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On the growth of proper polynomial mappings

by A. PŁOSKI (Kielce)

Abstract. Let $F = (F_1, \ldots, F_n)$: $C^n \to C^n$ be a proper polynomial mapping. The exponent $h(F)$ of F is the greatest number q such that $|F(z)| \geqslant C|z|^q$ with $C > 0$ for sufficiently large $|z|$. The aim of this paper is to give the estimation of $h(F)$ dependent on the geometric degree $d(F)$ of the mapping F and the degrees $\deg F_i$.

1. The main theorem. Let us introduce some notation. If P: $C^n \to C$ is a nonzero polynomial, we denote by $\deg P$ its degree, by P^+ the unique homogeneous form such that $\deg P^+ = \deg P$ and $\deg(P^+ - P) < \deg P$. By definition, we put $\deg 0 = -\infty$, $0^+ = 0$.

If $F = (F_1, \ldots, F_p)$: $C^n \to C^p$ is a polynomial mapping, we set $\deg F = \max_{j=1}^{p} (\deg F_j)$, $F^+ = (F_1^+, \ldots, F_p^+)$. For any $z = (z_1, \ldots, z_n) \in C^n$ we put $|z| = \max_{i=1}^{n} |z_i|$. The expression "for almost every $a \in C^n$" will mean: "there exists a Zariski open subset $U \subset C^n$ such that for every $a \in U$".

It is easy to check the following characterization of the degree:

(1.1) LEMMA. *Let F be a nonzero polynomial mapping. Then there exist a positive constant C such that $|F(z)| \leqslant C|z|^{\deg F}$ for $|z| \geqslant 1$. If $|F(z)| \leqslant D|z|^q$ with $q, D > 0$ for sufficiently large $|z|$, then $\deg F \leqslant q$.*

Now, let us suppose that $F = (F_1, \ldots, F_n)$: $C^n \to C^n$ is a dominating polynomial mapping. Then the field $C(Z)$ of rational functions in indeterminates $Z = (Z_1, \ldots, Z_n)$ is a finite extension of the field $C(F) = C(F_1, \ldots, F_n)$.

We define the geometric degree of F by putting $d(F) = (C(Z): C(F))$. Recall that the geometric degree $d(F)$ is equal to $\# F^{-1}(w)$ for almost every $w \in C^n$ (see e.g. [6]).

This characterization of $d(F)$ and the Bezout's theorem imply (cf. also A. Ostrowski [7]) the

(1.2) PROPOSITION. *Let $F = (F_1, \ldots, F_n)$: $C^n \to C^n$ be a dominating polynomial mapping. Then $d(F) \leqslant \prod_{i=1}^{n} \deg F_i$. The equality holds if $(F^+)^{-1}(0) = \{0\}$.*

Let us consider a proper polynomial mapping $F = (F_1, \ldots, F_n)$: $C^n \to C^n$ (i.e., such that inverse images of compact sets are compact). Since the mapping F is proper and polynomial, it is surjective, hence it is dominating. The converse is not true, however, we shall prove in Section 3 of this paper the following:

(1.3) PROPOSITION (cf. J. Chądzyński [2] for the case $n = 2$). *Let $F = (F_1, \ldots, F_n)$: $C^n \to C^n$ be a dominating polynomial mapping such that $d(F) > \prod_{i=1}^{n} \deg F_i - \min_{i=1}^{n}(\deg F_i)$. Then F is proper.*

Obviously the mapping $F\colon C^n \to C^n$ is proper if and only if $\lim_{|z| \to +\infty} |F(z)| = +\infty$. In fact, the polynomial mappings have a much stronger property due to L. Hörmander (cf. [3], [4]).

(1.4) PROPOSITION. *Let F: $C^n \to C^n$ be a proper polynomial mapping. Then there exist positive constants C, R, q such that $|F(z)| \geqslant C|z|^q$ for $|z| \geqslant R$.*

Inequality (1.4) allows us to make the following:

(1.5) DEFINITION. Let F: $C^n \to C^n$ be a proper polynomial mapping. The exponent $h(F)$ of F is the least upper bound of the set of all q which satisfy the condition: there exist positive constants C, R such that $|F(z)| \geqslant C|z|^q$ for $|z| \geqslant R$.

In Section 2 we shall prove the

(1.6) PROPOSITION (cf. [3]). *The least upper bound from* (1.5) *is attained, i.e., there is a constant $C > 0$ such that $|F(z)| \geqslant C|z|^{h(F)}$ for sufficiently large $|z|$. The exponent $h(F)$ is a rational number.*

Now we shall discuss some simple examples.

(1.7) EXAMPLE. For any proper polynomial mapping $F = (F_1, \ldots, F_n)$ we have $h(F) \leqslant \min_{i=1}^{n}(\deg F_i)$. If $(F^+)^{-1}(0) = \{0\}$, then $h(F) = \min_{i=1}^{n}(\deg F_i)$.

Proof. Let H: $C^n \to C$ be a polynomial of positive degree. From (1.1) and the first part of (1.6) we have

$$|H(F(z))| \leqslant C|F(z)|^{\deg(H \circ F)/h(F)} \quad \text{with } C > 0 \text{ for large } |z|.$$

Hence $|H(w)| \leqslant C_1 |w|^{\deg(H \circ F)/h(F)}$ for large $|w|$ and $\deg H \leqslant \deg(H \circ F)/h(F)$ by (1.1). Applying this inequality to the polynomials $H(w) = w_i$ $(i = 1, \ldots, n)$ we get the first assertion. Now, let $F = (F_1, \ldots, F_n)$ be a polynomial mapping such that $(F^+)^{-1}(0) = \{0\}$. Put $m = \min_{i=1}^{n}(\deg F_i)$. Replacing $F_1, \ldots, F_n$ by their suitable powers we may assume that $\deg F_1 = \ldots = \deg F_n = m$. Therefore $\deg(F^+ - F) < m$ and we have

$$|F(z)| \geqslant |F^+(z)| - |F^+(z) - F(z)|$$
$$\geqslant \big(\min_{|z|=1} |F^+(z)| - |F^+(z) - F(z)|\,|z|^{-m}\big)|z|^m \geqslant C|z|^m$$

with $C > 0$ for large $|z|$. ∎

As an immediate corollary of (1.1) we obtain

(1.8) EXAMPLE. If $F\colon C^n \to C^n$ is a polynomial automorphism, then

$$h(F) = 1/\deg (F^{-1}).$$

(1.9) EXAMPLE. For any rational number $r > 0$ there exists a proper polynomial mapping $F\colon C^2 \to C^2$ such that $h(F) = r$.

Proof. Take positive integers $a, b, c > 0$ such that $r = b/a$, $c > r$. Set $F(z) = \big(F_1(z), F_2(z)\big) = (z_1^c, z_1^{ac} + z_2^b)$ for $z = (z_1, z_2) \in C^2$. Therefore we have $z_1^c = F_1(z)$, $z_2^b = F_2(z) - F_1(z)^a$, hence $|z_1| \leqslant 2|F(z)|^{a/b}$, $|z_2| \leqslant 2|F(z)|^{a/b}$ if $|F(z)| \geqslant 1$. Then F is a proper mapping and $h(F) \geqslant b/a = r$. Now take positive numbers $C, R, q > 0$ such that $|F(z)| \geqslant C|z|^q$ for $|z| \geqslant R$, then

$$|F(z)|^b \geqslant C^b |z_2^{bq}| = C^b \left|\big(F_2(z) - F_1(z)^a\big)^q\right| \quad \text{for } |z| \geqslant R.$$

Hence there exist constants $C_1, R_1 > 0$ such that

$$|w|^b \geqslant C_1 |(w_2 - w_1^a)^q| \quad \text{for } |w| = \max(|w_1|, |w_2|) \geqslant R_1.$$

This gives $qa \leqslant b$, so $h(F) \leqslant b/a = r$. ∎

Now we shall present the main result of this paper. The expression $[x]$ denotes the greatest integer which does not exceed the real number x.

(1.10) THEOREM. *Let $F = (F_1, \ldots, F_n)\colon C^n \to C^n$ be a proper polynomial mapping. Then*

$$h(F) \geqslant \begin{cases} \dfrac{1}{\left[\dfrac{\prod_{i=1}^{n} \deg F_i - d(F) + 1}{\min_{i=1}^{n}(\deg F_i)}\right]} & \text{if } \dfrac{\prod_{i=1}^{n} \deg F_i - d(F) + 1}{\min_{i=1}^{n}(\deg F_i)} \geqslant 1, \\[4ex] d(F) - \prod_{i=1}^{n} \deg F_i + \min_{i=1}^{n}(\deg F_i) & \text{if } \dfrac{\prod_{i=1}^{n} \deg F_i - d(F) + 1}{\min_{i=1}^{n}(\deg F_i)} \leqslant 1. \end{cases}$$

The proof of (1.10) will be given in the last section. Now, let us mention two results which inspired our theorem. P. Tworzewski and T. Winiarski obtained in [12] an estimation of the growth of algebraic set ([12], Theorem 3). Applying this estimation to the graph of a proper polynomial mapping $F = (F_1, \ldots, F_n)\colon C^n \to C^n$ we get the inequality $h(F) \geqslant 1/\big(\prod_{i=1}^{n} \deg F_i - d(F) + 1\big)$. On the other hand, J. Chądzyński proved in [2] the second part of (1.10) in the case $n = 2$.

(1.11) EXAMPLE. Let $d_1, \ldots, d_n$, $d \geqslant 1$ be integers such that $d_1 \ldots d_n -$

$-\min_{i=1}^{n}(d_i) < d \leqslant d_1 \ldots d_n$. Then there exists a polynomial mapping $F = (F_1, \ldots, F_n)$: $C^n \to C^n$ such that $\deg F_i = d_i$ for $i = 1, \ldots, n$, $d(F) = d$ and $h(F) = d - \prod_{i=1}^{n} d_i + \min_{i=1}^{n}(d_i)$.

Proof. We may assume that $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n$. Let us define for every $n \geqslant 3$ a polynomial mapping F by the formula:

$$F(z) = (z_1^{d_1} + z_2^{d_1 - s}, z_3 z_2^{d_2 - 1}, F_3(z), \ldots, F_n(z))$$

where $F_i(z) = z_i^{d_i} + z_{i-1} z_2^{d_i - 1}$ for $2 < i < n$,

$$F_n(z) = z_n^{d_n} + z_1 z_2^{d_n - 1} \quad \text{and} \quad s = d_1 \ldots d_n - d.$$

In the case $n = 2$ we put $F(z_1, z_2) = (z_1^{d_1} + z_2^{d_1 - s}, z_1 z_2^{d_1 - 1})$. The mapping F is dominating because $F^{-1}(0) = \{0\}$.

To find the geometric degree of F we attach a weight $v(Z_i)$ (cf. Section 3) to each of the indeterminates Z_i $(i = 1, \ldots, n)$ by the formulae

$$v(Z_1) = (d_1 - s) d_3 \ldots d_n, \quad v(Z_2) = d_1 d_3 \ldots d_n,$$

$$v(Z_3) = d_1 d_3 \ldots d_n - s, \quad v(Z_{i+1}) = d_i v(Z_i) - (d_i - 1) v(Z_2) \quad \text{if} \quad 2 < i < n.$$

Thus the polynomial F_i is isobaric of weight $v(F_i) = d_i v(Z_i)$ for $i \neq 2$, $v(F_2) = d_1 \ldots d_n - s = d$. Now, it is easy to check using Proposition (1.2) that

$$d(F) = \frac{v(F_1) \ldots v(F_n)}{v(Z_1) \ldots v(Z_n)} = d.$$

Since $d > d_1 \ldots d_n - d_1$ the mapping F is proper in view of (1.3). Theorem (1.10) yields $h(F) \geqslant d - d_1 \ldots d_n + d_1$, on the other hand, we have $F(0, z_2, 0, \ldots, 0) = (z_2^{d_1 - s}, 0, \ldots, 0)$ so $h(F) = d - d_1 \ldots d_n + d_1$. ∎

Remark. I don't know if the first part of (1.10) gives the sharp estimation of $h(F)$ for given $\deg(F_i) = d_i$ $(i = 1, \ldots, n)$ and $d(F) = d$.

(1.12) Corollary. *Let* $F = (F_1, \ldots, F_n)$: $C^n \to C^n$ *be a proper polynomial mapping such that* $\deg F_1 = \ldots = \deg F_n = m > 1$ *and* $d(F) = m^n - 1$. *Then* $h(F) = m - 1$.

Proof. Let $a \in C^n \setminus (0)$ be such that $F^+(a) = 0$ (cf. Proposition (1.2)). Thus the polynomial mapping $t \to F(at)$ is of degree at most $m - 1$, so $h(F) \leqslant m - 1$. Indeed, it is easy to check that $h(F) \leqslant \deg_t F(p(t))/\deg_t p(t)$ for every polynomial mapping $C \ni t \to p(t) \in C^n$. On the other hand, we have by (1.10) the inequality $h(F) \geqslant m - 1$. ∎

(1.13) Corollary. *For any proper polynomial mapping* $F = (F_1, \ldots, F_n)$: $C^n \to C^n$ *we have* $h(F) \geqslant \min_{i=1}^{n}(\deg F_i) / \prod_{i=1}^{n} \deg F_i$. *If*

$$h(F) = \min_{i=1}^{n}(\deg F_i) / \prod_{i=1}^{n} \deg F_i,$$

then F *is a polynomial automorphism.*

Proof. By our theorem we have

$$h(F) \geqslant \frac{\min_{i=1}^{n}(\deg F_i)}{\prod_{i=1}^{n} \deg F_i - d(F)+1} \geqslant \frac{\min_{i=1}^{n}(\deg F_i)}{\prod_{i=1}^{n} \deg F_i}.$$

If $h(F) = \min_{i=1}^{n}(\deg F_i)/\prod_{i=1}^{n} \deg F_i$, then the above estimation gives $d(F) = 1$, so F is an automorphism (F^{-1} must be polynomial since it is a rational, locally bounded mapping cf. [11]). ∎

Now, suppose that F is a polynomial automorphism. By (1.8) and (1.13) we obtain $\deg(F^{-1}) \leqslant \prod_{i=1}^{n} \deg F_i/\min(\deg F_i)$. This implies the well-known inequality $\deg(F^{-1}) \leqslant (\deg(F))^{n-1}$ (cf. [1], [10]). In the case $n = 2$ we then have $\deg(F^{-1}) = \deg(F)$ and we may state the following:

(1.14) Corollary. *For any proper polynomial mapping* $F\colon \boldsymbol{C}^2 \to \boldsymbol{C}^2$ *we have* $h(F) \geqslant 1/\deg F$. *The equality holds if and only if F is an automorphism. If* $h(F) > 1/\deg F$, *then* $h(F) \geqslant 1/(\deg F - 1)$.

The second part of (1.14) follows from the observation: if $d_1 \leqslant d_2$, $1 < d \leqslant d_1 d_2$, then $\left[\frac{d_1 d_2 - d + 1}{d_1}\right] \leqslant d_2 - 1$.

2. A formula for the exponent of a proper polynomial mapping. The aim of this section is to prove Proposition (2.3) from which we infer Propositions (1.4) and (1.6) and a formula for $h(F)$ which generalizes Example (1.8).

For any monic polynomial $P(W, T) = T^d + a_1(W)T^{d-1} + \ldots + a_d(W)$ ($d > 0$) in indeterminates $(W, T) = (W_1, \ldots, W_n, T)$ we set

$$\delta(P) = \max_{i=1}^{d} \{\deg_w(a_i)/i\}.$$

Then, by definitions, $\delta(P) = -\infty$ if and only if $P = T^d$. We put $r^{-\infty} = 0$ for $r > 0$. The number $\delta(P)$ is the degree of the "multiple-valued function of variables w" defined by equation $P(w, t) = 0$. To be precise:

(2.1) Lemma (cf. [9]). *There exists a constant* $C > 0$ *such that if* $(w, t) \in \boldsymbol{C}^{n+1}$ *with* $|w| \geqslant 1$ *and* $P(w, t) = 0$, *then* $|t| \leqslant C|w|^{\delta(P)}$. *Suppose that there exist constants* $q, D, R > 0$ *such that* $\{(w, t)\colon P(w, t) = 0, |w| \geqslant R\} \subset \{(w, t)\colon |t| \leqslant D|w|^q, |w| \geqslant R\}$. *Then* $\delta(P) \leqslant q$.

Proof. By Lemma (1.1) there exists a constant $C_1 > 1$ such that for $i = 1, \ldots, d$ we have $|a_i(w)| \leqslant C_1 |w|^{\deg(a_i)}$ if $|w| \geqslant 1$. Hence $|t| \leqslant 2 \max|a_i(w)|^{1/i} \leqslant 2C_1|w|^{\delta(P)}$ if $P(w, t) = 0$. Next let us suppose that $\{(w, t)\colon P(w, t) = 0, |w| \geqslant R\} \subset \{(w, t)\colon |t| \leqslant D|w|^q, |w| \geqslant R\}$ with $q, D, R > 0$.

Fix $w \in \boldsymbol{C}^n$ such that $|w| \geqslant R$ and write $P(w, T) = \prod_{i=1}^{d} (T - t_i)$. Then

$|t_i| \leqslant D|w|^q$ for $i = 1, \ldots, d$ and we obtain the evaluation

$$|a_k(w)| = \Big| \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant d} t_{i_1} \ldots t_{i_k} \Big| \leqslant \binom{d}{k} D^k |w|^{q^k}.$$

Since $w \in C^n$, $|w| \geqslant R$ is arbitrary, we conclude that $\deg(a_k) \leqslant kq$ for $k = 1, \ldots, d$ and $\delta(P) \leqslant q$. ∎

Let us assume that $F = (F_1, \ldots, F_n)$ is a dominating polynomial mapping. For any $G \in C[W]$, $\deg G > 0$ we denote by $P_G(W, T)$ (resp. $Q_G(W, T)$) then unique monic polynomial from $C(W)[T]$ such that $P_G(F, T)$ (resp. $Q_G(F, T)$) is the characteristic polynomial of G (resp. the monic minimal polynomial of G) with respect to $C(Z)/C(F)$. Then $P_G = (Q_G)^{d/d_G}$, where $d = d(F) = (C(Z): C(F))$ and $d_G = (C(F, G): C(F))$.

(2.2) Lemma. *Let F: $C^n \to C^n$ be a dominating polynomial mapping. Then for each polynomial G, $\deg G > 0$ there exists a Zariski open set $U \subset C^n$ such that the coefficients of $P_G(W, T)$ are regular in U and for any $w \in U$ we have:*

$$P_G(w, T) = \prod_{z \in F^{-1}(w)} (T - G(z)).$$

Proof. Let $\tilde{Q}_G = \tilde{Q}_G(W, T) \in C[W, T]$ be a polynomial such that $\tilde{Q}_G(F, T)$ is a minimal polynomial of G with respect to $C(Z)/C(F)$. Then $\tilde{Q}_G$ is uniquely determined except for a constant factor; moreover, we have $\tilde{Q}_G(W, T) = c(W) Q_G(W, T)$ with nonzero $c(W) \in C[W]$.

Obviously, the mappings

$$(F, G): C^n \ni z \to (F(z), G(z)) \in \{(w, t) \in C^{n+1}: \tilde{Q}_G(w, t) = 0\}$$

and

$$\mathrm{pr}_1: \{(w, t) \in C^{n+1}: \tilde{Q}_G(w, t) = 0\} \ni (w, t) \to w \in C^n$$

are dominating of degree d/d_G and d_G respectively. By well-known properties of dominating regular mappings we conclude that there exists a Zariski open set $U \subset C^n$ such that:

(a) for any $w \in U$: $\# \mathrm{pr}_1^{-1}(w) = d_G$,

(b) for any $(w, t) \in \mathrm{pr}_1^{-1}(U)$: $\# (F, G)^{-1}((w, t)) = d/d_G$.

By (a) $c(w) \neq 0$ for $w \in U$ so the coefficients of Q_G and P_G are regular in U. Conditions (a), (b) imply $\# G(F^{-1}(w)) = d_G$, $\# (F^{-1}(w) \cap G^{-1}(t)) = d/d_G$ for $(w, t) \in \mathrm{pr}_1^{-1}(U)$.

Now, we have for any $w \in U$:

$$\prod_{z \in F^{-1}(w)} (T - G(z)) = \Big(\prod_{t \in G(F^{-1}(w))} (T - t) \Big)^{d/d_G} = Q_G(w, T)^{d/d_G} = P_G(w, T). \ ∎$$

(2.3) Proposition. *Let F: $C^n \to C^n$ be a proper polynomial mapping. Then for each polynomial G: $C^n \to C$, $\deg G > 0$ we have the following:*

(i) *the polynomial $P_G(W, T)$ has coefficients in $C[W]$.*

(ii) *There exists a constant* $C > 0$ *such that* $|G(z)| \leqslant C|F(z)|^{\delta(P_G)}$ *if* $|F(z)| \geqslant 1$.

(iii) *Suppose that there exist constants* $q, D, R > 0$ *such that if* $|F(z)| \geqslant R$, *then* $|G(z)| \leqslant D|F(z)|^q$. *Then* $\delta(P_G) \leqslant q$.

Proof. (i) Take a Zariski open set $U \subset C^n$ as in (2.2). Then $P_G(w, T) = T^d + \sum_{i=1}^{d} a_i(w) T^{d-i} = \prod_{z \in F^{-1}(w)} (T - G(z))$ for $w \in U$. Given $M > 0$ there exists $M^* > 0$ such that $|F(z)| \leqslant M$ implies $|z| \leqslant M^*$. Now, fix $w \in U$, $|w| \leqslant M$. A simple calculation like that from the second part of the proof of (2.1) yields

$$|a_i(w)| \leqslant \binom{d}{i} C^i \big(1 + (M^*)^{\deg G}\big)^i \quad \text{with } C > 0.$$

Then the coefficients of P_G are polynomials as locally bounded rational functions.

(ii) By the definition of P_G and by (i) we have $P_G\big(F(z), G(z)\big) = 0$ for all $z \in C^n$. Hence and from Lemma (2.1) we obtain the assertion.

(iii) From (2.2) we conclude that the conditions $P_G(w, t) = 0$, $|w| \geqslant R$ and $w \in U$ imply the inequality $|t| \leqslant D|w|^q$.

The set U being dense in C^n, the inequality $|t| \leqslant D|w|^q$ holds for all $(w, t) \in C^{n+1}$ such that $|w| \geqslant R$ and $P_G(w, t) = 0$. Now, from the second part of Lemma (2.1) we get $\delta(P_G) \leqslant q$. ∎

Let us suppose that $F\colon C^n \to C^n$ is a proper polynomial mapping. We write $P_i = P_i(W, T) = P_G(W, T)$ if $G = Z_i$ $(i = 1, \ldots, n)$. Then $P_i \in C[W][T]$ by (2.3) (i). Now, let us make the following observations:

(2.4) Property. *There exist constants* $C > 0$, $R_1 > 0$ *such that if* $|z| \geqslant R_1$, *then* $|z| \leqslant C|F(z)|^{\max_{i=1}^{n}(\delta(P_i))}$

Proof. By (2.3) (ii) there is a constant $C > 0$ such that $|z_i| \leqslant C|F(z)|^{\delta(P_i)}$ for $i = 1, \ldots, n$ if $|F(z)| \geqslant 1$. It suffices to take $R_1 > 0$ such that $|F(z)| \geqslant 1$ for $|z| \geqslant R_1$. ∎

(2.5) Property. *If there exist constants* $R, D, q > 0$ *such that* $|F(z)| \geqslant D|z|^q$ *for* $|z| \geqslant R$, *then* $q \leqslant 1/\max_{i=1}^{n}\big(\delta(P_i)\big)$.

Proof. We may suppose that $R \geqslant 1$. Let $A \geqslant 1$ be a constant such that $|F(z)| \leqslant A|z|^{\deg F}$ for $|z| \geqslant 1$. Then obviously the inequality $|F(z)| \geqslant AR^{\deg F}\cdot\big(1 + \max_{|z| \leqslant 1}|F(z)|\big)$ implies $|z| \geqslant R$, so by hypothesis we have $|F(z)| \geqslant D|z_i|^q$ for $i = 1, \ldots, n$ and by (2.3) (iii) applied to $G = Z_i$ $(i = 1, \ldots, n)$ we get $\delta(P_i) \leqslant 1/q$ $(i = 1, \ldots, n)$. ∎

Evidently (2.4) and (2.5) imply Propositions (1.4), (1.6) and the following formula:

(2.6) COROLLARY. *With the notation introduced above*:

$$h(F) = 1/\max_{i=1}^{n}(\delta(P_i)).$$

Let us note a simple corollary of (2.6) and of the definition of $\delta(P)$:

(2.7) COROLLARY. *The exponent $h(F)$ is of the form a/b, where a, b are integers such that $1 \leqslant b$, $1 \leqslant a \leqslant d(F)$. In particular $h(F) \leqslant d(F)$.*

Remark. Using Puiseux expansion one can prove in the case $n = 2$ the evaluation $h(F)\deg F \leqslant d(F)$.

3. Algebraic dependence of polynomials. For any polynomial $G(Z) = \sum a_{k_1 \ldots k_n} Z_1^{k_1} \ldots Z_n^{k_n}$ we set $\mathrm{supp}(G) = \{(k_1, \ldots, k_n) \in N^n \colon a_{k_1 \ldots k_n} \neq 0\}$. A weight v of the ring $C[Z]$ is a mapping of $C[Z] \setminus (0)$ into N satisfying the following conditions:

(a) $v(\sum(G_i) = \max(v(G_i)))$, where (G_i) is a finite family of nonzero polynomials such that $\mathrm{supp}\, G_i \cap \mathrm{supp}\, G_j = \emptyset$ if $i \neq j$.

(b) $v(GG') = v(G) + v(G')$.

(c) If $a \neq 0$ is an element of C, then $v(a) = 0$.

We assign to zero element of $C[Z]$ the value $-\infty$. Obviously if v is a weight of $C[Z]$, then $v(G) = \max(k_1 v(Z_1) + \ldots + k_n v(Z_n))$, where $(k_1, \ldots, k_n) \in \mathrm{supp}\, G$.

Suppose that $\min_{i=1}^{n}(v(Z_i)) > 0$, then $\deg G \leqslant v(G)/\min_{i=1}^{n}(v(Z_i))$ and obviously $\deg G \leqslant [v(G)/\min_{i=1}^{n}(v(Z_i))]$.

The aim of this section is to prove Proposition (3.3). First we recall a theorem from the classical algebra which is basic for our considerations.

(3.1) THEOREM (O. Perron [8], Theorem 57). *Let $F_1(Z), \ldots, F_n(Z)$, $F_{n+1}(Z) \in C[Z]$ be polynomials of positive degree, in n indeterminates $Z = (Z_1, \ldots, Z_n)$. Let v be the weight defined by conditions $v(W_i) = \deg F_i$ for $i = 1, \ldots, n+1$. Then there exists a nonzero polynomial $R \in C[W_1, \ldots, W_{n+1}]$ such that $R(F_1(Z), \ldots, F_{n+1}(Z)) = 0$ in $C[Z]$ and $v(R) \leqslant \prod_{i=1}^{n+1} \deg F_i$.*

(3.2) LEMMA. *Let $P_0(W)$, $P_1(W, A), \ldots, P_d(W, A)$ be polynomials in $n+N$ indeterminates (W, A) such that $P_0(W) \neq 0$ in $C[W]$. Suppose that the above polynomials are relatively prime in $C[W, A]$. Then for almost every $a \in C^N$ the polynomials $P_0(W)$, $P_1(W, a), \ldots, P_d(W, a)$ are relatively prime in $C[W]$.*

Proof. We may assume that $\deg(P_0) > 0$. Let $P_0(W) = \prod_i P_{0i}(W)$ be a factorization of P_0 into irreducible polynomials P_{0i}. By the assumption for each P_{0i} there is a polynomial $P_{k_i}(W, A)$ such that $P_{0i}(W)$ does not divide $P_{k_i}(W, A)$. Then by Hilbert's Nullstellensatz there exists $w^{(i)} \in C^n$ such that

$P_{0i}(w^{(i)}) = 0$ and $P_{k_i}(w^{(i)}, A) \neq 0$ in $C[A]$. It is easy to see that for any $a \in C^N$ such that $\prod_i P_{k_i}(w^{(i)}, a) \neq 0$ the polynomials $P_0(W)$, $P_1(W, a), \ldots, P_d(W, a)$ are relatively prime in $C[W]$. ∎

(3.3) PROPOSITION. *Let $F = (F_1, \ldots, F_n)$: $C^n \to C^n$ be a dominating polynomial mapping (we identify F and the sequence of polynomials $F_1, \ldots, F_n$ in indeterminates $Z = (Z_1, \ldots, Z_n)$). Let d be the degree of finite extension $C(Z)/C(F)$. Then for any polynomial $G = G(Z) \in C[Z]$ of positive degree there exists a polynomial $\tilde{P}_G(W, T) = P_0(W)T^d + P_1(W)T^{d-1} + \ldots + P_d(W) \in C[W][T]$ in indeterminates $(W, T) = (W_1, \ldots, W_n, T)$ such that*

(i) $\deg_T(\tilde{P}_G) = d$;

(ii) $\tilde{P}_G(F(Z), G(Z)) = 0$ *in* $C[Z]$;

(iii) $P_0(F(z))^{-1}\tilde{P}_G(F(Z), T) = P_G(F(Z), T)$ *(the characteristic polynomial of G with respect $C(Z)/C(F)$);*

(iv) *Let v be the weight of $C[W, T]$ defined by $v(W_i) = \deg F_i$ for $i = 1, \ldots, n$, $v(T) = \deg G$. Then $v(\tilde{P}_G) \leqslant \prod_{i=1}^{n} \deg F_i \deg G$. Let us suppose in addition that G is integral over $C[F]$. Then the characteristic polynomial $P_G(W, T)$ has the coefficients in $C[W]$ and $v(P_G) \leqslant \prod_{i=1}^{n} \deg F_i \deg G$.*

Proof. Let us fix an integer $l > 0$. We will prove (3.3) for all polynomials $G(Z)$ of degree less than l. Let $A = (A_{j_1, \ldots, j_n})_{j_1 + \ldots + j_n} \leqslant l$ be indeterminates, $N = \# \{(j_1, \ldots, j_n) \in N^n\colon j_1 + \ldots + j_n \leqslant l\}$ their number. Let $G(A, Z) = \sum A_{j_1 \ldots j_n} Z_1^{j_1} \ldots Z_n^{j_n}$.

Then each polynomial $G(Z) \in C[Z]$ of degree less than l is of the form $G(a, Z)$ with suitable $a \in C^N$.

Obviously the ring $C(Z)[A]$ is a free $C(F)[A]$-module of rank $d = (C(Z)\colon C(F))$. Multiplying the characteristic polynomial of $G(A, Z)$ with respect to this module by a suitable element from $C(F)$ we get a polynomial $P(W, A, T) \in C[W, A, T]$ with the following properties:

(a) $P(W, A, T) = P_0(W)T^d + P_1(W, A)T^{d-1} + \ldots + P_d(W, A) \in C[W, A, T]$ with $P_0(W) \neq 0$ in $C[W]$.

(b) The polynomials $P_0(W)$, $P_1(W, A), \ldots, P_d(W, A) \in C[W, A]$ are relatively prime.

(c) $P(F(Z), A, G(A, Z)) = 0$ in $C[A, Z]$.

For any $G(Z) = G(a, Z) \in C[Z]$ we define $\tilde{P}_G(W, T) = P(W, a, T) = P_0(W)T^d + P_1(W, a)T^{d-1} + \ldots + P_d(W, a)$. Obviously $\tilde{P}_G(W, T)$ has properties (i) and (ii). In order to check (iii) let us take a Zariski open set $U \subset C^n$ such that for any $w \in U$: $\# F^{-1}(w) = d$ and $P_0(w) \neq 0$. Then by (c) we get $P(w, A, T) = P_0(w) \prod_{z \in F^{-1}(w)} (T - G(A, z))$. Upon substituting $a \in C^N$ for A we get for $w \in U$: $P_0(w)^{-1} \tilde{P}_G(w, T) = P_0(w)^{-1} P(w, a, T) = \prod_{z \in F^{-1}(w)} (T - G(z))$. Then by Lemma (2.2) we have $P_0(W)^{-1} \tilde{P}_G(W, T) = P_G(W, T)$ which proves (iii).

Now let $D(W, A) = \mathrm{disc}_T P(W, A, T)$ be the discriminant of the polynomial $P(W, A, T) \in C[W, A][T]$. Obviously $D(W, A) \neq 0$ in $C[W, A]$. Hence and from Lemma (3.2) there exists a Zariski open set $\Omega \subset C^N$ such that:

(d) for every $a \in \Omega$ the coefficients $P_0(W)$, $P_1(W, a), \ldots, P_d(W, a)$ are relatively prime.

(e) For every $a \in \Omega$: $D(W, a) \neq 0$ in $C[W]$.

Let us fix $a \in \Omega$. From properties (d), (e) and (iii) it follows that the polynomial $P(W, a, T)$ is irreducible in the ring $C[W, T]$.

Then $P(W, a, T)$ is a generator of the ideal determined by $F_1(Z), \ldots, F_n(Z), G(Z)$ over C. According to (3.1) there is a nonzero polynomial $R(W, T) = C[W, T]$ such that $R(F(Z), G(Z)) = 0$ in $C[Z]$ and $v(R) \leqslant \prod_{i=1}^{n} \deg F_i \cdot \deg G$. The polynomial $P(W, a, T)$ divides $R(W, T)$ then we have $v(P(W, a, T)) \leqslant \prod_{i=1}^{n} \deg F_i \cdot \deg G$. Since the set Ω is open in C^N this estimation holds for every $a \in C^N$. This proves the first part of (iv). If G is integral over $C[F]$, then the characteristic polynomial $P_G(W, T)$ divides $\tilde{P}_G(W, T)$ in $C[W, T]$ so we have $v(P_G) \leqslant \prod_{i=1}^{n} \deg F_i \deg G$. ■

Now we will prove two corollaries of (3.3). Corollary (3.4) is an algebraic equivalent of (1.3).

(3.4) Corollary. *If* $d(F) > \prod_{i=1}^{n} \deg F_i - \min_{i=1}^{n}(\deg F_i)$, *then the ring* $C[Z]$ *is integral over* $C[F]$.

Proof. It suffices to check that every polynomial of degree 1 is integral over $C[F]$. Let $G \in C[Z]$ be a such polynomial and let $P_0(W)$ be the leading coefficient of $\tilde{P}_G(W, T)$. From property (iv) it follows that $v(P_0 T^d) \leqslant v(P) \leqslant \prod_{i=1}^{n} \deg F_i$ hence $v(P_0) \leqslant \prod_{i=1}^{n} \deg F_i - d(F)$ and $\deg_W(P_0) \leqslant (\prod_{i=1}^{n} \deg F_i - - d(F))/\min_{i=1}^{n}(\deg F_i) < 1$. Consequently $P_0(W)$ is a nonzero constant so G is integral over ring $C[F]$. ■

(3.5) Corollary (cf. [13], Proposition 6.2, p. 197). *For any polynomial* $H(W) \in C[W]$:

$$\deg(H) \leqslant \frac{\prod_{i=1}^{n} \deg F_i \cdot \deg(H \circ F)}{\min_{i=1}^{n}(\deg F_i)\, d(F)}.$$

Proof. Let us put $G(Z) = H(F(Z))$. Then obviously $P_G(W, T) = (T - H(W))^{d(F)}$. By (3.3) we get $d(F)v(H) \leqslant \prod_{i=1}^{n} \deg F_i \deg(H \circ F)$ hence follows (3.5). ∎

Remark. In all propositions of this section one may replace the field C of complex numbers by any field of characteristic zero.

4. Proof of the main result. We need a preliminary lemma.

(4.1) Lemma. *Let p, d, m be integers such that $d, m \geqslant 1$ and $p \geqslant d$. Let*

$$\delta = \max_{j=1}^{d} \left\{ \frac{1}{j} \left[\frac{p-d+j}{m} \right] \right\}.$$

Then

$$\delta = \frac{1}{d-p+m} \quad \text{if} \quad \frac{p-d+1}{m} \leqslant 1 \quad \text{and} \quad \delta = \left[\frac{p-d+1}{m}\right] \quad \text{if} \quad \frac{p-d+1}{m} \geqslant 1.$$

Proof. Suppose that

$$\frac{p-d+1}{m} \leqslant 1,$$

i.e., $d-p+m > 0$. Put $j_0 = d-p+m$, then

$$\frac{p-d+j}{m} = \frac{m-j_0+j}{m}.$$

If $j < j_0$, then

$$\left[\frac{m-j_0+j}{m}\right] = 0.$$

For $j \geqslant j_0$ we have

$$\frac{1}{j}\left[\frac{m-j_0+j}{m}\right] \leqslant \frac{m-j_0+j}{mj} \leqslant \frac{1}{j_0}$$

with equality for $j = j_0$. Consequently $\delta = 1/j_0$.

Now, let us consider the case $\frac{p-d+1}{m} \geqslant 1$. We will check that for every $j > 1$: $\left[\frac{p-d+1}{m}\right] \geqslant \frac{1}{j}\left[\frac{p-d+j}{m}\right]$.

It is obvious if $m = 1$, then we assume $m > 1$. We have

$$\frac{1}{j}\left[\frac{p-d+j}{m}\right] = \frac{1}{j}\left(\left[\frac{p-d+1}{m}+\frac{j-1}{m}\right]\right)$$
$$= \frac{1}{j}\left(\left[\frac{p-d+1}{m}\right]+\left[\frac{j-1}{m}\right]+\varepsilon\right), \qquad \varepsilon \in \{0, 1\}.$$

Hence it suffices to show that for every $j > 1$

$$\left[\frac{p-d+1}{m}\right] \geqslant \frac{1}{j-1}\left(\left[\frac{j-1}{m}\right]+\varepsilon\right), \qquad \text{where } \varepsilon \in \{0, 1\}.$$

It is obvious if $j = 2$, so let $j > 2$.
Therefore

$$\frac{1}{j-1}\left(\left[\frac{j-1}{m}\right]+\varepsilon\right) \leqslant \frac{1}{j-1}\left(\frac{j-1}{m}+1\right) \leqslant 1 \leqslant \left[\frac{p-d+1}{m}\right]$$

since $m \geqslant 2$, $j-1 \geqslant 2$ and $\frac{p-d+1}{m} \geqslant 1$. ∎

Now, let $F = (F_1, \ldots, F_n)$: $C^n \to C^n$ be a proper polynomial mapping of degree $d = d(F)$, and let $G \in C[Z]$, $\deg G = 1$ be given. Then the characteristic polynomial $P_G(W, T) = T^d + P_1(W) T^{d-1} + \ldots + P_d(W)$ has the coefficients in $C[W]$ and by Proposition (3.3) we have

$$v(P_j T^{d-j}) \leqslant v(P) \leqslant \prod_{i=1}^{n} \deg F_i.$$

Hence

$$v(P_j) \leqslant \prod_{i=1}^{n} \deg F_i - d + j$$

and

$$\deg_W(P_j) \leqslant \left[\frac{\prod_{i=1}^{n} \deg F_i - d + j}{\min_{i=1}^{n}(\deg F_i)}\right]$$

for $j = 1, \ldots, d$. By definition of $\delta(P_G)$ we get

$$\delta(P_G) \leqslant \max_{j=1}^{d}\left\{\frac{1}{j}\left[\frac{\prod_{i=1}^{n} \deg F_i - d + j}{\min_{i=1}^{n}(\deg F_i)}\right]\right\}.$$

Now, Lemma (1.4) and the above estimation imply

$$\delta(P_G) \leqslant \left[\frac{\prod_{i=1}^{n} \deg F_i - d + 1}{\min_{i=1}^{n}(\deg F_i)}\right] \quad \text{if} \quad \frac{\prod_{i=1}^{n} \deg F_i - d + 1}{\min_{i=1}^{n}(\deg F_i)} \geqslant 1$$

and $$\delta(P_G) \leqslant \frac{1}{d - \prod_{i=1}^{n} \deg F_i + \min_{i=1}^{n}(\deg F_i)} \quad \text{if} \quad \frac{\prod_{i=1}^{n} \deg F_i - d + 1}{\min_{i=1}^{n}(\deg F_i)} \leqslant 1.$$

Hence and from (2.6) follows the theorem.

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INSTYTUT MECHANIKI STOSOWANEJ
POLITECHNIKA ŚWIĘTOKRZYSKA
KIELCE, POLAND

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