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THE AUTOMORPHISMS GROUPS OF SYMMETRIC ALGEBRAS

BY

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In this paper the following problem is considered. Let A be a set (finite or infinite) and let $\boldsymbol{P}$ be a sequence of groups P_k, $1 \leqslant k < n$ (n finite or infinite), where P_k is a subgroup of the permutation group S_k. We are interested in the transformation groups G acting on the set A such that there exist families $\boldsymbol{F}^{(k)}$, $1 \leqslant k < n$, of functions

$$f\colon \underbrace{A \times A \times \ldots \times A}_{k \text{ times}} \to A$$

such that

$$(*) \qquad f(x_1, x_2, \ldots, x_k) = f(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma k}) \quad \text{for all } \sigma \in P_k,\ 1 \leqslant k < n,$$

and

$$(**) \qquad G = \{g\colon g^{-1}f(gx_1, \ldots, gx_k) = f(x_1, \ldots, x_k) \quad \text{for all } f \in \boldsymbol{F}^{(k)},\ g \in G,\ 1 \leqslant k < n\}.$$

In [1] Jónsson investigated the problem of characterization of G such that (*) and (**) are satisfied with $P_k = \{e\}$, $1 \leqslant k < n$. Necessary and sufficient conditions on G for which (*) and (**) hold and $P_k = \{e\}$, $1 \leqslant k < n$, are given in [4].

The problem of describing G such that (*) and (**) hold for a prescribed sequence of groups P_k, $1 \leqslant k < n$, has a natural algebraic formulation. In fact, if (**) is satisfied, then G is the group of automorphisms of the algebra $\mathfrak{A} = \langle A; \bigcup_{1 \leqslant k < n} \boldsymbol{F}^{(k)} \rangle$ (cf. [2]) the fundamental operations of which are subdued to the symmetry conditions (*).

1. Terminology and notions. We fix a set A with $|A| > 1$. Let S_A denote the group of all one-to-one transformations of A onto itself. We shall write A^k for $\underbrace{A \times A \times \ldots \times A}_{k \text{ times}}$. For a function $f\colon A^k \to A$ by $s(f)$ we denote the group of all permutations σ such that

$$f(x_1, x_2, \ldots, x_k) = f(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma k})$$

holds for all $x_1, x_2, \ldots, x_k \in A$.

Clearly, if $P_k \subset s(f)$, then (*) holds; we then say that f is *P_k-symmetric.*

If for a sequence $\boldsymbol{P}$ and for a transformation group G ($G \subset S_A$) there exists an algebra $\mathfrak{A}$ such that (**) and (*) are satisfied, we call it a ***GP*-*algebra*** or say that the condition $a(G, \boldsymbol{P})$ holds.

If G acts on A and $x = \langle x_1, x_2, \ldots, x_k\rangle \in A^k$, let

$$gx = \langle gx_1, gx_2, \ldots, gx_k\rangle, \tag{1}$$

and, for $\sigma \in S_k$, let

$$\sigma x = \langle x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma k}\rangle. \tag{2}$$

For $G \subset S_A$ and $P_k \subset S_k$ let

$$\boldsymbol{F}_{GP_k} = \{f\colon s(f) \supset P_k, f(gx) = gf(x) \text{ for all } x \in A^k \text{ and } g \in G\}. \tag{3}$$

For $x \in A^k$ we put

$$G_x = \{g\colon\ gx \in P_k(x)\}$$

and

$$\boldsymbol{D}_{GP_k}(x) = \{a\colon\ a \in A,\ ga = a \text{ for } g \in G_x\}. \tag{4}$$

2. Lemmas. Observe first that transformations g and σ of A^k as defined in (1) and (2) commute with each other and that, for any $\sigma \in P_k$,

$$\boldsymbol{D}_{GP_k}(x) = \boldsymbol{D}_{GP_k}(\sigma x).$$

The role of sets $\boldsymbol{D}_{GP_k}(x)$ is explained by the following

LEMMA 1. *Suppose that for every k, $1 \leqslant k < n$, and for all $x \in A^k$ we have $\boldsymbol{D}_{GP_k}(x) \neq \emptyset$. Then there exists a function $f \in \boldsymbol{F}_{GP_k}$ with $f(x) = a$ if and only if $a \in \boldsymbol{D}_{GP_k}(x)$.*

Proof. Suppose that $f \in \boldsymbol{F}_{GP_k}$ and $f(x) = a$. Let $gx = \sigma x$ for some $g \in G$ and $\sigma \in P_k$. Then

$$ga = gf(x) = f(gx) = f(\sigma x) = f(x) = a,$$

and therefore $a \in \boldsymbol{D}_{GP_k}(x)$.

Now let $a \in \boldsymbol{D}_{GP_k}(x)$ and $\boldsymbol{D}_{GP_k}(y) \neq \emptyset$ for all $y \in A^k$. Decompose A^k into GP_k orbits,

$$GP_k(y),\ GP_k(z),\ \ldots,$$

and let s be a selector from the orbits such that

$$s\colon GP_k(x) \to x.$$

Put

$$f(y) = \begin{cases} a \text{ if } y = x, \\ \text{an arbitrary element } b \text{ in } \boldsymbol{D}_{GP_k}(y) \text{ if } y \neq x \text{ and } y = s\big(G\boldsymbol{P}(y)\big), \\ gf(z) \text{ if } y = g\sigma z \text{ for } g \in G \text{ and } \sigma \in P_k. \end{cases} \tag{5}$$

Clearly, y can be presented as $y = g\sigma z$ and $y = g'\sigma' z$ for different $g, g' \in G$ and $\sigma, \sigma' \in P_k$, and $z \in s(GP_k(z))$; therefore we have to prove that $gf(z) = g'f(z)$ for such g and g'. In fact, we have $g'^{-1}gz = \sigma^{-1}\sigma' z$, whence, by $f(y) \in \boldsymbol{D}_{GP_k}(y)$, we obtain

$$g'f(y) = gf(y).$$

By definition (5), we have $f(g\sigma y) = gf(\sigma y) = gf(y)$ for all $g \in G$, $\sigma \in P_k$, and $y \in A^k$. Consequently, $f \in \boldsymbol{F}_{GP_k}$.

Thus the proof of Lemma 1 is completed.

Inclusion $\boldsymbol{F} \subset \bigcup_{P_k \in \boldsymbol{P}} \boldsymbol{F}_{GP_k}$ valid for every $G\boldsymbol{P}$-algebra $\mathfrak{A} = \langle A; \boldsymbol{F} \rangle$ yields

LEMMA 2. *Condition $a(G, \boldsymbol{P})$ is equivalent to the equality*

$$\mathrm{Aut}\,\langle A;\ \bigcup_{P_k \in \boldsymbol{P}} \boldsymbol{F}_{GP_k} \rangle = G.$$

Let us now consider the following two properties of a pair $\langle \varphi, x \rangle$, where $\varphi \in S_A$, $x \in A^k$:

$W_1(\varphi, x)$. *For all $g \in G$ and $\sigma \in P_k$ the equality $\varphi x = g\sigma x$ implies $\varphi | \boldsymbol{D}_{GP_k}(x) = g | \boldsymbol{D}_{GP_k}(x)$.*

$W_2(\varphi, x)$. *$\varphi x \notin GP_k(x)$ implies $|\boldsymbol{D}_{GP_k}(x)| = 1$ and $\boldsymbol{D}_{GP_k}(\varphi x) = \varphi \boldsymbol{D}_{GP_k}(x)$.*

LEMMA 3. *A permutation φ is an automorphism of the algebra $\langle A;\ \bigcup_{P_k \in \boldsymbol{P}} \boldsymbol{F}_{GP_k} \rangle$ if and only if for every k, $1 \leqslant k < n$, and every $x \in A^k$ either both $W_1(\varphi, x)$ and $W_2(\varphi, x)$ are satisfied or $\boldsymbol{D}_{GP_k}(y) = \emptyset$ for some $y \in A^k$.*

Proof. Let us suppose that $\varphi \in \mathrm{Aut}\,\langle A;\ \bigcup_{P_k \in \boldsymbol{P}} \boldsymbol{F}_{GP_k} \rangle$, $x \in A^k$, and the sets $\boldsymbol{D}_{GP_k}(y)$ are non-empty for all $y \in A^k$. We shall show that $W_1(\varphi, x)$ and $W_2(\varphi, x)$ are satisfied.

In fact, if $\varphi x = g\sigma x$ and $a \in \boldsymbol{D}_{GP_k}(x)$, then, by Lemma 1, there exists a function $f \in \boldsymbol{F}_{GP_k}$ with $f(x) = a$. Hence

$$\varphi a = f(x) = f(\varphi x) = f(g\sigma x) = gf(\sigma x) = ga.$$

Thus $W_1(\varphi, x)$ is satisfied.

Contrary to our statement, suppose that $W_2(\varphi, x)$ does not hold. By Lemma 1, we have

$$\varphi \boldsymbol{D}_{GP_k}(x) = \boldsymbol{D}_{GP_k}(\varphi x)$$

and, therefore, we may assume that $|\boldsymbol{D}_{GP_k}(x)| \neq 1$. Let $a \neq b$ be two elements from $\boldsymbol{D}_{GP_k}(x)$. Applying Lemma 1 again, we infer that there exist two functions f_a and f_b in $\boldsymbol{F}_{GP_k}$ such that $f_a(x) = a$ and $f_b(x) = b$. Moreover, since $\varphi x \in GP_k(x)$, one can choose f_a and f_b such that

$$f_a(\varphi x) = f_b(\varphi x).$$

This, however, gives

$$\varphi a = \varphi f_a(x) = f_a(\varphi x) = f_b(\varphi x) = \varphi f_b(x) = \varphi b,$$

which is a contradiction, since $a \neq b$.

In order to prove the converse implication, assume that $f \in \boldsymbol{F}_{GP_k}$ and $x \in A^k$. Note that $\boldsymbol{D}_{GP_k}(y) \neq \emptyset$ for every $y \in A^k$ and consider two cases (a) and (b).

(a) $\varphi x \in GP_k(x)$. If $\varphi x = g\sigma x$, then using the condition $W_1(\varphi, x)$ we get

$$f(\varphi x) = f(g\sigma x) = gf(x) = \varphi f(x),$$

since $f(x)$ is an element of $\boldsymbol{D}_{GP_k}(x)$.

(b) $\varphi x \notin GP_k(x)$. Then, by $W_2(\varphi, x)$, we obtain

$$\varphi \boldsymbol{D}_{GP_k}(x) = \boldsymbol{D}_{GP_k}(\varphi x).$$

Hence and from the equality

$$f(x) = \boldsymbol{D}_{GP_k}(x),$$

being a consequence of $|\boldsymbol{D}_{GP_k}(x)| = 1$, follows the proof.

3. Theorem. *For a family $\boldsymbol{P}$ and a transformation group G acting on a set A there exists a family of functions $\boldsymbol{F} = \bigcup_{1 \leqslant k < n} \boldsymbol{F}_k$ such that $(*)$ and $(**)$ are satisfied if and only if for every one-to-one transformation φ of A, if conditions $W_1(\varphi, x)$ and $W_2(\varphi, x)$ are satisfied for every x in A^k, $1 \leqslant k < n$, for which $\boldsymbol{D}_{GP_k}(x) \neq \emptyset$, then $\varphi \in G$.*

Proof follows immediately from lemmas 2 and 3.

4. Discusion and examples. If all groups in the family $\boldsymbol{P}$ are trivial, then $a(G, \boldsymbol{P})$ can be reduced to the condition $a_n(G)$ (cf. [1]). It is not hard to see that, in this case, the theorem yields the theorem from [4].

To get an answer to a question of E. Marczewski concerning conditions on transformation group G which would ensure the existence of a symmetric algebra for which G is the group of automorphisms, it is enough to put $P_k = S_k$ for $k = 1, 2, \ldots$ and apply the theorem.

As is shown in [1], if $2 \neq n < \omega$, then $a_{n+1}(G)$ need not imply $a_n(G)$. The following two examples give some more information on $a_n(G)$ (cf. [1], examples).

Example 1 shows that $a_\omega(G)$ need not imply $a_n(G)$ for every finite n.

Let A_k be the alternating group on a k-element set X_k, $k \geqslant 2$, let X be the disjoint union of X_k, $k \geqslant 2$, and let G be the direct product of A_k, $k \geqslant 2$, acting on X. Since for every $\varphi \notin G$ there exists a finite set Y, $Y \subset X$, such that $\varphi | Y \neq g | Y$ for all $g \in G$, the condition $\beta_{\aleph_0}(G)$ of [1] holds. Therefore $a_\omega(G)$ holds.

For $a, b \in X_{n+2}$, $a \neq b$, we construct a permutation φ on the set X, which does not belong to G, setting

$$\varphi(x) = \begin{cases} a & \text{if } x = b, \\ b & \text{if } x = a, \\ x & \text{otherwise.} \end{cases}$$

Clearly, for arbitrary Y with $|Y| = n-1$ there exists a $g \in G$ which agrees with φ on Y. Since $C_G(Y) = Y$ (cf. [4]), we see that $a_n(G)$ does not hold.

Example 2. $a_3(G)$ need not imply $a_2(G)$. The only symmetric operations which commute with each $g \in G$ are the unary operations.

Consider $X = \bigcup_{k=2}^{\infty} X_k$, where $X_2 = \{a_1, a_2\}$, $X_3 = \{b_1, b_2, b_3\}, \ldots,$ $X_n = \{c_1, \ldots, c_n\}, \ldots$; and let G be the direct product

$$G = S_2 \times A_3 \times A_4 \times \prod_{k=5}^{\infty} S_k.$$

We have

$$\begin{aligned} C_G(\{a_1\}) &= C_G(\{a_2\}) = \{a_1, a_2\}, \\ C_G(\{b_1\}) &= C_G\{(b_2)\} = C_G(\{b_3\}) = \{b_1, b_2, b_3\}, \\ C_G(\{x\}) &= \{x\} \text{ if } x \in X_k,\ k \geqslant 4, \\ C_G(\{a_1, x\}) &= C_G(\{a_2, x\}) = \{a_1, a_2, x\} \text{ for all } x \in X, \\ C_G(\{x, y\}) &= X_k \text{ if } x, y \in X_k \text{ and } k = 3 \text{ or } 4, \\ C_G(\{b_i, x\}) &= \{b_1, b_2, b_3, x\} \text{ if } x \notin \{a_1, a_2\} \text{ and } i = 1, 2, 3, \\ C_G(\{x, y\}) &= X_4 \text{ if } x, y \in X_4 \text{ and } x \neq y, \\ C_G(\{x, y\}) &= \{x, y\} \text{ if } x \in X_k,\ y \in X_l \text{ and } k \geqslant 4,\ l > 4. \end{aligned}$$

Let φ be a permutation of the set X, and suppose that for every 2-element set Y the permutation φ agrees on $C_G(Y)$ with some $g \in G$. Then, as is not difficult to check, φ is in G. Consequently, $a_3(G)$ holds.

In order to prove that $a_2(G)$ does not hold it is sufficient to consider permutation φ defined as follows:

$$\varphi x = \begin{cases} c_3 & \text{if } x = c_4, \\ c_4 & \text{if } x = c_3, \\ x & \text{otherwise.} \end{cases}$$

It is clear that for every $x \in X$ there exists an element $g \in G$ such that $\varphi | C_G(\{x\}) = g | C_G(\{x\})$ and $\varphi \notin G$. Therefore, by the theorem of [4], there is no unary G-algebra.

Finally, we shall show that $\boldsymbol{D}_{GS_n}(\langle c_1, \ldots, c_n\rangle) = \emptyset$ for $n \geqslant 2$. In fact, if $a \notin X_n$, then a cannot belong to $\boldsymbol{D}_{GS_n}$, for there exists a $g \in G$ which moves a. Since for every $x \in X_n$ there exists a $g \in G$ such that $gx \neq x$, then x is not in $\boldsymbol{D}_{GS_n}(\langle c_1, \ldots, c_n\rangle)$; thus $\boldsymbol{D}_{GS_n}$ must be empty.

COROLLARY. *There exists a transformation group G such that there is no symmetric G-algebra although a binary G-algebra exists.*

Following E. Marczewski (see [3]) we say that a k-ary operation f is *quasi-symmetric* if for any l, m, $1 \leqslant m \leqslant l \leqslant k$, there exists a $\sigma \in s(f)$ such that $\sigma(l) = m$ and $\sigma(m) = l$. An algebra $\langle A; \boldsymbol{F}\rangle$ is called *quasi-symmetric* if all operations from $\boldsymbol{F}$ are quasi-symmetric. Observe that if $s(f)$ is a doubly transitive group, then f is quasi-symmetric and, therefore, our theorem yields a sufficient condition on G which ensures the existence of a quasi-symmetric G-algebra.

A simple modification of the theorem yields a necessary and sufficient condition.

Example 3 shows that for a group G and all $n \geqslant 4$ there exists n-ary quasi-symmetric G-algebra, and does not exist an n-ary symmetric G-algebra.

To this purpose take $X = \{0, 1, 2, \ldots, n+1\}$, $G = \{g\colon g(0) = 0, g \in A_{n+2}\}$, and $\boldsymbol{P} = \langle S_1, S_2, S_3, A_4, \ldots, A_n\rangle$ for $n \geqslant 4$. Every $G\boldsymbol{P}$-algebra is, of course, a quasi-symmetric G-algebra. Let us note that

$$\boldsymbol{D}_{GA_n}(\langle 1, 2, \ldots, n\rangle) = \{0, n+1, n+2\},$$

$$\boldsymbol{D}_{GS_n}(\langle 1, 2, \ldots, n\rangle) = \{0\}.$$

If φ is a permutation on X with $\varphi(0) = 0$, $\varphi \in G$, then $\varphi\langle 1, 2, \ldots, n\rangle = g\langle 1, 2, \ldots, n\rangle$ for some $g \in G$ and, in addition, $\varphi|\boldsymbol{D}_{GA_n}(\langle 1, 2, \ldots, n\rangle) \neq g|\boldsymbol{D}_{GA_n}(\langle 1, 2, \ldots, n\rangle)$. Since the unary operation $f(x) \equiv 0$ commutes with each $g \in G$, $\alpha(G, \boldsymbol{P})$ holds.

Now let us consider a permutation $\varphi \in S_X$ such that $\varphi(0) = 0$ and φ is odd. Observe that for every k, $1 \leqslant k \leqslant n$, and every $\langle x_1, x_2, \ldots, x_k\rangle \in A^k$ we have the inclusion

$$\boldsymbol{D}_{GS_k}(\langle x_1, \ldots, x_k\rangle) \subset \{0, x_1, \ldots, x_k\}.$$

It follows that $W_1(\varphi, x)$ and $W_2(\varphi, x)$ are satisfied for all x. Since $\varphi \notin G$ and $\boldsymbol{D}_{GS_k}(\langle x_1, \ldots, x_k\rangle) \neq \emptyset$, the symmetric n-ary G-algebra does not exist.

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