ANNALES
POLONICI MATHEMATICI
LII (1991)

On foliations with coregular factor mapping

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Abstract. The notion of a foliation in the category of R. Sikorski's differential spaces (see [2]) was introduced in [6]. In this paper, we give a sufficient, and for a foliation of a locally connected d.s. also a necessary, condition ensuring that the foliation has a coregular factor mapping; see [4] and [5].

0. Introduction. For any d.s. N, we denote the set of all points of N, the differential structure of N and the topology of N by $\underline{N}$, $\mathrm{F}(N)$ and $\operatorname{top}N$, respectively. For any $S \subset \underline{N}$, the d.s. induced by N on the set S will be denoted by N_S, i.e., $\mathrm{F}(N_S) = \mathrm{F}(N)_S$ (see [1]).

In [1], it has been pointed out that in the category of d.s., as in the category of differentiable manifolds, a submersion $f\colon M \to N$ (called also a *coregular mapping*) foliates M; to be more precise, the connected components of f-preimages of one point sets do. On the other hand, to any foliation F on a d.s. M we can associate the factor mapping

$$(0.1) \qquad \underline{F}.\colon M \to M/F$$

assigning to any point p of M the set $\underline{Fp}$ such that $p \in \underline{Fp}$ and $Fp \in F$. M/F denotes the d.s. coinduced (see [5]) from M on the set of all $\underline{Fp}$, $p \in \underline{M}$.

Let us recall the notion of a foliation on a d.s. The set F of d.s. is said to be *locally homogeneous* iff for any two $K, L \in F$ and points $p \in \underline{K}$ and $q \in \underline{L}$ there exist $A \in \operatorname{top}K$ and $B \in \operatorname{top}L$, $p \in A$, $q \in B$, and a diffeomorphism $h\colon K_A \to L_B$ such that $h(p) = q$. A set F of locally homogeneous pairwise disjoint d.s. with $M = \bigcup\{\underline{L};\ L \in F\}$ is a *foliation* of the d.s. M iff the following conditions are satisfied:

(a) any $L \in F$ is connected and regularly lying (see [5]) in M,

(b) for any $p \in M$ there exist a neighbourhood V of p in Fp, a d.s. H and a diffeomorphism

$$(0.2) \qquad k\colon M_U \to Fp_V \times H$$

such that for any $L \in F$, any connected (in $\operatorname{top}M$) component of $\underline{L} \cap U$ is of the form $k^{-1}(V \times \{b\})$, $b \in \underline{H}$, $p \in U \in \operatorname{top}M$.

THEOREM. *Let F be a foliation on a d.s. M. Then its factor mapping is coregular if the following two conditions are satisfied*:

(i) *for any point p of M and any open neighbourhood of p there exists a smaller open neighbourhood U of p such that*

$$(0.3)\qquad (\underline{F}.\ U)^{*-1}(\mathrm{F}(M)_U) \subset (\underline{F}.^{*-1}\mathrm{F}(M))_{\underline{F}.U},$$

(ii) *for any point p of M and any open neighbourhood of p there exists a smaller open neighbourhood U of p such that the sets of the form $U \cap \underline{L}$, where $L \in F$, are connected.*

Moreover, if M is locally connected, these two conditions are necessary.

For any mapping

$$(0.4)\qquad f\colon S \to T$$

we define the pull-back $f^*\colon \boldsymbol{R}^T \to \boldsymbol{R}^S$ from the set $\boldsymbol{R}^T$ of all $\beta\colon T \to \boldsymbol{R}$ to $\boldsymbol{R}^S$ by $f^*(\beta) = \beta \circ f$.

1. (M, F, p)**-charts.** Let F be a foliation on a d.s. M. Let us consider a diffeomorphism

$$(1.1)\qquad j\colon M_U \to Fp_V \times M_W,$$

where $p \in W \subset U \in \operatorname{top} M$ and $p \in V \in \operatorname{top} Fp$. If any connected component of $U \cap \underline{L}$ for $L \in F$ is of the form $j^{-1}(V \times \{b\})$ for some $b \in W$, and

$$(1.2)\qquad j(w) = \big(j_1(w), w\big) \quad \text{for } w \in W,$$

then j is called an (M, F, p)*-chart.*

1.1. LEMMA. *For any $p \in M$ there exists an (M, F, p)-chart.*

Proof. From the definition of a foliation of a d.s., for any $p \in M$ there exist $U \in \operatorname{top} M$, $V \in \operatorname{top} Fp$ and a diffeomorphism (0.2), where $p \in U \cap V$, H is a d.s. and any connected component of $U \cap \underline{L}$, $L \in F$, is of the form $k^{-1}(V \times \{b\})$, $b \in H$. Setting for $u \in U$

$$(1.3)\qquad k(u) = \big(k_1(u), k_2(u)\big), \quad j(u) = \big(k_1(u), k^{-1}(k_1(p), k_2(u))\big)$$

and

$$(1.4)\qquad W = \{k^{-1}(k_1(p), k_2(u));\ u \in U\},$$

we have the smooth mapping (1.1) and $W \subset U$.

First we check (1.2). Take any $w \in W$. Then, by (1.4), $w = k^{-1}(k_1(p), k_2(u))$, where $u \in U$. So, $k_1(w) = k_1(p)$ and $k_2(w) = k_2(u)$ and by (1.3), $j(w) = \big(k_1(w), k^{-1}(k_1(w), k_2(w))\big) = \big(k_1(w), w\big)$. Now, take any $v \in V$ Then, setting $x = k^{-1}(v, k_2(w))$ we get $x \in U$ and $k(x) = \big(v, k_2(w)\big)$. Thus, $k_1(x) = v$, $k_2(x) = k_2(w)$ and $j(x) = \big(v, k^{-1}(k_1(p), k_2(w))\big) = \big(v, k^{-1}(k_1(w), k_2(w))\big) = (v, w)$. Therefore, $jU = V \times W$. From (1.3) it follows also that j is one-to-one and

$j^{-1}(v, w) = k^{-1}(v, k_2(w))$ for $(v, w) \in V \times W$. This yields that (1.1) is a diffeomorphism. Similarly, it is easy to verify that for any $h \in \underline{H}$ we have $k^{-1}(V \times \{h\}) = j^{-1}(V \times \{b\})$, where $b = k^{-1}(k_1(p), h)$. This ends the proof. ∎

The proofs of the following two propositions are not very difficult.

1.2. PROPOSITION. *If j is an (M, F, p)-chart, $p \in W_1 \in \operatorname{top} M_W$, then setting $U_0 = j^{-1}(V \times W_1)$ we get an (M, F, p)-chart of the form*

$$j|U_0\colon M_{U_0} \to Fp_V \times M_{W_1}. \tag{1.5}$$

1.3. PROPOSITION. *If j is an (M, F, p)-chart, then $Fx = Fj_2(x)$, where $j(x) = (j_1(x), j_2(x))$ for $x \in U$.*

2. Remarks on coinducing of differential structures. Let us consider a mapping (0.4) with $T = fS$ and $S = \underline{M}$ where M is any d.s. We recall that (cf. [5]) $f^{*-1}\mathrm{F}(M)$ is a differential structure on T called the *structure coinduced from* $\mathrm{F}(M)$ *on the set* T. The d.s. $(T, f^{*-1}\mathrm{F}(M))$ may also be defined as the d.s. N such that $f\underline{M} = \underline{N}$ and for any d.s. P a mapping g from $\underline{N}$ into $\underline{P}$ is a smooth mapping $g\colon N \to P$ iff the mapping $g \circ f\colon M \to P$ is smooth. For any $A \subset M$ we have a mapping

$$f|A\colon A \to fA \tag{2.1}$$

and a d.s. $(fA, (f|A)^{*-1}(\mathrm{F}(M)_A))$ coinduced from M by (2.1).

2.1. PROPOSITION. *Let M be a d.s. and let $f\colon M \to T$ be any mapping. For any $A \subset \underline{M}$ the condition*

$$(f|A)^{*-1}\mathrm{F}(M)_A \subset (f^{*-1}\mathrm{F}(M))_{fA} \tag{2.2}$$

ensures that

$$(f|A)^{*-1}\mathrm{F}(M)_A = (f^{*-1}\mathrm{F}(M))_{fA}.$$

For the proof, see [3].

As an immediate corollary from 2.1 we get

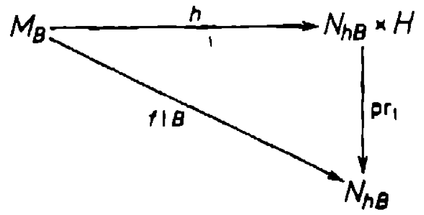
2.2. PROPOSITION. *For any d.s. M, any mapping $f\colon M \to T$ and any sets $A \subset B \subset S$ satisfying* (2.2) *and $fA = fB$ we have*

$$(f|A)^{*-1}\mathrm{F}(M)_A = (f|B)^{*-1}\mathrm{F}(M)_B. \tag{2.3}$$

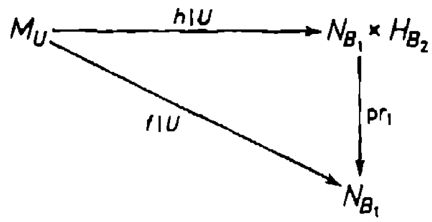
3. Proof of the theorem.

3.1. LEMMA. *If for a foliation F on a locally connected d.s. M the factor mapping is coregular, then condition* (ii) *holds.*

Proof. Let $p \in A \in \operatorname{top} M$. Then from the coregularity of the factor mapping $f\colon M \to M/F$, $f(q) = \underline{Fq}$ for $q \in \underline{M}$, there exist $B \in \operatorname{top} M$, a d.s. H and a diffeomorphism $h\colon M_B \to N_{hB} \times H$, where $p \in B \in \operatorname{top} M$, $hB \in \operatorname{top} N$, $N = M/F$ such that we have the commutative diagram



By local connectedness of M we have the locally connected d.s. M_B. Thus, both N_{hB} and H are locally connected. We have $p \in A \cap B \in \operatorname{top} M$. Therefore $h(A \cap B) \in \operatorname{top}(N_{hB} \times H) = (\operatorname{top} N|hB) \times \operatorname{top} H$. Then there exist $B_1 \in \operatorname{top} N|hB$ and $B_2 \in \operatorname{top} H$ such that $h(p) \in B_1 \times B_2 \subset h(A \cap B)$, B_1 and B_2 are connected in N_{hB} and H, respectively. Setting $U = h^{-1}(B_1 \times B_2)$ we get $p \in U \in \operatorname{top} M$, $U \subset A$ and the commutative diagram



Let $K \in \underline{M/F}$ and $U \cap K \neq \emptyset$. We then have $(f|U)^{-1}\{K\} = U \cap f^{-1}\{K\}$, and $f^{-1}\{K\} = K$. This yields $U \cap K = (f|U)^{-1}\{K\} = (\mathrm{pr}_1 \circ h|U)^{-1}\{K\} = (h|U)^{-1}\mathrm{pr}_1^{-1}\{K\} = (h|U)^{-1}(\{K\} \times B_2)$. Hence, by connectedness of B_2 in H, it follows that $U \cap K$ is connected. ∎

3.2. Lemma. *If a foliation F on a d.s. M satisfies* (i) *and* (ii), *then* (0.1) *is coregular.*

Proof. Let $p \in \underline{M}$. We have, by 1.1, an (M, F, p)-chart j. From (ii) it follows that there exists $U_1 \in \operatorname{top} M$ such that $p \in U_1 \subset U$ and $U_1 \cap \underline{L}$ is connected for $L \in F$. Let us set $W_1 = W \cap U_1$ and $U_0 = j^{-1}(V \times W_1)$. Assuming that $U_0 \cap \underline{L} \neq \emptyset$, $L \in F$, we have $u_0 \in U_0 \cap \underline{L}$ and $j(u_0) = (j_1(u_0), j_2(u_0)) \in V \times W_1$. Setting $b = j_2(u_0)$, according to 1.3, we get $L = Fu_0 = Fb$, $b \in W_1$. Let C be a connected component of the set $U \cap \underline{L}$ such that $b \in C$. Then

$$C = j^{-1}(V \times \{b\}). \tag{3.1}$$

Recall that $C \subset j^{-1}(V \times W_1) = U_0$ and $C \subset \underline{L}$. Therefore, $C \subset U_0 \cap \underline{L}$. Taking any $x \in U_0 \cap \underline{L}$ we have $j(x) = (j_1(x), j_2(x)) \in V \times W_1$. Hence, by 1.3 it follows that $L = Fx = Fj_2(x)$. This yields $j_2(x) \in W_1 \cap \underline{L} \subset U_1 \cap \underline{L}$. On the other hand, $b \in U_1 \cap \underline{L} \subset U \cap \underline{L}$ and $U_1 \cap \underline{L}$ is connected. Hence $U_1 \cap \underline{L} \subset C$. Thus, $j_2(x) = b$. This implies that $j(x) \in V \times \{b\}$, and $x \in j^{-1}(V \times \{b\}) = C$. Then $U_0 \cap \underline{L} \subset C$, so $U_0 \cap \underline{L} = C$. Therefore the set $U_0 \cap \underline{L}$ is connected in $\operatorname{top} M$.

Since $W_1 \in \operatorname{top} M_W$, $U_0 \in \operatorname{top} M$ and $p \in U_0$. Applying 1.3, we check that the images by the factor mapping of the sets U_0 and W_1 are equal, i.e.,

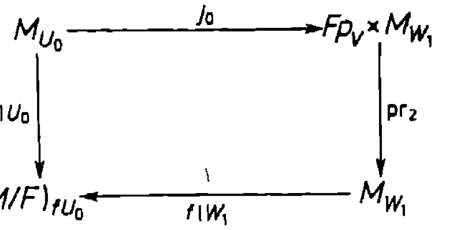
$$fU_0 = fW_1, \quad \text{where } f(q) = \underline{Fq} \text{ for } q \in M. \tag{3.2}$$

Now, we verify that the mapping

$$f|W_1\colon W_1 \to fU_0 \tag{3.3}$$

is one-to-one. Indeed, according to 1.2 we have the (M, F, p)-chart j of the form (1.5). Let $\underline{L} = f(w)$, $w \in W_1$. The set $U_0 \cap \underline{L}$ is connected in M and $w \in U_0 \cap \underline{L}$. Let C_0 be the connected component of the set $U_0 \cap \underline{L}$ such that $w \in C_0$. According to 1.2 $C_0 = j_0^{-1}(V \times \{b\})$, for some $b \in W_1$. Then we have $j(w) = (j_1(w), w) \in V \times \{b\}$. This yields $w = b$. Thus, $C_0 = j_0^{-1}(V \times \{w\})$. From the inclusion $C_0 \subset U \cap \underline{L}$ and connectedness of $U_0 \cap \underline{L}$ it follows that $C_0 = U_0 \cap \underline{L}$. Consequently, $U_0 \cap \underline{L} = j^{-1}(V \times \{w\})$, $V \times \{w\} = j(U_0 \cap \underline{L})$, and $\{w\} = \operatorname{pr}_2 j(U_0 \cap \underline{L})$. Therefore the mapping (3.3) is one-to-one.

We have the commutative diagram



According to (i), there exists a neighbourhood A of p in M such that $A \subset U_0$ and $(f|A)^{*-1}\mathrm{F}(M)_A \subset (f^{*-1}\mathrm{F}(M))_{fA}$.

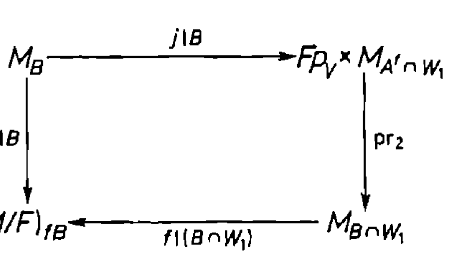
Let us set $A^f = f^{-1}fA$. It is easy to check that

$$jB = V \times (A^f \cap W_1), \quad \text{where } B = A^f \cap U_0.$$

Similarly, by a direct verification we get

$$fB = f(B \cap W_1)$$

and the commutative diagram



From the second of equalities (3.2) it follows that $fA = fB$. This, together with 2.2, yields equality (2.3). The differential structure of the space $(M/F)_{fB}$ is coinduced (see [4]) from M_B by the maping $f|B$: $B \to fB$. To prove that the mapping

$$(f|(B \cap W_1))^{-1}\colon (M/F)_{fB} \to M_{B \cap W_1}$$

is smooth it is sufficient to notice (see [5]) that there is a smooth mapping

$$(f|(B \cap W_1))^{-1} \circ f|B\colon M_B \to M_{B \cap W_1}.$$

This is so because of $(f|(B \cap W_1))^{-1} \circ f|B = \mathrm{pr}_2 \circ j|B$. Thus, the mapping

$$f|(B \cap W_1)\colon M_{B \cap W} \to (M/F)_{fB}$$

is a diffeomorphism. The superposition of coregular mappings (see [4]) is coregular. This gives the coregularity of the mapping f: $M_B \to (M/F)_{fB}$, which means that the factor mapping is coregular. ∎

Proof of the theorem. Lemma 3.2 ensures that conditions (i) and (ii) are sufficient for the coregularity of the factor mapping. It results from 3.1 that the local connectedness of M and the coregularity of the factor mapping imply condition (ii). Let us recall that for a coregular mapping f: $M \to N$ the differential structure $\mathrm{F}(N)$ is coinduced (see [5]) from M, i.e., $\mathrm{F}(N) = f^{*-1}\mathrm{F}(M)$. Then for any $U \in \mathrm{top}M$, from the coregularity of the mapping $f|U$: $M_U \to N_{fU}$ we get $(f^{*-1}\mathrm{F}(M))_{fU} = \mathrm{F}(N)_{fU} = (f|U)^{*-1}\mathrm{F}(M_U) = f^{*-1} \times (\mathrm{F}(M)_U)$. Therefore condition (i) results from the coregularity of the factor mapping. ∎

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Reçu par la Rédaction le 30.08.1988