ANNALES
POLONICI MATHEMATICI
XXIII (1970)

# A generalization of Hermite collineation to the $n$-dimensional space

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A projective transformation transforming the two-dimensional plane into itself is called *Hermite collineation*, if a conic belonging to this plane is transformed into itself. A special case of a collineation is a harmonic homology. Hermite collineation may be generalized in a natural way to the three-dimensional space, replacing the conic by a space curve of third order [6].

In this paper there are formulated fundamental theorems on Hermite collineation generalized to the $n$-dimensional projective space; the discussion of kinds of projectivities which could be a priori Hermite transformations, is not performed.

Notations.

$P^{n-k}$ — $(n-k)$-dimensional projective space,

$A_1 \ldots A_{n-k+1}$ — a simplex of the space $P^{n-k}$,

$\alpha_k^{n-1}$ — an $(n-1)$-dimensional hyperplane contained in $P^n$ and spanned by a simplex with vertices $A_1 \ldots A_{k-1} A_{k+1} \ldots A_{n+1}$,

$\alpha_{k,l}^{n-2}$ — $(n-2)$-dimensional hyperplane contained in $P^n$ and spanned by the simplex $A_1 \ldots A_{k-1} A_{k+1} \ldots A_{l-1} A_{l+1} \ldots A_{n+1}$,

$\beta_1^k$ — $k$-dimensional hyperplane spanned by the simplex $A_2 \ldots A_{k+2}$,

$\beta_{n+1}^k$ — $k$-dimensional hyperplane spanned by the simplex $A_n \ldots A_{n-k}$,

$C^{n-k}$ — a normal curve in the space $P^{n-k}$ [1] and [3],

${}^lC^m$ — a curve of order $m$, possessing an $l$-tuple point and belonging to the space $P^{m-l+1}$,

$\underset{i}{\gamma}^k$ — $k$-dimensional hyperplane, touching with the order $k$ to a curve $C^n$ $(k = 1, \ldots, n-1)$ at a point $A_i$,

$B^k$ — a projection of the point $B$, from the point $A_k$ on the hyperplane $\alpha_k^{n-1}$,

$B^{k,l}$ — a projection of the point $B^k$, from the point $A_l$ on the hyperplane $\alpha_{k,l}^{n-2}$,

$X_{a_k}$ — a common point of the straight line $X_1X_2$ ($X_2$ is the point corresponding to $X_1$ in the projective transformation under consideration) and the hyperplane $a_k^{n-1}$,

$\lambda_{k,k+1}$ — the cross ratio of the four points $X_1, X_2, X_{a_k}, X_{a_{k+1}}$,

$\Phi(W, C^{n-k-1})$ — the conical surface lying in the space $P^{n-k}$, with a directrix $C^{n-k-1}$ and vertex $W$ not belonging to the space in which the directrix is contained; the elements of the surface are straight lines passing through $W$ and intersecting the curve $C^{n-k-1}$.

The above definition of a conical surface differs essentially from the well-known definition of a cone contained in $P^{n-k}$, whose directrix is a hyperquadric contained in $P^{n-k-1}$.

Properties of the above introduced surface, needed in the sequel, will be explained by a number of lemmas.

LEMMA 1. *The curve $C^m$ has at most $k+1$ common points (not always different) with a $k$-dimensional hyperplane* $(1 \leqslant k \leqslant m-1)$.

Let us suppose, a $k$-dimensional hyperplane cuts $C^m$ in more than $k+1$ points. Then there exists an $(m-1)$-dimensional hyperplane containing the $k$-dimensional hyperplane and intersecting the curve $C^m$ at more than $k+1+[m-(k+1)]$ points. This hyperplane would have more than $m$ common points with the curve $C^m$, but this is impossible.

Arguing similarly we may verify easily the following

LEMMA 2. *The curve ${}^lC^m$ has at most $k+l$ common points (not always different) with a $k$-dimensional hyperplane* $(1 \leqslant k \leqslant m-1)$.

LEMMA 3. *A $k$-dimensional hyperplane* $(k = 1, \ldots, n-2)$ *which does not contain the vertex of the surface $\Phi(W, C^{n-1})$ has at most $k+1$ common points (not always different) with this surface.*

To prove this lemma we shall apply the following notation:

$\delta^{n-1}$ — $(n-1)$-dimensional hyperplane containing the curve $C^{n-1}$;

$\varepsilon^{k+1}$ — $(k+1)$-dimensional hyperplane containing the $k$-dimensional hyperplane under consideration and the point $W$,

$\varphi^k$ — the common part of hyperplanes $\delta^{n-1}$ and $\varepsilon^{k+1}$.

Let us suppose the $k$-dimensional hyperplane intersects the surface $\Phi(W, C^{n-1})$ at more than $k+1$ points. Then the hyperplane $\varepsilon^{k+1}$ would intersect this surface at more than $k+1$ elements, and hence, the curve $C^{n-1}$ at more than $k+1$ points. But this is impossible, for these points would belong to the hyperplane $\varphi^k$, and by Lemma 1, this hyperplane may intersect the curve $C^{n-1}$ at most at $k+1$ points.

LEMMA 4. *An $(n-1)$-dimensional hyperplane not passing through the point $W$ intersects the surface $\Phi(W, C^{n-1})$ along the curve $C_1^{n-1}$.*

We denote by $\delta^{n-1}$ the hyperplane containing the curve $C^{n-1}$, and

by $\varepsilon^{n-2}$ an arbitrary $(n-2)$-dimensional hyperplane contained in the hyperplane of intersection.

The order of the curve $C_1^{n-1}$ is equal to the number of common points of this curve and the hyperplane $\varepsilon^{n-2}$, and this number is equal in turn to the number of common points of the curve $C^{n-1}$ and the $(n-1)$-dimensional hyperplane passing through $W$ and containing $\varepsilon^{n-2}$. This hyperplane has an $(n-2)$-dimensional hyperplane, intersecting the curve $C^{n-1}$ at $n-1$ points, common with $\delta^{n-1}$. Thus the curve $C_1^{n-1}$ is of order $n-1$. It is a normal curve, since its projection from the point $W$ on the hyperplane $\delta^{n-1}$ is the curve $C^{n-1}$, and the projection of a degenerated curve of order $n-1$ could not be a normal curve of order $n-1$.

Remark. If an $(n-1)$-dimensional hyperplane of intersection passes through the point $W$, then it cuts the surface $\Phi(W, C^{n-1})$ at $n-1$ elements.

LEMMA 5. *There is a curve $C^n$ on the surface $\Phi(W, C^{n-1})$ passing through the point $W$ and intersecting the elements of the surface once more.*

Let $\delta^{n-1}$ denote the hyperplane containing the curve $C^{n-1}$. Let us choose $n+2$ points on the surface $\Phi$, which together with the point $W$ define the curve $C^n$. Let us suppose, the curve $C^n$ does not lie on $\Phi(W, C^{n-1})$. Its projection from the point $W$ on the hyperplane $\delta^{n-1}$ would be a normal curve of order $n-1$, possessing $n+2$ common points with the curve $C^{n-1}$. This is impossible, because these points define a normal curve of order $n-1$ uniquely. Since the curve $C^{n-1}$ is a projection of the curve $C^n$ from the point $W$, every element of the surface $\Phi$ intersects the curve $C^n$ at one point besides the point $W$, only. The proof of the lemma is concluded.

LEMMA 6. *There exists a curve ${}^{N-n+1}C^{N-1}$ on the surface $\Phi(W, C^{n-2})$ $(N \geqslant n)$ having an $(N-n+1)$-tuple point at the point $W$ and intersecting every element at $N-n+2$ points.*

By Lemma 5, we may choose the conical surface $\Phi_1(W, C_1^{N-2})$ with the curve $C^{N-1}$ lying on it. We choose a point $S_1 \neq W$ on the surface $\Phi_1$, $S_1$ not lying on the curve $C^{N-1}$. The projection of the curve $C^{N-1}$ from the point $S_1$ on the hyperplane $\delta^{N-2}$ (containing the curve $C_1^{N-2}$) is the curve ${}^2C^{N-1}$, because the element $S_1W$ intersects the curve $C^{N-1}$ at two points. The curve ${}^2C^{N-1}$ has a double point at the point $W_2 \in C_1^{N-2}$ ($W_2$ is the projection of the point $W$ on $\delta^{N-2}$). The projections of the elements of the surface $\Phi_1$ are straight lines passing through the point $W_2$ and intersecting also the curves $C_1^{N-2}$, ${}^2C^{N-1}$ once more. Hence it follows that the curve ${}^2C^{N-1}$ lies on the surface $\Phi_2(W_2, C_2^{N-3})$.

Repeating the above argument yet $N-n-1$ times we obtain $\Phi_{N-n+1}(W_{N-n+1}, C_{N-n+1}^{n-2}) = \Phi(W, C^{n-2})$ and a curve ${}^{N-n+1}C^{N-1}$ lying on this surface and having an $(N-n+1)$-tuple point at the point $W$. Every element of the surface $\Phi$ intersects the curve ${}^{N-n+1}C^{N-1}$ at $N-n+2$ points.

LEMMA 7. *Let the curves* $C_1^{n-1}$ *and* ${}^{N-n+1}C^{N-1}$ $(n \geqslant 4, N \geqslant n)$ *lie on the surface* $\Phi(W, C^{n-2})$ *and pass through the point* $W$; *moreover, let* $W$ *be a multiple point of the second of these curves. If the two curves do not possess a common tangent line at the point* $W$, *then they intersect themselves at* $N$ *points different from the point* $W$.

For $n = 4$ we have $\Phi(W, C^2)$, $C_1^3$ and ${}^{N-3}C^{N-1}$ having an $(N-3)$-tuple point at the point $W$.

We choose on the curve $C_1^3$ an arbitrary point $S$ not lying on the curve ${}^{N-3}C^{N-1}$, and we project both curves from this point on the plane of the conic $C^2$. We obtain a curve $C_1^2$ passing through the point $W'$ (the projection of the point $W$), and a curve ${}^{N-2}C^{N-1}$ possessing an $(N-2)$-tuple point at the point $W'$, because the straight line $SW$ intersects the curve ${}^{N-3}C^{N-1}$ at $N-2$ points. The curves $C_1^2$, ${}^{N-2}C^{N-1}$ intersect at $2(N-1)$ points, and among them at $2(N-1)-(N-2)$ points different from $W'$, since by the assumption, the curves are not touching at the point $W'$. From the assumption $S \in \Phi$ it follows that the curves $C_1^3$, ${}^{N-3}C^{N-1}$ intersect also at $N$ points different from $W$, and the lemma is proved in the case $n = 4$.

Assuming the lemma to be true for $n = k$ we prove it to be true for $n = k+1$.

Let us choose a conical surface $\Phi(W, C^{k-1})$ with the curves $C_1^k$ and ${}^{N-k}C^{N-1}$ lying on it, where the point $W$ is a multiple point of the curve ${}^{N-k}C^{N-1}$. Let us suppose, both the curves do not possess a common tangent line at the point $W$. We choose a point $S$ on the curve $C_1^k$, which does not lie on the curve ${}^{N-k}C^{N-1}$. We project both curves and the elements of the conical surface from the point $S$ on the hyperplane containing the curve $C^{k-1}$. We obtain two curves $C^{k-1}$, $C_1^{k-1}$ passing through the point $W'$, and the curve ${}^{N-k+1}C^{N-1}$ having a multiple point at the point $W'$, where the curves $C_1^{k-1}$, ${}^{N-k+1}C^{N-1}$ do not possess a common tangent line at the point $W'$. Both curves lie on the conical surface with vertex $W'$ and directrix $C^{k-1}$, i.e. on the surface $\Phi_1(W', C^{k-2})$. By the assumption, the curves $C_1^{k-1}$, ${}^{N-k+1}C^{N-1}$ possess $N$ common points different from $W'$. The curves $C_1^{k-1}$, ${}^{N-k+1}C^{N-1}$ are projections (from the point $S$) of the curves $C_1^k$, ${}^{N-k}C^{N-1}$ lying on the surface $\Phi(W, C^{k-1})$. Since $S \in \Phi$, we obtain by Lemma 3, that the curves $C_1^k$, ${}^{N-k}C^{N-1}$ possess also $N$ common points different from $W$. Thus the lemma is true for an arbitrary $n \geqslant 4$.

In the three-dimensional space, the surfaces $\Phi_1(W_1, C_1^2)$, $\Phi_2(W_2, C_2^2)$ intersect always along a curve of order 4, i.e. along a curve of order $n+1$. An arbitrary straight line intersects each of these surfaces at two points. For $n \geqslant 4$ the straight line cannot intersect the surface $\Phi(W, C^{n-1})$, and consequently, two conical surfaces cannot possess a common line. This explains

LEMMA 8. *Conical surfaces* $\Phi_1(W_1, C_1^{n-1})$, $\Phi_2(W_2, C_2^{n-1})$ $(W_1 \in C_2^{n-1}$, $W_2 \in C_1^{n-1})$ *with a common element* $W_1W_2$, *and different tangent planes to the surfaces* $\Phi_1$ *and* $\Phi_2$ *along that element* $W_1W_2$ *have for* $n \geqslant 4$

1° *a finite number of common points, if the curve* $C_1^{n-1}$ *does not lie on the conical surface* $\Phi_3(W_2, C_2^{n-2})$, *where* $C_2^{n-2}$ *is the projection of the curve* $C_2^{n-1}$ *from the point* $W_1$ *on the hyperplane containing the curve* $C_1^{n-1}$,

2° *a common curve of order* $n+1$, *if the curve* $C_1^{n-1} \in \Phi_3(W_2, C_2^{n-2})$.

Let us intersect both conical surfaces by means of an $(n-1)$-dimensional hyperplane not passing through the points $W_1$, $W_2$. By Lemma 4, we obtain curves $C_{\mathrm{I}}^{n-1}$, $C_{\mathrm{II}}^{n-1}$ as intersections.

Let us project the surfaces $\Phi_1(W_1, C_1^{n-1})$, $\Phi_2(W_2, C_2^{n-1})$ together with the curves $C_{\mathrm{I}}^{n-1}$, $C_{\mathrm{II}}^{n-1}$ from the point $W_1$ on the $(n-1)$-dimensional hyperplane containing the curve $C_1^{n-1}$. We obtain: a surface $\Phi_3(W_2, C_2^{n-2})$ (because $W_1 \in C_2^{n-1}$, and $W_2$ do not belong to the space in which the curve $C_2^{n-1}$ is contained), the curve $C_{\mathrm{II}}'^{n-1}$ lying on this surface, and the curve $C_{\mathrm{I}}'^{n-1} = C_1^{n-1}$.

Case 1°. The curve $C_{\mathrm{I}}^{n-1}$ intersects the surface $\Phi_3(W_2, C_2^{n-2})$ at a finite number of points. These points (and only these) are projections of common points of surfaces $\Phi_1(W_1, C_1^{n-1})$, $\Phi_2(W_2, C_2^{n-1})$, and this proves the first part of the lemma.

Case 2°. The curves: $C_1^{n-1} = C_{\mathrm{I}}'^{n-1}$, $C_{\mathrm{II}}'^{n-1}$ lie on the surface $\Phi_3(W_2, C_2^{n-2})$. From the assumption it follows that both curves do not possess a common tangent line at the point $W$. Hence one may apply Lemma 7. We obtain that both curves have $n+1$ common points. From Lemma 3 and from the assumption that the point $W_1 \in \Phi_2(W_2, C_2^{n-1})$ it follows that the curves $C_{\mathrm{I}}^{n-1}$, $C_{\mathrm{II}}^{n-1}$ intersect also at $n+1$ points. This means that the common line of the surfaces $\Phi_1(W_1, C_1^{n-1})$, $\Phi_2(W_2, C_2^{n-1})$ is a curve of order $n+1$. This curve degenerates into the single straight line $W_1W_2$ and a curve of order $n$ which cannot be degenerated. This curve passes through the vertices of the conical surfaces, and is a normal curve.

LEMMA 9. *Let* $\underset{i}{\gamma}^k$ *be a* $k$*-dimensional hyperplane touching with the order* $k$ *to the curve* $C^n$ *at the point* $A_i$ $(k = 1, \ldots, n-1)$, *and let* $A_l$ *be the common point of the hyperplane* $\alpha_i^{n-1}$ *and the tangent line* $A_iA_i$ *to the curve* $C^n$. *The projection of the curve* $C^n$ *from the point* $A_i$ *on the hyperplane* $\alpha_i^{n-1}$ *is the curve* $C^{n-1}$ *touching with the order* $k-1$ *to the hyperplane* $\underset{i}{\gamma}^{k-1}$ *(the projection of the hyperplane* $\underset{i}{\gamma}^k$*) at the point* $A_l$.

Proof. By Lemma 1, the hyperplane $\underset{i}{\gamma}^{k-1}$ cannot be touching with the order greater than $k-1$ to the curve $C^{n-1}$. Let us suppose, this order is less than $k-1$. Two cases are possible:

1° $\underset{i}{\gamma}^{k-1}$ has a common point different from $A_l$ (e.g. $A_m^i$) with the curve $C^{n-1}$,

2° $\underset{i}{\gamma}^{k-1}$ has only the point $A_l$ common with the curve $C^{n-1}$, but it is touching to $C^{n-1}$ with the order $m < k-1$.

Case 1°. The point $A_m^i$ would be the projection of the point $A_m$ of the curve $C^n$ different from $A_i$. But this would mean that $\underset{i}{\gamma}^k$ passes through $A_i$ and $A_m$, intersecting the curve $C^n$ at the $k+1$ points coincident with the point $A_i$ and at the point $A_m$. By Lemma 1, this is impossible.

Case 2°. There would exist an $(n-2)$-dimensional hyperplane $\delta^{n-2}$ in the hyperplane $a_i^{n-1}$, containing $\underset{i}{\gamma}^{k-1}$ and intersecting the curve $C^{n-1}$ besides the point $A_l$ at further $n-1-(m+1)$ points. Consequently, the $(n-1)$-dimensional hyperplane containing the point $A_i$ and $\delta^{n-2}$ (whence also $\underset{i}{\gamma}^k$) would intersect the curve $C^n$ at $k+1+(n-m-2)$ points, but this is impossible.

LEMMA 10. *Let $\underset{i}{\gamma}^k$ be a $k$-dimensional hyperplane touching with the order $k$ to the curve $C^n$ at the point $A_i$ $(k = 1, \ldots, n-1)$. Let $A_l \in C^n$, and let $a_l^{n-1}$ contain $\underset{i}{\gamma}^k$. The projection of the curve $C^n$ from the point $A_l$ on the hyperplane $a_l^{n-1}$ is a curve $C^{n-1}$ which is also touching with the order $k$ to the hyperplane $\underset{i}{\gamma}^k$* at the point $A_i$ $(k = 1, \ldots, n-2)$.

The proof is to be performed for $k = 1, \ldots, n-2$, since for $k = n-1$ there is $a_l^{n-1} = \underset{i}{\gamma}^{n-1}$ and the hyperplane $\underset{i}{\gamma}^{n-1}$ contains the curve $C^{n-1}$.

Since the central projection does not change the incidence, and by the projection from the point $A_l$ on the hyperplane $a_l^{n-1}$, the hyperplane $\underset{i}{\gamma}^k$ is left unchanged, we see that $\underset{i}{\gamma}^k$ has a common point $A_i$ with the curve $C^{n-1}$ of multiplicity greater than or equal to the number of common points of $\underset{i}{\gamma}^k$ and $C^n$. On the other hand, by Lemma 1, $\underset{i}{\gamma}^k$ may have at most $k+1$ common points with the curve $C^{n-1}$, and the proof of the lemma is finished.

THEOREM 1. *If the points $A_1, \ldots, A_{n+1}$ are vertices of a simplex in the space $P^n$, and if the point $B$ belonging to $P^n$ does not belong to any of the hyperfaces of the simplex, then through the point $B$ there passes exactly one curve $C^n$ touching with the order $k$ at points $A_1$ and $A_{n+1}$ to the hyperplanes $\underset{1}{\gamma}^k$, $\underset{n+1}{\gamma}^k$, spanned over the simplexes:*

$$(*) \qquad \begin{aligned} &\underset{1}{\gamma}^k\colon A_1 A_2 \ldots A_{k+1} \\ &\underset{n+1}{\gamma}^k\colon A_{n+1} A_n \ldots A_{n-k+1} \end{aligned} \qquad \textit{for } k = 1, 2, \ldots, n-1.$$

The above theorem is true in two- and three-dimensional spaces [6]. Supposing that it is true in $(n-2)$- and $(n-1)$-dimensional spaces we shall prove it to be true in the $n$-dimensional space. We shall perform the proof in two steps:

1° we shall prove the existence of a curve $C^n$ satisfying the above conditions,

2° we shall show that the above conditions determine the curve $C^n$ uniquely.

Ad 1°. In the hyperplanes $\alpha_1^{n-1}$ and $\alpha_{n+1}^{n-1}$ defined by means of simplexes

$$A_2A_3\ldots A_{n+1}, \qquad A_1A_2\ldots A_n$$

let us construct the curves $C_1^{n-1}$, $C_{n+1}^{n-1}$ passing through the points $B^1$ and $B^{n+1}$ (the projections of the point $B$ from the points $A_1$ and $A_{n+1}$), respectively, and touching with the order $k$ $(k=1,\ldots,n-2)$ at the points $A_2$ and $A_{n+1}$ resp. $A_1$ and $A_n$ to the hyperplanes $\underset{2}{\gamma}^k$, $\underset{n+1}{\gamma}^k$ and $\underset{1}{\gamma}^k$, $\underset{n}{\gamma}^k$ spanned over the simplexes:

$$(**)\qquad \begin{array}{ll} \underset{2}{\gamma}^k\colon\ A_2A_3\ldots A_{k+2} & \\ & \text{for } C_1^{n-1}; \\ \underset{n+1}{\gamma}^k\colon\ A_{n+1}A_n\ldots A_{n-k+1} & \\ \underset{1}{\gamma}^k\colon\ A_1A_2\ldots A_{k+1} & \\ & \text{for } C_{n+1}^{n-1}. \\ \underset{n}{\gamma}^k\colon\ A_nA_{n-1}\ldots A_{n-k} & \end{array}$$

By the assumption, such curves on $(n-1)$-dimensional hyperplanes exist.

Now, we prove that there exists a common line of the conical surfaces $\Phi_1(A_1, C_1^{n-1})$, $\Phi_2(A_{n+1}, C_{n+1}^{n-1})$. For this purpose let us project the curve $C_1^{n-1}$ and the surface $\Phi_2(A_{n+1}, C_{n+1}^{n-1})$ from the point $A_1$ on $\alpha_1^{n-1}$ — a hyperplane containing the curve $C_1^{n-1}$. We obtain $\Phi_3(A_{n+1}, C_{n+1}^{n-1})$ and $C_1^{n-1}$, where the curve $C_{n+1}^{n-2}$ passes through the point $B^{n+1,1}$ (the projection of the hyperplane $\alpha_{n+1}^{n-1}$ on the hyperplane $\alpha_1^{n-1}$ from the point $A_1$ is indeed the projection on the hyperplane $\alpha_{1,n+1}^{n-2}$) and, moreover, by Lemmas 9 and 10, is touching with the order $k$ $(k=1,\ldots,n-3)$ to the hyperplanes $\underset{2}{\gamma}^k$ and $\underset{n}{\gamma}^k$ spanned over the simplexes:

$$A_2A_3\ldots A_{k+2}, \qquad A_nA_{n-1}\ldots A_{n-k}.$$

On the other hand, the projection of the curve $C_1^{n-1}$ (passing through $B_1$) from the point $A_{n+1}$ on the $(n-2)$-dimensional hyperplane $\alpha_{1,n+1}^{n-2}$ is the curve $C_1^{n-2}$ passing through the point $B^{1,n+1}$ and touching with the order $k$ $(k=1,\ldots,n-3)$ at the points $A_2$ and $A_n$ to the hyperplanes spanned over the same simplexes as the curve $C_{n+1}^{n-2}$.

Let us yet remark that $B^{n+1,1} = B^{1,n+1}$, because the plane $A_1A_{n+1}B$, containing the points $B^1$ and $B^{n+1}$, intersects $a_{1,n+1}^{n-2}$ at one point, which is simultaneously the point $B^{1,n+1}$ and $B^{n+1,1}$.

The above formulated conditions define (by the assumption) a curve in the $(n-2)$-dimensional space, uniquely, and $C_1^{n-2} = C_{n+1}^{n-2}$. This means that the curve $C_1^{n-1}$ lies on the surface $\Phi_3(A_{n+1}, C_{n+1}^{n-2})$, and condition 2° of Lemma 8 with $W_1, W_2, C_2$ replaced by $A_1, A_{n+1}, C_{n+1}$ is satisfied.

Thus we have shown that the common line of the conical surfaces $\Phi_1(A_1, C_1^{n-1})$, $\Phi_2(A_{n+1}, C_{n+1}^{n-1})$ exists, and it consists of the straight line $A_1A_{n+1}$ and the curve $C^n$ passing through the points $A_1, A_{n+1}$.

We have still to show that the curve $C^n$ satisfies conditions (*).

The hyperplane $a_{n+1}^{n-1}$ intersects the conical surface with vertex $A_{n+1}$ at the curve $C_{n+1}^{n-1}$, and the surface with vertex $A_1$ at the element $A_1A_2$, since the hyperplane $a_{1,n+1}^{n-2}$ is touching with the order $n-2$ to the curve $C_1^{n-1}$ at the point $A_2$. The straight line $A_1A_2$ is touching to the curve $C_{n+1}^{n-1}$ at the point $A_1$, and by Lemma 1, this is the only common point of $a_{n+1}^{n-1}$ and $C^n$.

Hence and from the assumption that the hyperplane spanned over the simplex $A_2 \dots A_{k+2}$ is touching with the order $k$ to the curve $C_1^{n-1}$ at the point $A_2$ $(k = 1, \dots, n-2)$ it follows that the hyperplane spanned over the simplex $A_1 \dots A_{k+1}$ is touching with the order $k$ to the curve $C^n$ at the point $A_1$ $(k = 1, \dots, n-1)$. Similarly we prove that the hyperplane spanned over the simplex $A_{n+1} \dots A_{n-k+1}$ is touching with the order $k$ to the curve $C^n$ at the point $A_{n+1}$ $(k = 1, \dots, n-1)$.

Ad 2°. Should there exist two curves of order $n$ passing through $B$ and satisfying conditions (*), there projections from the point $A_1$ on the hyperplane $a_1^{n-1}$ would be the curves $C_1^{n-1}$ and $\bar{C}_1^{n-1}$ satisfying conditions (**) at the points $A_2$ and $A_{n+1}$, by Lemmas 9 and 10. From the assumption of existence of exactly one curve of order $n-1$ lying on the $(n-1)$-dimensional hyperplane and satisfying conditions (**) at the points $A_2$ and $A_{n+1}$ it follows that $C_1^{n-1} = \bar{C}_1^{n-1}$. Similarly, the projection of both curves from the point $A_{n+1}$ on the hyperplane $a_{n+1}^{n-1}$ would be the curve $C_{n+1}^{n-1}$. We obtain a contradiction, since both curves of order $n$ would lie on two conical surfaces of order $n-1$, simultaneously, but this is impossible, by Lemma 8.

THEOREM 2. *There exists such a projective transformation $H$ of the space $P^n$ into itself, which transforms a curve $C^n$ into itself.*

Let the curve $C^n$ satisfy conditions (*). We define the transformation $H$ by means of fixed points $A_1, \dots, A_{n+1}$ and a pair of points $X_1, X_2$ of the curve $C^n$ which correspond to themselves.

Would a point $Y_2$ not lying on the curve $C^n$ correspond to a point $Y_1 \neq X_1$ of the curve $C^n$ by means of the transformation $H$, then to the

curve $C^n$ there would correspond by means of $H$ a curve $C_1^n \neq C^n$ having a common point $X_2$ with the curve $C^n$, and both curves would satisfy conditions (*). By Theorem 1, this is impossible.

The transformation $H$ will be called the *Hermite transformation.*

THEOREM 3. *If at the projective transformation the curve $C^n$ goes into itself, then one may mark the fixed points of the transformation in such a manner that the curve $C^n$ satisfies conditions* (*).

In that transformation a range of points lying on the curve $C^n$ corresponds projectively to a range of points of the same curve. Let $A_1$ and $A_{n+1}$ be fixed points of both ranges. The projective transformation associates with the $k$-dimensional hyperplanes $\underset{1}{\gamma}^k$, $\underset{n+1}{\gamma}^k$ $(k = 1, \ldots, n-1)$ touching with the order $k$ to the curve $C^n$ at the points $A_1$ and $A_{n+1}$, the same hyperplanes. These hyperplanes are coincident.

The intersection of the hyperplanes $\underset{1}{\gamma}^{n-1}$, $\underset{n+1}{\gamma}^{n-1}$ is a hyperplane $\beta_{1,n+1}^{n-2}$ spanned over the simplex with vertices $A_2, \ldots, A_n$, the fixed points of the projective transformation. Next, the hyperplanes $\underset{1}{\gamma}^{n-2}$, $\underset{n+1}{\gamma}^{n-2}$ intersect $\beta_{1,n+1}^{n-2}$ at hyperplanes $\beta_1^{n-3}$ and $\beta_{n+1}^{n-3}$, respectively. These hyperplanes cannot be identical, for in other case the $(n-1)$-dimensional hyperplane determined by the hyperplanes $\beta_1^{n-3} = \beta_{n+1}^{n-3}$ and points $A_1, A_{n+1}$, would contain $\underset{1}{\gamma}^{n-2}$ and $\underset{n+1}{\gamma}^{n-2}$ intersecting the curve $C^n$ at $2(n-1)$ points. But this is impossible for $n \geqslant 3$, and in order that the hyperplanes $\beta_1^{n-3}, \beta_{n+1}^{n-3}$ be non-void sets it must be $n \geqslant 3$.

Thus, these hyperplanes are spanned over the simplexes

$$\begin{aligned} &\beta_1^{n-3}\colon\ A_2 A_3 \ldots A_{n-1}, \\ &\beta_{n+1}^{n-3}\colon\ A_n A_{n-1} \ldots A_3. \end{aligned} \tag{1}$$

Arguing in a similar manner we can state that the hyperplanes $\underset{1}{\gamma}^k$, $\underset{n+1}{\gamma}^k$ intersect the hyperplanes $\beta_1^k, \beta_{n+1}^k$ at the hyperplanes $\beta_1^{k-1}, \beta_{n+1}^{k-1}$ spanned over the simplexes

$$\begin{aligned} &\beta_1^{k-1}\colon\ A_2 A_3 \ldots A_{k+1}, \\ &\beta_{n+1}^{k-1}\colon\ A_n A_{n-1} \ldots A_{n-k+1}. \end{aligned} \tag{2}$$

But this means that the hyperplanes $\underset{1}{\gamma}^k$, $\underset{n+1}{\gamma}^k$ are of form (*).

First, let us suppose that $k \leqslant \frac{1}{2}(n-1)$.

We prove that the hyperplanes $\underset{1}{\gamma}^k$ and $\underset{n+1}{\gamma}^k$ intersect the hyperplanes $\beta_1^k, \beta_{n+1}^k$ at the hyperplanes $\beta_1^{k-1}, \beta_{n+1}^{k-1}$ spanned over disjoint simplexes.

Let us suppose that these simplexes possess at least one common vertex. The hyperplane spanned over the simplex possessing not more

than $k+(k-1)+2$ vertices (a hyperplane of dimension at most $2k$, passing through $A_1$ and $A_{n+1}$ and containing the hyperplanes $\beta_1^{k-1}, \beta_{n+1}^{k-1}$) would contain the curve, for in other case it would intersect this curve at $2(k+1)$ points. But by Lemma 1, this is impossible. Hence it follows, that there would be $2k \geqslant n$, but this is a contradiction to the assumption.

If $k \leqslant \frac{1}{2}(n-1)$, the vertices of simplexes which span the hyperplanes $\beta_1^{k-1}$ and $\beta_{n+1}^{k-1}$ belong to disjoint sets, and the choice of these vertices (the notation) is not essential.

Let us suppose that $k > \frac{1}{2}(n-1)$.

The simplexes which span the hyperplanes $\beta_1^{k-1}, \beta_{n+1}^{k-1}$ must differ by $2(n-k-1)$ vertices.

Should both simplexes differ by a greater number of vertices, i.e. by $2(n-k+h)$ vertices, where $h = 0, 1, \ldots$, then there would have $k-(n-k+h)$ common vertices. The hyperplanes $\beta_1^{k-1}$ and $\beta_{n+1}^{k-1}$ would be spanned over a simplex of $k-(n-k+h)+2(n-k+h) = n+h$ vertices. But this is impossible, since these hyperplanes do not contain points $A_1$ and $A_{n+1}$.

Should the simplexes differ by a smaller number of vertices, then the hyperplane spanned over a simplex of a number of vertices not greater than $2+k+\dfrac{2(n-k-2)}{2}$, i.e. a hyperplane of dimension $\leqslant n-1$, passing through the vertices $A_1$ and $A_{n+1}$, could contain the hyperplanes $\underset{1}{\gamma}^k$, $\underset{n+1}{\gamma}^k$. This would mean, that the hyperplane intersects the curve $C^n$ at $2(k+1)$ points. But this is possible only if $2k+2 \leqslant n$. We have obtained a contradiction, since we have supposed that $2k+1 > n$.

If $k > \frac{1}{2}(n-1)$, the simplexes which span the hyperplanes $\beta_1^{k-1}$ and $\beta_{n+1}^{k-1}$ must differ by $2(n-k-1)$ vertices. Hence they must be of form (2) (if we do not take into account the notation).

Thus we have proved that the curve $C^n$ is touching with the order $k$ to the hyperplanes $\underset{1}{\gamma}^k$, $\underset{n+1}{\gamma}^k$ (which must be given as in conditions (*)) at the points $A_1, A_{n+1}$, respectively.

THEOREM 4. *If the transformation $H$ transforms a curve $C^n$ satisfying conditions (*) into itself, then every curve $\bar{C}^n$ satisfying conditions (*) is transformed by means of the transformation $H$ into itself.*

The theorem is true for $n = 2$. Assuming it to be true in an $(n-1)$-dimensional space, we prove it to be true in the $n$-dimensional space.

Let us denote the projections of the curves $C^n, \bar{C}^n$ from the point $A_1$ $(A_{n+1})$ on the hyperplane $\alpha_1^{n-1}$ $(\alpha_{n+1}^{n-1})$ by $C_1^{n-1}$ $(C_{n+1}^{n-1})$ and $\bar{C}_1^{n-1}$ $(\bar{C}_{n+1}^{n-1})$, respectively.

Since the transformation $H$ transforms the hyperplane $\alpha_1^{n-1}$ and the curve $C^n$ into itself, it transforms also the curve $C_1^{n-1}$ into itself. Similarly,

the curve $C_{n+1}^{n-1}$ is transformed into itself by means of the transformation of the hyperplane $a_{n+1}^{n-1}$. The transformation $H$ is an Hermite transformation on the hyperplanes $a_1^{n-1}$ and $a_{n+1}^{n-1}$.

The curves $\bar{C}_1^{n-1}$, $\bar{C}_{n+1}^{n-1}$ satisfy conditions (**), and by the assumption, each of both the curves is transformed into itself. Next, arguing similarly as in Theorem 1 we may state that the curve $\bar{C}^n$ is transformed into itself by means of the transformation $H$. Thus the theorem is proved.

Let a projective transformation of the space $P^n$ into itself, defined by means of the fixed points $A_1, \ldots, A_{n+1}$, and by a pair of corresponding points $X_1, X_2$, be given. The straight line $X_1X_2$ intersects two $(n-1)$-dimensional hyperplanes $a_k^{n-1}$, $a_{k+1}^{n-1}$ at points $X_{a_k}$, $X_{a_{k+1}}$ such that the cross ratio of the four points $X_1, X_2, X_{a_k}, X_{a_{k+1}}$ — a property of a projective transformation — is a constant value, independent of the choice of the point $X_1$ [5]. This cross ratio will be denoted by $\lambda_{k,k+1}$.

In case of an Hermite collineation we have:

THEOREM 5. *A necessary and sufficient condition in order that a projective transformation be an Hermite collineation is that*

$$\lambda_{1,2} = \ldots = \lambda_{k,k+1} = \ldots = \lambda_{n,n+1}.$$

Necessity. We assume that the projective transformation is an Hermite collineation, that is the curve $C^n$ passing through the point $X_1$ and satisfying conditions (*) passes through the point $X_2$.

Let us project the points $X_1, X_2, A_{n+1}, A_1$ of the curve $C^n$ on the straight line $X_1X_2$ from $(n-2)$-dimensional hyperplanes containing the $(k-2)$-dimensional hyperplanes $\underset{1}{\gamma}^{k-2}$ (spanned over the simplexes $A_1A_2 \ldots \ldots A_{k-1}$) and the $(n-k-1)$-dimensional hyperplanes $\underset{n+1}{\gamma}^{n-k-1}$ (spanned over the simplex $A_{n+1}A_n \ldots A_{k+2}$) for $k = 1, 2, \ldots, n$.

The projections of the points $X_1, X_2$ are the same points, but the projection of the point $A_{n+1}$ is the point of intersection of the straight line $X_1X_2$ and the $(n-1)$-dimensional hyperplane spanned by the simplex $A_1 \ldots A_{k-1}A_{n+1} \ldots A_{k+1}$, for this hyperplane must be touching to the curve $C^n$ with the order $n-k$ at the point $A_{n+1}$, while the $(n-2)$-dimensional hyperplane under consideration is touching to $C^n$ with the order $n-k-1$ at the point $A_{n+1}$. Thus, the projection of the point $A_{n+1}$ is the point $X_{a_k}$. Arguing similarly we may state, the projection of the point $A_1$ is the point $X_{a_{k+1}}$. Hence the projections of the points $X_1, X_2, A_{n+1}, A_1$ from the $(n-2)$-dimensional hyperplane under consideration are the points $X_1, X_2, X_{a_k}, X_{a_{k+1}}$.

The above defined $(n-2)$-dimensional hyperplanes (for $k = 1, \ldots, n$) intersect the curve $C^n$ at the $k-1$ points coincident with the point $A_1$ and the $n-k$ points coincident with the point $A_{n+1}$, i.e. at $n-1$ points. Hence we may apply the following theorem [3]:

*The cross ratio of four $(n-1)$-dimensional hyperplanes passing through four fixed points of the curve $C^n$ and possessing a common $(n-2)$-dimensional hyperplane which intersects the curve $C^n$ at $n-1$ points, is a constant value.*

From this theorem it follows that:

$$(X_1X_2X_{a_1}X_{a_2}) = \ldots = (X_1X_2X_{a_k}X_{a_{k+1}}) = \ldots = (X_1X_2X_{a_n}X_{a_{n+1}}),$$

and the proof of necessity is finished.

Sufficiency. Let $\lambda_{1,2} = \ldots = \lambda_{k,k+1} = \ldots = \lambda_{n,n+1}$. We have to prove that the curve $C^n$ passing through $X_1$ and satisfying conditions $(*)$ passes through $X_2$, i.e. that the projective transformation is an Hermite collineation.

The theorem is true for $n = 2$. Assuming it to be true in an $(n-1)$-dimensional space we shall prove it to be true in an $n$-dimensional space.

Should the point $X_2$ not lie on the curve $C^n$, then at least one of the straight lines $A_1X_2$, $A_{n+1}X_2$ would not intersect the curve $C^n$ at points different from $A_1, A_{n+1}$, since the two-dimensional plane $A_1A_{n+1}X_2$ cannot intersect the curve $C^n$ at four points. Let us suppose, the straight line $A_{n+1}X_2$ has no other common point with the curve $C^n$ besides the point $A_{n+1}$. Projecting the curve $C^n$ and the range of points $X_1, X_2, X_{a_1}, \ldots$ $\ldots, X_{a_n}$ from the point $A_{n+1}$ on the hyperplane $a_{n+1}^{n-1}$, we obtain a curve $C_{n+1}^{n-1}$ satisfying conditions $(**)$ at points $A_1$ and $A_n$, $X_1' \in C_{n+1}^{n-1}$, $X_2' \notin C_{n+1}^{n-1}$, $X_{a_k}'$ $(k = 1, \ldots, n)$ belonging to hyperplanes spanned over simplexes $A_1 \ldots$ $\ldots A_i \ldots A_n$, where $A_i \neq A_k$, and $(X_1'X_2'X_{a_1}'X_{a_2}') = \ldots = (X_1'X_2'X_{a_{n-1}}'X_{a_n}')$.

The above condition is sufficient in order that a projective transformation defined in an $(n-1)$-dimensional space $a_{n+1}^{n-1}$ be an Hermite transformation. Hence $X_2'$ must lie on the curve $C_{n+1}^{n-1}$, if only $X_1' \in C_{n+1}^{n-1}$. Thus we obtained a contradiction.

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*Reçu par la Rédaction le 1. 4. 1968*