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Remarks on some generalizations of asymptotic periodicity in dynamical systems on metric spaces

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Abstract. Dynamical systems on metric spaces are considered. The notion of the asymptotic periodicity (and the asymptotic pseudoperiodicity as well) is generalized and some theorems extending certain known results are proved.

The purpose of the paper is to present some generalizations of the previous author's results (cf. [2], [3]). In [2], the condition of the asymptotic periodicity of motions in dynamical systems on metric spaces has been introduced and investigated. Here, we propose some generalization of that condition, and prove corresponding theorems generalizing simultaneously results presented in [2] and [3]. Certain observations concerning dynamical systems on the real plane are presented in the last Section 8.

1. We shall consider dynamical systems in metric spaces. In order to exclude any misfits we would like to recall the usual terminology (see, for instance, [1], [4]).

Let (X, ϱ) be a metric space. A mapping $\pi\colon \boldsymbol{R}\times X \to X$ is said to be a *dynamical system on* X (or, using another terminology: the triplet $(X, \boldsymbol{R}; \pi)$ is called a *dynamical system*) if and only if

$\pi(0, x) = x$ for every $x \in X$,

$\pi(t, \pi(s, x)) = \pi(t+s, x)$ for $s, t \in \boldsymbol{R}$, $x \in X$,

π is continuous.

For a fixed $x \in X$, we denote by π^x the mapping

$$\boldsymbol{R} \ni t \to \pi^x(t) := \pi(t, x) \in X$$

and we call it the *motion of* x.

The positive limit set of the point x is the set

$$\Lambda^+(x) := \{y\colon \text{there is a sequence } \{t_m\} \text{ of real numbers such that } t_m \to \infty \text{ and } \pi(t_m, x) \to y \text{ as } m \to \infty\}.$$

The negative limit set $\Lambda^-(x)$ is defined by substituting $t_m \to -\infty$ in the place of $t_m \to \infty$.

2. Let $(X, \boldsymbol{R}, \pi)$ be an arbitrary dynamical system, fixed throughout Sections 2–6, and let

$$\alpha\colon (0, \infty) \to (0, \infty), \qquad \gamma, \tau\colon [0, \infty) \to [0, \infty)$$

be three functions fixed throughout Sections 2–5. In Sections 7 and 8, some additional conditions will be assumed with respect to X and functions α and γ.

Let x be a given point of X. We say that the motion π^x *satisfies the condition* $P^+[\alpha, \gamma]$ (resp. $P^-[\alpha, \gamma]$) if and only if

(2.1) for every $\varepsilon > 0$ there exists $s \geqslant 0$ $(s \leqslant 0)$ such that

$$\varrho\big(\pi(t+\alpha(\varepsilon), x), \pi(t, x)\big) \leqslant \gamma(\varepsilon) \quad \text{for } t \geqslant s \text{ (resp. } t \leqslant s).$$

We say that π^x *satisfies the condition* $S^+[\tau]$ (resp. $S^-[\tau]$) if and only if

(2.2) for every $\varepsilon > 0$ there exist $\delta > 0$ and $s \geqslant 0$ (resp. $s \leqslant 0$) such that

$$\varrho(x, y) < \delta \Rightarrow \varrho\big(\pi(t, x), \pi(t, y)\big) \leqslant \tau(\varepsilon) \quad \text{for } t \geqslant s \ (t \leqslant s).$$

EXAMPLES. I. Assume that $\alpha(\varepsilon) := \alpha = \text{const} > 0$ and $\gamma(\varepsilon) := \eta + \varepsilon$, where η is some fixed nonnegative constant. In this case the condition $P^+[\alpha, \gamma]$ is equivalent to the following one:

(2.3) for every $\varepsilon > 0$ there is $s \geqslant 0$ such that

$$\varrho\big(\pi(t+\alpha, x), \pi(t, x)\big) \leqslant \eta + \varepsilon \quad \text{for } t \geqslant s.$$

That condition (called *positive asymptotic* (η, α)*-pseudoperiodicity*) has been investigated in [3] together with positive η*-pseudostability* which is equivalent to $S^+[\tau]$, where $\tau(\varepsilon) := \eta + \varepsilon$.

II. A special case of the condition $P^+[\alpha, \gamma]$ considered in Example I is that which corresponds to $\eta = 0$. It was discussed in [2] for $\gamma(\varepsilon) = \varepsilon$ as *positive asymptotic periodicity*.

III. In the case of $\alpha(\varepsilon) := \alpha = \text{const} > 0$ and $\gamma(\varepsilon) := 0$ we obtain $P^+[\alpha, \gamma]$ (and $P^-[\alpha, \gamma]$ as well) as the classical *periodicity* of the motion π^x; α is a period in that case.

IV. The classical *Lyapunov positive* (*negative*) *stability of* π^x is equivalent to $S^+[\tau]$ $(S^-[\tau])$ with $\tau(\varepsilon) := \varepsilon$ and $s = 0$ in (2.2).

V. If η is not a constant function, then putting $\gamma(\varepsilon) = \varepsilon$ we obtain the condition $P^+[\alpha, \gamma]$ being a generalization of the *almost periodicity*.

3. THEOREM 1. *Assume that* $x \in X$. *If* $y \in \Lambda^+(x)$ $(y \in \Lambda^-(x))$ *and* π^x *satisfies the condition* $P^+[\alpha, \gamma]$ $(P^-[\alpha, \gamma])$, *then for every* $\varepsilon > 0$ *and every* t *the following inequality*

$$(3.1) \qquad \varrho\big(\pi(t+\alpha(\varepsilon), y), \pi(t, y)\big) \leqslant \gamma(\varepsilon)$$

holds true.

Proof. Assume that $y \in \Lambda^+(x)$ and that π^x satisfies the condition $P^+[\alpha, \gamma]$. Let $t \in \boldsymbol{R}$ be arbitrarily fixed. Let $\{t_m\}$ be a sequence of real numbers such that for $m \to \infty$

$$t_m \to \infty$$

and

$$(3.2) \qquad \pi(t_m, x) \to y.$$

We have for every fixed $\varepsilon > 0$

$$(3.3) \qquad \varrho\big(\pi(t+\alpha(\varepsilon), y), \pi(t, y)\big) \leqslant \varrho\big(\pi(t+\alpha(\varepsilon), y), \pi(t+\alpha(\varepsilon)+t_m, x)\big) + \varrho\big(\pi(t+\alpha(\varepsilon)+t_m, x), \pi(t+t_m, x)\big) + \varrho\big(\pi(t+t_m, x), \pi(t, y)\big).$$

The mapping π is continuous, and so, from (3.2), we get

$$\pi(t+t_m, x) = \pi\big(t, \pi(t_m, x)\big) \to \pi(t, y)$$

and similarly

$$\pi\big(t+\alpha(\varepsilon)+t_m, x\big) \to \pi\big(t+\alpha(\varepsilon), y\big)$$

as $m \to \infty$.

Thus the first and the third terms on the right-hand side of (3.3) tend to zero as $m \to \infty$. In order to estimate the second term we observe that for m sufficiently large we have $t+t_m \geqslant s$, where s is chosen according to the condition $P^+[\alpha, \gamma]$ in such a way that (2.1) holds for $t \geqslant s$.

Hence we have proved that

$$\varrho\big(\pi(t+\alpha(\varepsilon), y), \pi(t, y)\big) \leqslant \gamma(\varepsilon) + a(t_m),$$

where

$$a(t_m) := \varrho\big(\pi(t+\alpha(\varepsilon), y), \pi(t+\alpha(\varepsilon)+t_m, x)\big) + \varrho\big(\pi(t+t_m, x), \pi(t, y)\big)$$

and so $a(t_m) \to 0$ as $m \to \infty$. Thus (3.1) holds true. The proof is completed in the case $y \in \Lambda^+(x)$ and π^x satisfying $P^+[\alpha, \gamma]$.

The proof in the case of the condition $P^-[\alpha, \gamma]$ satisfied by π^x and y belonging to $\Lambda^-(x)$ is clearly similar.

4. THEOREM 2. *Suppose that $x \in X$, $y \in \Lambda^+(x)$ ($y \in \Lambda^-(x)$) and π^y satisfies the conditions $S^+[\tau]$ ($S^-[\tau]$) and $P^+[\alpha, \gamma]$ ($P^-(\alpha, \gamma]$). Then π^x satisfies the condition $P^+[\alpha, \tilde{\gamma}]$ (resp. $P^-[\alpha, \tilde{\gamma}]$), where*

$$\tilde{\gamma}(\varepsilon) := \gamma(\varepsilon) + 2\tau(\varepsilon).$$

Proof. Assume that $y \in \Lambda^+(x)$ and suppose that π^y satisfies the conditions

$S^+[\tau]$ and $P^+[\alpha, \gamma]$. Let $\{s_m\}$ be such a sequence of real numbers tending to the infinity for which

$$y = \lim \pi(s_m, x). \tag{4.1}$$

We have to prove that for every $\varepsilon > 0$ there is $s \geqslant 0$ such that if $t \geqslant s$ then

$$\varrho\big(\pi(t+\alpha(\varepsilon), x), \pi(t, x)\big) \leqslant \tilde{\gamma}(\varepsilon).$$

Assume the contrary; then there is $\varepsilon > 0$, say $\varepsilon = \varepsilon^0$, such that for every $s \geqslant 0$ there is $t \geqslant s$ for which

$$\varrho\big(\pi(t+\alpha(\varepsilon^0), x), \pi(t, x)\big) > \gamma(\varepsilon^0)+2\tau(\varepsilon^0). \tag{4.2}$$

So there exists a sequence $\{t_m\}$ of non-negative numbers such that

$$\varrho\big(\pi(t_m+s_m+\alpha(\varepsilon^0), x), \pi(t_m+s_m, x)\big) > \gamma(\varepsilon^0)+2\tau(\varepsilon^0). \tag{4.3}$$

We may assume without loss of generality that $t_m \to \infty$. By the triangle inequality we get

$$\begin{aligned}\varrho\big(\pi(t_m+s_m+\alpha(\varepsilon^0), x), \pi(t_m+s_m, x)\big) \\ \leqslant \varrho\big(\pi(t_m+\alpha(\varepsilon^0), \pi(s_m, x)), \pi(t_m+\alpha(\varepsilon^0), y)\big) \\ +\varrho\big(\pi(t_m+\alpha(\varepsilon^0), y), \pi(t_m, y)\big)+\varrho\big(\pi(t_m, y), \pi(t_m,\pi(s_m, x))\big).\end{aligned} \tag{4.4}$$

The motion π^y satisfies the condition $S^+[\tau]$. So the first and the third terms on the right-hand side of (4.4) are, for m sufficiently large, estimated by $\tau(\varepsilon^0)$. The second term is for m large enough not greater than $\gamma(\varepsilon^0)$. Thus we have obtained a contradiction with (4.3). The proof is completed.

5. Remarks. I. It is not difficult to observe (compare examples in Section 2) that Theorem 1 of [2] and Theorem 1 of [3] can be deduced directly from Theorem 1 of the present paper, while Theorems 2 and 3 of [2] and Theorem 2 of [3] can be obtained as simple corollaries of Theorem 2 in Section 4.

II. It is easy to observe that in the proof of Theorem 1 we need essentially the continuity of π only with respect to the second variable.

For further remarks and some corollaries concerning dynamical systems on the real plane, see Section 8 below.

6. Assume that $(X, \boldsymbol{R}, \pi)$ is a dynamical system as in Sections 2–5.

Let β be a positive real functions defined on $(0, \infty)\times \boldsymbol{R}$ and let γ be, as previously, a real nonnegative function defined on $[0, \infty)$. We shall consider the (formally extended) conditions $P^+[\beta, \gamma]$ and $P^-[\beta, \gamma]$ as in Section 2: a motion π^x satisfies the condition $P^+[\beta, \gamma]$ ($P^-[\beta, \gamma]$) if and only if for every $\varepsilon > 0$ there is $s \geqslant 0$ ($s \leqslant 0$) such that

$$\varrho\big(\pi(t+\beta(\varepsilon, t), x), \pi(t, x)\big) \leqslant \gamma(\varepsilon) \quad \text{for } t \geqslant s \ (t \leqslant s). \tag{6.1}$$

THEOREM 3. *Assume that the function β introduced above is such that for every $\varepsilon > 0$ there exists the limit*

$$(6.2)\qquad \alpha(\varepsilon) := \lim \beta(\varepsilon, s) \quad \textit{as} \quad s \to +\infty \ (\textit{as}\ s \to -\infty)$$

and it is a positive number.

Let $x \in X$, $y \in \Lambda^+(x)$ ($y \in \Lambda^-(x)$). If π^x satisfies the condition $P^+[\beta, \gamma]$ ($P^-[\beta, \gamma]$), then for every $\varepsilon > 0$ and every $t \in \boldsymbol{R}$ inequality (3.1) *holds true with α defined by* (6.2).

Proof. We apply the same method as in the proof of Theorem 1. Assume that π^x satisfies $P^+[\beta, \gamma]$ and $y \in \Lambda^+(x)$. We have $y = \lim \pi(t_n, x)$ for some $\{t_n\}$ such that $t_n \to \infty$. Using the triangle inequality, we get

$$(6.3)\qquad \begin{aligned} &\varrho\big(\pi(t+\beta(\varepsilon, t+t_n), y), \pi(t, y)\big) \\ &\quad \leqslant \varrho\big(\pi(t+\beta(\varepsilon, t+t_n), y), \pi(t+\beta(\varepsilon, t+t_n)+t_n, x)\big) \\ &\qquad + \varrho\big(\pi(t+\beta(\varepsilon, t+t_n)+t_n, x), \pi(t+t_n, x)\big) + \varrho\big(\pi(t+t_n, x), \pi(t, y)\big). \end{aligned}$$

The third term on the right-hand side of the above inequality tends clearly to zero as $n \to \infty$ because of the convergence $\pi(t_n, x) \to y$ and the continuity of π (there we need, similarly as in Theorem 1, the continuity of the mapping $z \to \pi(t, z)$ for every fixed t; compare Remark II in Section 5). The second term is for sufficiently large n dominated by $\gamma(\varepsilon)$ because of the condition $P^+[\beta, \gamma]$. The first term tends to zero as $n \to \infty$, since $\beta(\varepsilon, t+t_n) \to \alpha(\varepsilon)$, $\pi(t_n, x) \to y$ as $n \to \infty$ and so the sequence $\{r_n\}$, where

$$r_n := \pi\big(t+\beta(\varepsilon, t+t_n)+t_n, x\big) = \pi\big(t+\beta(\varepsilon, t+t_n), \pi(t_n, x)\big),$$

is convergent to $\pi\big(t+\alpha(\varepsilon), y\big)$ as well as the sequence $\{u_n\}$ with $u_n := \pi\big(t+\beta(\varepsilon, t+t_n), y\big)$. The left-hand side of inequality (6.3) converges to $\varrho\big(\pi(t+\alpha(\varepsilon), y), \pi(t, y)\big)$. So, passing to the limit as $n \to \infty$, we get from (6.3) the inequality required in the assertion:

$$\varrho\big(\pi(t+\alpha(\varepsilon), y), \pi(t, y)\big) \leqslant \gamma(\varepsilon).$$

The proof in the case of $P^-[\beta, \gamma]$ and $y \in \Lambda^-(x)$ is similar.

Remark III. Observe that in the proof of Theorem 3 we need the continuity of π, and so we cannot modify (formally: generalize) our theorem replacing the continuity of π by the weaker condition mentioned in Remark II in Section 5. In the theory of classical (continuous) dynamical systems, however, Theorem 3 is a generalization of Theorem 1 proved in Section 2.

7. Let us consider α and γ introduced in Section 2, assuming that

$$(7.1)\qquad \gamma(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0,\ \varepsilon > 0.$$

PROPOSITION 1. *Suppose that π^y satisfies the inequality*

$$(7.2) \qquad \varrho\big(\pi(t+\alpha(\varepsilon), y), \pi(t, y)\big) \leqslant \gamma(\varepsilon)$$

for every $\varepsilon > 0$ *and* $t \in \boldsymbol{R}$ (see (3.1)). *If*

$$(7.3) \qquad 0 < \liminf_{\varepsilon\to 0^+} \alpha(\varepsilon) \leqslant \limsup_{\varepsilon\to 0^+} \alpha(\varepsilon) < \infty,$$

then the motion π^y *is periodic with a period* $\alpha_0 \in [\liminf\alpha, \limsup\alpha]$.

Proof. Consider a sequence $\{\varepsilon_n\}$ tending to zero and the corresponding sequence $\{\alpha_n\} = \{\alpha(\varepsilon_n)\}$ which is obviously bounded because of (7.3). Without loss of generality we may assume that $\{\alpha_n\}$ is convergent to some α_0 being between $\liminf\alpha$ and $\limsup\alpha$. Passing to the limit as $n \to \infty$ in the both sides of the inequalities

$$(7.4) \qquad \varrho(\pi(t+\alpha_n, y), \pi(t, y)) \leqslant \gamma(\varepsilon_n) \quad (n = 1, 2, \ldots),$$

we get the equality

$$\varrho(\pi(t+\alpha_0, y), \pi(t, y)) = 0$$

valid for every $t \in \boldsymbol{R}$. The proof is finished.

PROPOSITION 2. *Assume that* π^y *fulfils* (7.2) *for every* $\varepsilon > 0$ *and* $t = 0$. *Suppose that there is a sequence* $\{\varepsilon_n\}$ *such that* $\varepsilon_n \to 0$ *and*

$$(7.5) \qquad \alpha(\varepsilon_n) \to \infty \quad (\alpha(\varepsilon_n) \to -\infty) \text{ as } n \to \infty.$$

Then the motion π^y *is positively (negatively) Poisson stable (which means that* $y \in \Lambda^+(y)$ *[resp.* $y \in \Lambda^-(y)$*]).*

Proof. Putting $\alpha_n = \alpha(\varepsilon_n)$, we now obtain from (7.4), considered for $t = 0$, the relation

$$\lim \pi(\alpha_n, y) = y$$

as $n \to \infty$ which finishes the proof.

Remarks. IV. It is clear that in the assumptions of Proposition 1 instead of (7.1) we may only assume that there exists a sequence $\{\varepsilon_n\}$ tending to zero such that $\gamma(\varepsilon_n) \to 0$ as $n \to \infty$.

V. It is not excluded that the assertion of Proposition 1 is fulfilled by trivial periodicity of π^y in the case of y being a stationary point. In paper [2], essential periodicity and stationary points were considered as separate cases; here we consider both the cases simultaneously.

As a simple corollary of Theorem 1 and Propositions 1 and 2 we obtain the following lemma.

LEMMA. *If* $x \in X$, $y \in \Lambda^+(x)$ $(y \in \Lambda^-(x))$, π^x *satisfies* $P^+[\alpha, \gamma]$ $(P^-[\alpha, \gamma])$, *where* α *is such that*

$$(7.6) \qquad 0 < \eta \leqslant \alpha(\varepsilon) \quad \textit{for every } \varepsilon,$$

with some fixed η, *and* γ *satisfies* (7.1), *then the motion* π^y *is positively* (*negatively*) *Poisson stable. If, moreover,* α *is bounded from above* (see (7.3)), *then* π^y *is periodic with a period* $\alpha^0 \in [\liminf_{\varepsilon \to 0^+} \alpha(\varepsilon), \limsup_{\varepsilon \to 0^+} \alpha(\varepsilon)]$.

It is known (see for instance [1]) that if $X = \boldsymbol{R}^2$ then each Poisson stable motion must be periodic. So we have the following theorem.

THEOREM 4. *Let* $(\boldsymbol{R}^2, \boldsymbol{R}, \pi)$ *be a dynamical system. If* $x \in \boldsymbol{R}^2$, $y \in \Lambda^+(x)$ $(y \in \Lambda^-(x))$, π^x *satisfies the condition* $P^+[\alpha, \gamma]$ $(P^-[\alpha, \gamma])$ *with* α *satisfying* (7.6) *and* γ *fulfilling* (7.1), *then the motion* π^y *is periodic. If, moreover,* α *satisfies* (7.3), *then there is a period* α_0 *of* π^y *belonging to the interval* $[\liminf_{\varepsilon \to 0^+} \alpha(\varepsilon), \limsup_{\varepsilon \to 0^+} \alpha(\varepsilon)]$.

8. In the present section we shall assume that $X = \boldsymbol{R}^2$. First of all we recall that if $(X, \boldsymbol{R}, \pi)$ is a dynamical system with X being a locally compact metric space, then every positive (negative) limit set which is compact must be connected. Here, we shall need slightly stronger results valid for some special cases in $X = \boldsymbol{R}^2$. So let us assume that $(\boldsymbol{R}^2, \boldsymbol{R}, \pi)$ is a dynamical system fixed throughout this section. We have the following obvious proposition.

PROPOSITION 3. *If* $x, y \in \boldsymbol{R}^2$ *are such that*: (a) y *is not a stationary point,* (b) π^y *is periodic,* (c) $y \in \Lambda^+(x)$ $(y \in \Lambda^-(x))$, *then*

$$\pi(y) = \Lambda^+(x) \qquad (\pi(y) = \Lambda^-(x)).$$

Proof (outline). Observe, first of all, that if $\pi(y)$ is a connected component of $\Lambda^+(x)$ $(\Lambda^-(x))$, then the assertion of the proposition holds true.

Indeed, $\pi(y)$ is compact and so there exists $\varepsilon > 0$ such that putting

$$B_\varepsilon := \{z: \inf\{\varrho(z, w): w \in \pi(y)\} < \varepsilon\} \tag{8.1}$$

we have

$$\bar{B}_\varepsilon \cap (\Lambda^+(x) \setminus \pi(y)) = \emptyset \qquad (\bar{B}_\varepsilon \cap (\Lambda^-(x) \setminus \pi(y)) = \emptyset). \tag{8.2}$$

So we may use the standard arguments applied in the classical proof of the well-known theorem about the connectedness of compact limit sets in dynamical systems in locally compact metric spaces: if there exists a point z belonging to $\Lambda^+(x) \setminus \pi(y)$ $(z \in \Lambda^-(x) \setminus \pi(y))$, we may find a sequence $\{s_m\}$ of real numbers such that $s_m \to \infty$ $(s_m \to -\infty)$ and $\pi(s_m, x) \in \partial B_\varepsilon$ (with ε such that (8.2) is fulfilled); without loss of generality we can suppose that $\{\pi(s_m, x)\}$ is convergent to some w belonging necessarily to ∂B_ε and simultaneously to $\Lambda^+(x)$ $(\Lambda^-(x))$ which is impossible, because of (8.2).

So assume now that the assertion does not hold true and suppose that for every $\delta > 0$ the set

$$B_\delta \cap (\Lambda^+(x) \setminus \pi(y)) \qquad (B_\delta \cap (\Lambda^-(x) \setminus \pi(y)))$$

(compare the notation (8.1)) is not empty.

Thus there is a sequence $\{z_m\}$ of elements of the set $\Lambda^+(x)\setminus\pi(y)$ $(\Lambda^-(x)\setminus\pi(y))$ such that

$$\varrho(z_m, \pi(y)) \to 0 \quad \text{as } m \to \infty;$$

without loss of generality we may assume that

$$z_m \to z \quad \text{as } m \to \infty,$$

where z is an element of $\pi(y)$.

Since y is not a stationary point, we have the following statement (cf., for instance, [1], Chapter 1): for every $w \in \pi(y)$ there exist an open neighbourhood U_w of w, a subset S_w of U_w and a positive number τ_w such that

$$w \in S_w, \quad \{\pi(t, v): |t| < \tau_w, v \in S_w\} \subset U_w$$

and for every $v \in U_w$ there is exactly one $\tau(v)$ belonging to the interval $(-\tau_w, \tau_w)$ for which

$$\pi(\tau(v), v) \in S_w.$$

Such a neighborhood U_w is called a *tube*, the set S_w is a section. The geometrical interpretation is presented in Fig. 1.

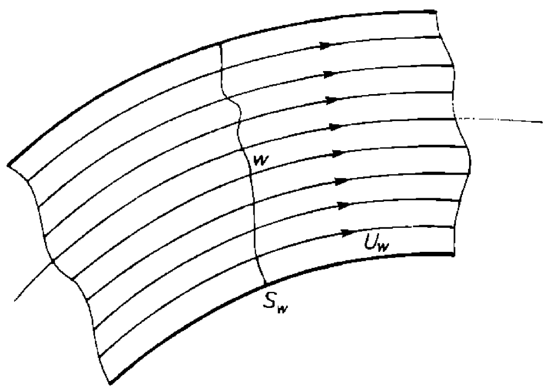

Fig. 1

It is obvious that the arrows on the trajectories passing through U_w (showing the direction of motions) are compatible, as it is indicated in Fig. 1 since the mapping $v \to \tau(v)$ is continuous, which is provable by using the classical arguments (see also [1]). Because of the compactness of $\pi(y)$ we can find a finite set of points $w_1, \ldots, w_k \in \pi(y)$ such that

$$\pi(y) \subset U_{w_1} \cup \ldots \cup U_{w_k}.$$

This means that we may be sure that for sufficiently small $\delta > 0$ all trajectories passing through points of the set B_δ (see (8.1)) are such that arrows are compatible and, in particular, the positive (negative) semitrajectory of x approaching $\pi(y)$ is a spiral curve; if, for instance, we have the first case, that is, $y \in \Lambda^+(x)$, then we have exactly one of the following two qualitative situations

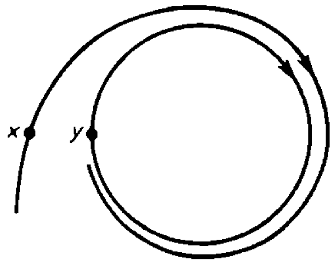

Fig. 2

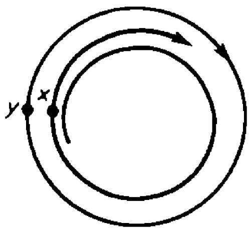

Fig. 3

presented in Figs. 2 and 3, respectively (up to the simple change of the direction of all arrows). Let now m_0 be so large that

$$\varrho(z_{m_0}, z) < \delta.$$

Take $\delta^0 = \frac{1}{2}\varrho(z_{m_0}, z)$. It is clear that in both the cases considered above (Figs. 2 and 3) we have:

$$\pi(t, x) \in B_{\delta^0}$$

for t sufficiently large (say for $t \geqslant t_0$). This means that z_{m_0} does not belong to $\Lambda^+(x)$. This contradiction finishes the proof in the case $y \in \Lambda^+(x)$; the proof in the case $y \in \Lambda^-(x)$ is clearly similar.

Remark VI. Assumption (a) of Proposition 3 is essential as is easy to observe considering the last example presented in Section 8 of paper [2], where a stationary point q belongs to the positive limit set being a circle which is the union of $\{q\}$ and another trajectory. It is easy to see that one can construct a more "drastic" example: $\Lambda^+(x)$ is equal to a circle which is, however, the union of trivial trajectories of its points, being – all of them – stationary points.

Using Proposition 3, we can establish the next one:

PROPOSITION 4. *If* $y \in \Lambda^+(x)$ $(y \in \Lambda^-(x))$, y *is not a stationary point, and* π^x *satisfies the condition* $P^+[\alpha, \gamma]$ $(P^-[\alpha, \gamma])$ *with* α *satisfying* (7.6) (*with some* $\eta > 0$) *and* γ *fulfilling* (7.1), *then* $\Lambda^+(x) = \pi(y)$ $(\Lambda^-(x) = \pi(y))$.

Proof. Assume that $y \in \Lambda^+(x)$, y is not a stationary point, and π^x satisfies $P^+[\alpha, \gamma]$. Theorem 4 gives the periodicity of the motion π^y; applying Proposition 3, we finish the proof. Similar reasoning gives the proof for $y \in \Lambda^-(x)$ and π^x satisfying $P^-[\alpha, \gamma]$.

As a corollary of Theorem 4 and Proposition 4 we can prove, using results of [2], the following theorem.

THEOREM 5. *If* $x \in \boldsymbol{R}^2$, $y \in \Lambda^+(x)$ $(y \in \Lambda^-(x))$, π^x *satisfies the condition* $P^+[\alpha, \gamma]$ $(P^-[\alpha, \gamma])$, *with* α *satisfying* (7.6) *and* γ *fulfilling* (7.1), *then* π^x *is* *positively (negatively) asymptotically periodic which means that there is a positive number* $\tilde{\alpha}$ *such that for every* $\varepsilon > 0$ *there is* $s \geqslant 0$ $(s \leqslant 0)$ *such that*

$$\varrho(\pi(t+\tilde{\alpha}, x), \pi(t, x)) < \varepsilon \quad \textit{for } t \geqslant s \ (t \leqslant s). \tag{8.3}$$

Proof. Assume that $y \in \Lambda^+(x)$ and π^x satisfies the condition $P^+[\alpha, \gamma]$. Applying Theorem 5 from Section 7, we get periodicity of π^y and so — by using Proposition 4 — we obtain the equality $\Lambda^+(x) = \pi(y)$.

Now we may use Theorem 4 from Section 3 of [2]:

If $(X, \boldsymbol{R}, \pi)$ is a dynamical system, X is locally compact, $\Lambda^+(x) = \pi(y)$ $(\Lambda^-(x) = \pi(y))$, where π^y is periodic or $\pi(y) = \{y\}$, then the motion π^x is positively (negatively) asymptotically periodic.

This finishes the proof in the case $y \in \Lambda^+(x)$ and π^x satisfying $P^+[\alpha, \gamma]$. The second case is clearly similar.

Let us notice that for $X = \boldsymbol{R}^2$ Theorem 5 reduces problems concerning every — a priori very general (with α, γ being some functions of ε) — condition of "asymptotic periodicity type" to suitable problems concerning the simple "proper" asymptotic periodicity in the sense of [2].

References

[1] N. P. Bhatia, G. P. Szegö, *Stability Theory of Dynamical Systems*, Springer Verlag, Berlin–Heidelberg–New York 1970.

[2] A. Pelczar, *Asymptotically Periodic Motions*, Bull. Acad. Polon. Sci. Math. 33 (1985), 314–319.

[3] —, *Pseudoperiodic and Asymptotically Pseudoperiodic Motions in Dynamical Systems on Metric Spaces* (to be published in Proc. of the XI-th International Conference on Nonlinear Oscillations, Budapest 1987).

[4] G. R. Sell, *Topological Dynamics and Ordinary Differential Equations*, Van Nostrand Reinhold Co., London–New York–Cincinnati–Toronto–Melbourne 1971.

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