}$, by Theorem 1 there exist $n \geqslant 6$ and two adjacent vertices $x, y \in V(G)$ such that $\deg x = 5$, $\deg y = n-1$ and $G_{x,y} = G(n)$. For example, if $G_{x,y} = G(7)$, then the graph G has the representation shown in Fig. 2.

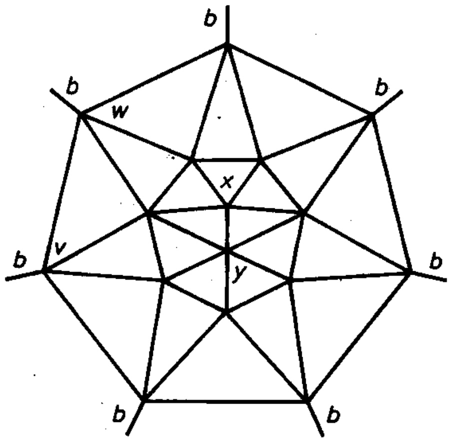

Fig. 2. $G_{x,y} = G(7)$

If $n \geqslant 7$, it is easy to note that there exist vertices $v, w \in V(G)$ such that

(i_1) $\deg v = \deg w = 5$;

(i_2) $G_{v,w} \in \mathscr{M}_5$;

(i_3) $G_{v,w}$ has four vertices of degree $\geqslant 6$.

Since G is minimal in the family $\mathscr{M}_5 \setminus \mathscr{S}$, by ($i_1$), ($i_2$) and Theorem 1 there exists $m \geqslant 6$ such that $G_{v,w} = G(m)$. We have a contradiction with (i_3) since $G(m)$ has two vertices of degree $\geqslant 6$.

If $n = 6$, then obviously $G = A$.

(ii) $\Rightarrow$ (i). It is enough to note that if for any $a, b \in V(A)$, a adj b, $A_{a,b}$ is in $\mathscr{M}$, then $A_{a,b} = G(6)$.

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