ANNALES
POLONICI MATHEMATICI
XXVII (1972)

Absolute Nörlund summability factors of power series and Fourier series*

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1.1. DEFINITIONS. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0+p_1+p_2+\ldots+p_n; \quad P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation

$$t_n = (P_n)^{-1}\sum_{\nu=0}^{n} p_{n-\nu}s_\nu \quad (P_n \neq 0) \tag{1.1.1}$$

defines the sequence $\{t_n\}$ of Nörlund means ([14], [19]) of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$.

The series $\sum a_n$ is said to be *absolutely summable* (N, p_n), or *summable* $|N, p_n|$, if

$$\sum_n |t_n - t_{n-1}| < \infty, \quad [13].$$

In the special case in which

$$p_n = A_n^{a-1} = \frac{\Gamma(n+a)}{\Gamma(a)\Gamma(n+1)} \quad (a > -1), \tag{1.1.2}$$

the Nörlund mean reduces to the familiar (C, a)-mean, ([7], § 5.13). The summability $|N, p_n|$, with p_n defined by (1.1.2), is thus the same as summability $|C, a|$, ([6], [10]). Similarly, in the case in which

$$\begin{cases} p_n = 1/(n+1) \quad (n = 0, 1, 2, \ldots), \\ P_n = 1+\frac{1}{2}+\ldots+\dfrac{1}{n+1} \sim \log n, \quad \text{as } n \to \infty, \end{cases} \tag{1.1.3}$$

* This work was executed under the Senior Research Fellowship Scheme of the Council of Scientific and Industrial Research, at the Department of Postgraduate Studies and Research in Mathematics, University of Jabalpur, Jabalpur, India.

the Nörlund mean reduces to 'harmonic mean' [7], and the summability $|N, p_n|$ is then the same as 'absolute harmonic summability'. It is well known that $|N, (n+1)^{-1}| \subset |C, \alpha|$ for every positive α, [12].

1.2. If

$$\sum_{\nu=1}^{n} |s_\nu| = O(n), \tag{1.2.1}$$

as $n \to \infty$, the series $\sum a_n$ is said to be *strongly bounded by Cesàro means of order* 1, or *bounded* $[C, 1]$. If

$$\sum_{\nu=1}^{n} \frac{|s_\nu|}{\nu} = O(\log n), \tag{1.2.2}$$

as $n \to \infty$, the series $\sum a_n$ is said to be *strongly bounded* by 'logarithmic means' with index 1, or bounded $[R, \log n, 1]$, [15].

Let σ_n and τ_n denote the nth $(C, 1)$-means of the sequences $\{s_n\}$ and $\{na_n\}$ respectively, viz.,

$$\sigma_n = \frac{1}{n+1} \sum_{\nu=0}^{n} s_\nu; \qquad \tau_n = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu a_\nu.$$

Then, since by an identity of Kogbetliantz [10], $n(\sigma_n - \sigma_{n-1}) = \tau_n$, the *nth total variation* of the sequence $\{\sigma_n\}$ is given by

$$\sum_{\nu=1}^{n} |\sigma_\nu - \sigma_{\nu-1}| = \sum_{\nu=1}^{n} \frac{|\tau_\nu|}{\nu}. \tag{1.2.3}$$

1.3. Let $f(t)$ be a periodic function, with period 2π, integrable in the sense of Lebesgue over $(-\pi, \pi)$, and let

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t) \tag{1.3.1}$$

denote the Fourier series of the function $f(t)$.

1.4. For any sequence $\{f_n\}$, we write

$$\Delta^0 f_n = f_n, \qquad \Delta f_n = \Delta^1 f_n = f_n - f_{n+1}, \tag{1.4.1}$$

and

$$\Delta^\varrho f_n = \sum_{\nu=0}^{\infty} A_\nu^{-\varrho-1} f_{n+\nu}, \tag{1.4.2}$$

provided this series is convergent.

If h and k are positive integers, we have

$$\Delta^h \Delta^k f_n = \Delta^{h+k} f_n. \tag{1.4.3}$$

A sequence $\{\lambda_n\}$ is said to be *convex* ([20], p. 93) if $\Delta^2\lambda_n \geqslant 0$, for $n = 0, 1, 2, \dots$

If $\{\lambda_n\}$ is a convex sequence such that the series $\sum n^{-1}\lambda_n$ is convergent, then it is known that

(1.4.4)
(i) $\log n\, \lambda_n = o(1)$, as $n \to \infty$ ([5], Lemma IV);
(ii) $\sum n\,\Delta^2\lambda_n < \infty$ ([5], Lemma IV);
(iii) $\sum n\,(\log n)\,\Delta^2\lambda_n$ ([16], Lemma 5).

We write

$$P_{n,\nu} = P_n p_{n-\nu} - P_{n-\nu} p_n,$$

for $\nu \geqslant 1$, $n \geqslant 1$.

2.1. Introduction. It was proved by McFadden ([12], Theorem 2.28), that, if (i) p_n is non-negative and non-increasing, with $p_0 > 0$, and (ii) p_{n+1}/p_n is non-decreasing, then $|N, p_n| \subset |C, 1|$.

We raise the question as to what type of sequences of factors $\{\varepsilon_n\}$ can be chosen so that the series $\sum \varepsilon_n a_n$ may be summable $|N, p_n|$, with p_n more general than that characterized above, whenever the series $\sum a_n$ is not summable $|C, 1|$, but the total variation of the $(C, 1)$-mean of $\sum a_n$ is of a certain order μ_n, say, where $\{\mu_n\}$ is a positive non-decreasing sequence.

As an answer to this question we establish, in Theorem 1, a resutl on the absolute Nörlund summability factors for general infinite series, which, in view of (1.4.4), includes the following results as special cases:

THEOREM A. *If $\sum a_n$ is bounded $[R, \log n, 1]$ and λ_n is a convex sequence such that $\sum n^{-1}\lambda_n < \infty$, then*

(i) *$\sum a_n\lambda_n$ is summable $|C, 1|$,* ([15], Theorem 1);

(ii) *$\sum n^{-1}(\overline{\log n + 1})\lambda_n a_n$ is summable $|N, (n+1)^{-1}|$* ([11], Theorem 2).

THEOREM B. *If $\sum a_n$ is summable $|C, 1|$, then*

(i) *$\sum n^{-1}(\overline{\log n + 1})\lambda_n a_n$ is summable $|N, (n+1)^{-1}|$,* [18];

(ii) *$\sum n^{-1}P_n\lambda_n a_n$ is summable $|N, p_n|$,* [9].

Our main object in the present paper is to establish, with the help of our Theorem 1, some general theorems (Theorems 2, 3, 4 and 5) on the absolute Nörlund summability factors of power series and Fourier series. It is interesting to note that, in view of (1.4.4), our theorems contain, as special cases, the following known results in this line.

THEOREM C. *If $f(z) = \sum c_n z^n$ is a power series of the complex class L, such that*

(2.1.1)
$$\int_0^t |f(e^{i\theta})|\,d\theta = O(|t|),$$

as $t \to +0$, and $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n < \infty$, then

(i) $\sum\lambda_n c_n$ *is summable* $|C, 1|$ ([17], Theorem I);

(ii) $\sum n^{-1}(\overline{\log n+1})\lambda_n c_n$ *is summable* $|N, (n+1)^{-1}|$ ([11], Theorem 1).

THEOREM D. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\sum n^{-1}\lambda_n < \infty$, *then*

(i) *the series* $\sum\lambda_n A_n(x)$ *is summable* $|C, 1|$ *for almost all values of* x, [5];

(ii) *the series* $\sum n^{-1}(\overline{\log n+1})\lambda_n A_n(x)$ *is summable* $|N, (n+1)^{-1}|$ *for almost all values of* x ([11], Theorem 3).

THEOREM E ([17], Theorem II). *If* $F(x)$ *is even*, $F(x)\in L^2(-\pi, \pi)$,

$$\int_0^t |F(x)|^2 dx = O(t), \tag{2.1.2}$$

as $t \to +0$, *and if* $\{\lambda_n\}$ *is a convex sequence such that* $\sum n^{-1}\lambda_n < \infty$, *then the sequence* $\{A_n\}$ *of Fourier coefficients of* $F(x)$ *has the propery that* $\sum\lambda_n A_n$ *is summable* $|C, 1|$.

THEOREM F ([17], Theorem III). *If* $F(x)$ *is even*, $F(x)\in L(-\pi, \pi)$,

$$\int_0^t |F(x)| dx = O(t), \tag{2.1.3}$$

as $t \to +0$, *and if* $\{\lambda_n\}$ *is a convex sequence such that* $\sum n^{-1}\lambda_n < \infty$, *then the sequence* $\{A_n\}$ *of Fourier coefficients of* $F(x)$ *has the property that* $\sum(\log n+1)^{-1/2}\lambda_n A_n$ *is summable* $|C, 1|$.

2.2. We establish the following theorems:

THEOREM 1. *Let* $p_0 > 0$, $p_n \geqslant 0$ $(n = 1, 2, \ldots)$, *and let* $\{p_n\}$ *be non-increasing. If*

$$\sum_{\nu=1}^{n} \frac{|\tau_\nu|}{\nu} = O(\mu_n),$$

where $\{\mu_n\}$ *is a positive non-decreasing sequence, and if the sequence* $\{\varepsilon_n\}$ *is such that*

(i) $\varepsilon_n\mu_n = O(1)$, $n\Delta\mu_n = O(\mu_n)$,

(ii) $\sum n\mu_n|\Delta^2\varepsilon_n| < \infty$,

then the series $\sum(n+1)^{-1}P_n\varepsilon_n a_n$ *is summable* $|N, p_n|$.

THEOREM 2. *Let* p_n *be the same as in Theorem* 1. *If* $\{\varepsilon_n\}$ *is such that*

(i) $\log n\, \varepsilon_n = O(1)$,

(ii) $\sum n\log n|\Delta^2\varepsilon_n| < \infty$,

then the series $\sum(n+1)^{-1}P_n\varepsilon_n A_n(x)$ *is summable* $|N, p_n|$ *for almost all values of* x.

THEOREM 3. *Let p_n be the same as in Theorem 1. If $F(x)$ is even, $F(x) \in L^2(-\pi, \pi)$,*

$$\int_0^t |F(x)|^2 dx = O(t), \tag{2.2.1}$$

as $t \to +0$, and if $\{\varepsilon_n\}$ satisfies the same conditions as in Theorem 2, then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that $\sum (n+1)^{-1} P_n \varepsilon_n A_n$ is summable $|N, p_n|$.

THEOREM 4. *Let p_n be the same as in Theorem 1. If $F(x)$ is even, $F(x) \in L(-\pi, \pi)$,*

$$\int_0^t |F(x)| dx = O(t), \tag{2.2.2}$$

as $t \to +0$, and if $\{\varepsilon_n\}$ satisfies the same conditions as in Theorem 2, then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that $\sum (n+1)^{-1} (\log n)^{-1/2} P_n \varepsilon_n A_n$ is summable $|N, p_n|$.

THEOREM 5. *If $f(z) = \sum c_n z^n$ is a power series of the complex class L, such that*

$$\int_0^t |f(e^{i\theta})| d\theta = O(|t|), \tag{2.2.3}$$

as $t \to +0$, and if $\{\varepsilon_n\}$ satisfies the same conditions as in Theorem 2, then $\sum (n+1)^{-1} P_n \varepsilon_n c_n$ is summable $|N, p_n|$.

2.3. We use the following lemmas in the sequel.

LEMMA 1 [1]. *Let $p_0 > 0$, $p_n \geqslant 0$ $(n = 1, 2, \ldots)$, and let $\{p_n\}$ be non-increasing. Then, for $\nu \geqslant 1$,*

(a) $$\sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} p_{n-\nu} \leqslant \frac{K}{\nu} \text{ (1)};$$

(b) $$\sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu}) \leqslant K;$$

(c) $$\sum_{n=\nu}^{\infty} \frac{(p_{n-\nu} - p_n)}{P_{n-1}} \leqslant K;$$

(d) $$\sum_{n=\nu}^{\infty} \frac{|\Delta_\nu p_{n-\nu}|}{P_{n-1}} \leqslant \frac{K}{P_\nu} + \frac{K}{\nu}.$$

(1) Throughout, K denotes an absolute constant, not necessarily the same at each occurrence.

LEMMA 2. *If p_n satisfies the same conditions as in Lemma 1, then, for $\nu \geqslant 1$,*

(a)
$$\sum_{n=\nu}^{\infty} \frac{P_{n,\nu}}{P_n P_{n-1}} \leqslant K,$$

(b)
$$\sum_{n=\nu}^{\infty} \frac{|\Delta_\nu P_{n,\nu}|}{P_n P_{n-1}} \leqslant \frac{K}{P_\nu}.$$

Proof. Since

$$\sum_{n=\nu}^{\infty} \frac{P_{n,\nu}}{P_n P_{n-1}} = \sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu}) + \sum_{n=\nu}^{\infty} \frac{(p_{n-\nu} - p_n)}{P_{n-1}},$$

(a) follows from Lemma 1(b) and (c).

Again, since

$$\Delta_\nu P_{n,\nu} = P_n \cdot \Delta_\nu p_{n-\nu} - p_{n-\nu} p_n,$$

we have

$$\sum_{n=\nu}^{\infty} \frac{|\Delta_\nu P_{n,\nu}|}{P_n P_{n-1}} \leqslant \sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} p_{n-\nu} + \sum_{n=\nu}^{\infty} \frac{|\Delta_\nu p_{n-\nu}|}{P_{n-1}}$$

$$\leqslant \frac{K}{\nu} + \frac{K}{P_\nu} \quad \text{(by Lemma 1(a) and (d))}$$

$$= K \cdot \frac{\nu+1}{\nu} \cdot \frac{1}{P_\nu} \cdot \frac{P_\nu}{\nu+1} + \frac{K}{P_\nu} \leqslant \frac{K}{P_\nu},$$

by hypothesis, since $P_\nu/(\nu+1)$ is non-increasing.

This completes the proof of the lemma.

LEMMA 3 ([4], Lemma 1; see also [2], [3]). *If $\varepsilon_n = O(1)$, then*

$$\Delta^{\alpha+\beta} \varepsilon_n = \sum_{\nu=n}^{\infty} A_{\nu-n}^{-\beta-1} \Delta^\alpha \varepsilon_n,$$

for $\alpha \geqslant 0$, $\beta \geqslant -1$, $\alpha+\beta > 0$. If $\varepsilon_n = o(1)$, the last condition may be replaced by $\alpha+\beta \geqslant 0$.

LEMMA 4. *Let $\{\mu_n\}$ be a positive non-decreasing sequence such that $n\Delta\mu_n = O(\mu_n)$, as $n \to \infty$. If the sequence $\{\varepsilon_n\}$ is such that*

(i) $\mu_n \varepsilon_n = O(1)$, *as* $n \to \infty$,

(ii) $\sum n\mu_n |\Delta^2 \varepsilon_n| < \infty$,

then

(a) $\sum \mu_n |\Delta \varepsilon_n| < \infty$,

and

(b) $n\mu_n \Delta \varepsilon_n = O(1)$, *as* $n \to \infty$.

Proof. (a) Since, by hypothesis (i), $\varepsilon_n = O(1)$, applying Lemma 3, we have

$$\Delta\varepsilon_n = \sum_{\nu=n}^{\infty} \Delta^2\varepsilon_\nu,$$

and hence,

$$\sum_{n=1}^{\infty} \mu_n |\Delta\varepsilon_n| = \sum_{n=1}^{\infty} \mu_n \Big|\sum_{\nu=n}^{\infty} \Delta^2\varepsilon_\nu\Big| \leqslant \sum_{n=1}^{\infty} \mu_n \sum_{\nu=n}^{\infty} |\Delta^2\varepsilon_\nu|$$

$$= \sum_{\nu=1}^{\infty} |\Delta^2\varepsilon_\nu| \sum_{n=1}^{\nu} \mu_n \leqslant \sum_{\nu=1}^{\infty} \nu\mu_\nu |\Delta^2\varepsilon_\nu| \leqslant K < \infty,$$

by hypothesis.

(b) We see that

$$\sum_{n=1}^{\infty} |\Delta(n\mu_n \Delta\varepsilon_n)| \leqslant \sum_{n=1}^{\infty} n\mu_n |\Delta^2\varepsilon_n| + \sum_{n=1}^{\infty} \mu_n |\Delta\varepsilon_{n+1}| + \sum_{n=1}^{\infty} (n+1)|\Delta\mu_n| |\Delta\varepsilon_{n+1}|$$

$$\leqslant K + K\sum_{n=1}^{\infty} \mu_n |\Delta\varepsilon_{n+1}| \leqslant K < \infty,$$

by hypothesis and (a). This means that $\{n\mu_n \Delta\varepsilon_n\}$ is a sequence of bounded variation, which implies that it is bounded, that is,

$$n\mu_n \Delta\varepsilon_n = O(1), \quad \text{as } n \to \infty.$$

LEMMA 5. *If* $\bar{\varepsilon}_n = (n+1)^{-1} P_n \varepsilon_n$, *then under the hypotheses of Theorem* 1,

(i) $\displaystyle\sum_{\nu=1}^{\infty} \frac{|\bar{\varepsilon}_\nu|}{P_\nu} |\tau_\nu| \leqslant K,$

(ii) $\displaystyle\sum_{\nu=1}^{\infty} |\Delta\bar{\varepsilon}_\nu| \, |\tau_\nu| \leqslant K.$

Proof. (i) We have, as $m \to \infty$,

$$\sum_{\nu=1}^{m} \frac{|\bar{\varepsilon}_\nu|}{P_\nu} |\tau_\nu| = O\Big(\sum_{\nu=1}^{m} |\varepsilon_\nu| \frac{|\tau_\nu|}{\nu}\Big)$$

$$= O\Big(\sum_{\nu=1}^{m} \mu_\nu |\Delta\varepsilon_\nu|\Big) + O(|\varepsilon_m| \mu_m) = O(1),$$

by hypotheses and Lemma 4 (a).

(ii) Since

$$\Delta\bar{\varepsilon}_n = \frac{P_n}{n+1} \Delta\varepsilon_n - \frac{p_{n+1}}{n+1} \varepsilon_{n+1} + \frac{P_{n+1}}{n+2} \frac{\varepsilon_{n+1}}{n+1},$$

we have

$$|\Delta\bar{\varepsilon}_n| = O\Big(|\Delta\varepsilon_n| \frac{P_n}{n}\Big) + O\Big(\frac{|\varepsilon_{n+1}|}{n}\Big).$$

Hence, as $m \to \infty$,

$$\sum_{\nu=1}^{m} |\Delta\bar{\varepsilon}_\nu|\,|\tau_\nu| = O\Big(\sum_{\nu=1}^{m} P_\nu |\Delta\varepsilon_\nu| \frac{|\tau_\nu|}{\nu}\Big) + O\Big(\sum_{\nu=1}^{m} |\varepsilon_{\nu+1}| \frac{|\tau_\nu|}{\nu}\Big)$$

$$= O\Big(\sum_{\nu=1}^{m-1} \nu\mu_\nu |\Delta^2\varepsilon_\nu|\Big) + O\Big(\sum_{\nu=1}^{m-1} \mu_\nu |\Delta\varepsilon_{\nu+1}|\Big) + $$

$$+ O(m\mu_m |\Delta\varepsilon_m|) + O(\mu_{m+1}|\varepsilon_{m+1}|)$$

$$= O(1),$$

by hypotheses and Lemma 4(a) and (b).

This completes the proof of the lemma.

LEMMA 6 [17]. *If $f(z) = \sum c_n z^n$ is a power series of complex class L, such that*

$$\int_0^t |f(e^{i\theta})|\,d\theta = O(|t|),$$

as $t \to +0$, then $\sum c_n$ is bounded $[R, \log n, 1]$ (²).

LEMMA 7 ([15], p. 294). *If $\sum a_n$ is bounded $[R, \log n, 1]$, then*

$$\sum_{\nu=1}^{n} \frac{|\tau_\nu|}{\nu} = O(\log n),$$

as $n \to \infty$.

LEMMA 8. *If $\sum a_n$ is bounded $[C, 1]$, then it is bounded $[R, \log n, 1]$.*

The proof is easy.

LEMMA 9. ([5], Lemma 2). *Let*

$$\tau_n(x) = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu A_\nu(x).$$

Then

$$\sum_{\nu=1}^{n} |\tau_\nu(x)| = o(n),$$

for almost all values of x.

LEMMA 10. *If $\tau_n(x)$ is defined as in Lemma 9, then*

$$\sum_{\nu=1}^{n} \frac{|\tau_\nu(x)|}{\nu} = o(\log n),$$

as $n \to \infty$, for almost all values of x.

(²) This is the 'O' version of a previous result of Hardy and Littlewood [8].

Proof. By Abel's transformation and Lemma 9, we have

$$\sum_{\nu=1}^{n} \frac{|\tau_\nu(x)|}{\nu} = \sum_{\nu=1}^{n} \frac{1}{\nu(\nu+1)} \sum_{\mu=1}^{\nu} |\tau_\mu(x)| + \frac{1}{n+1} \sum_{\mu=1}^{n} |\tau_\mu(x)|$$
$$= o(\log n) + o(1) = o(\log n),$$

as $n \to \infty$, for almost all values of x.

LEMMA 11 ([17], Lemma 4). *Let $F(x)$ be even, $F(x) \in L^2(-\pi, \pi)$, and let S_n denote the nth partial sum of its Fourier series at the origin. Then, if*

$$\int_0^{\theta} |F(x)|^2 dx = O(\theta),$$

as $\theta \to +0$, $\{S_n\}$ will be summble $[C, 1]$.

LEMMA 12 [17]. *Let $F(x)$ be even, $F(x) \in L(-\pi, \pi)$, and let S_n denote the nth partial sum of its Fourier series at the origin. Then, if*

$$\int_0^{\theta} |F(x)| dx = O(\theta),$$

as $\theta \to +0$, then

$$\sum_{\nu=1}^{n} |S_\nu| = O\{n(\log n)^{1/2}\}.$$

2.4. Proof of Theorem 1. Let $\bar{\varepsilon}_n = (n+1)^{-1} P_n \varepsilon_n$, and let t_n^* denote the nth Nörlund mean of the series $\sum \bar{\varepsilon}_\nu a_\nu$. Then, by definition, we have

$$t_n^* = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} \sum_{\mu=0}^{\nu} \bar{\varepsilon}_\mu a_\mu = \frac{1}{P_n} \sum_{\nu=0}^{n} P_{n-\nu} \bar{\varepsilon}_\nu a_\nu$$

and

$$t_n^* - t_{n-1}^* = \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} (P_n p_{n-\nu} - P_{n-\nu} p_n) \bar{\varepsilon}_\nu a_\nu$$
$$= \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{n,\nu} \bar{\varepsilon}_\nu a_\nu$$
$$= \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{n,\nu} \frac{\bar{\varepsilon}_\nu}{\nu} \tau_\nu + \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{n,\nu} \Delta \bar{\varepsilon}_\nu \tau_\nu +$$
$$+ \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} \Delta_\nu P_{n,\nu} \bar{\varepsilon}_{\nu+1} \tau_\nu = \frac{1}{P_n P_{n-1}} (\Sigma_1 + \Sigma_2 + \Sigma_3),$$

say.

Therefore, in order that $\sum_n |t_n^* - t_{n-1}^*| \leqslant K$, it is sufficient to show that

$$\sum_n \frac{1}{P_n P_{n-1}} |\Sigma_r| \leqslant K \qquad (r = 1, 2, 3).$$

Now, first, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} |\Sigma_1| &\leqslant \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{n,\nu} |\bar{\varepsilon}_\nu| \frac{|\tau_\nu|}{\nu} \\ &= \sum_{\nu=1}^{\infty} |\bar{\varepsilon}_\nu| \frac{|\tau_\nu|}{\nu} \sum_{n=\nu}^{\infty} \frac{P_{n,\nu}}{P_n P_{n-1}} \\ &\leqslant K \sum_{\nu=1}^{\infty} |\bar{\varepsilon}_\nu| \frac{|\tau_\nu|}{\nu} \qquad \text{(by Lemma 2(a))} \\ &\leqslant K \sum_{\nu=1}^{\infty} \frac{|\bar{\varepsilon}_\nu|}{P_\nu} |\tau_\nu| \leqslant K,\end{aligned}$$

by hypotheses and Lemma 5(i).

Next, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} |\Sigma_2| &\leqslant \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{n,\nu} |\Delta \bar{\varepsilon}_\nu| \, |\tau_\nu| \\ &= \sum_{\nu=1}^{\infty} |\Delta \bar{\varepsilon}_\nu| |\tau_\nu| \sum_{n=\nu}^{\infty} \frac{P_{n,\nu}}{P_n P_{n-1}} \\ &\leqslant K \sum_{\nu=1}^{\infty} |\Delta \bar{\varepsilon}_\nu| \, |\tau_\nu| \qquad \text{(by Lemma 2 (a))} \\ &\leqslant K,\end{aligned}$$

by hypotheses and Lemma 5 (ii).

Finally, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} |\Sigma_3| &\leqslant \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} |\Delta_\nu P_{n,\nu}| \, |\bar{\varepsilon}_{\nu+1}| \, |\tau_\nu| \\ &\leqslant K \sum_{\nu=1}^{\infty} \frac{|\bar{\varepsilon}_{\nu+1}|}{P_\nu} |\tau_\nu| \qquad \text{(by Lemma 2 (b))} \\ &\leqslant K,\end{aligned}$$

by hypotheses and Lemma 5(i).

This terminates the proof of Theorem 1.

2.5. Proof of Theorems 2, 3, 4 and 5. We obtain Theorem 2 from Theorem 1, with $\mu_n = \log n$, by an appeal to Lemma 10.

Theorem 3 can easily be obtained from Theorem 1, with $\mu_n = \log n$, by successive applications of Lemmas 11, 8 and 7.

We get Theorem 4 from Theorem 1 with $\mu_n = (\log n)^{3/2}$, and with $\varepsilon_n/(\log n)^{1/2}$ in place of ε_n, by an appeal to Lemma 12 and by using the fact that

$$\sum_{\nu=1}^{n} |S_\nu| = O\{n(\log n)^{1/2}\}$$

implies

$$\sum_{\nu=1}^{n} \frac{|\tau_\nu|}{\nu} = O\{(\log n)^{3/2}\}.$$

Finally, we obtain Theorem 5 from Theorem 1 with $\mu_n = \log n$, by appealing to Lemmas 6 and 7.

I take this opportunity to express my sincere thanks to Professor T. Pati for his kind encouragement and to the Council of Scientific and Industrial Research for their financial support during the preparation of this paper.

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Reçu par la Rédaction le 2. 3. 1970