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On the fundamental solution of the equation $\Delta^p u(X)+k(r)u(X)=0$

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In this paper we construct the fundamental solution of the equation

$$\Delta^p u(X)+k(r)u(X)=0\,, \tag{1}$$

where $X=(x_1,x_2,x_3)$, $r^2=x_1^2+x_2^2+x_3^2$, p is a positive integer greater than 1 and k is a real entire function of the variable r. In previous papers [1] and [3] we dealt with this equation when the function $k(r)$ is constant.

1. Let $X=(x_1,x_2,x_3)$ and $Y=(y_1,y_2,y_3)$ denote two different points of the Euclidean 3-space, let $r=\overline{XY}=\big(\sum_{i=1}^{3}(x_i-y_i)^2\big)^{1/2}$ denote their distance.

Definition. A function $u(r)$ is called a *fundamental solution* of equation (1) in a bounded domain D if 1° u is a function of class C^{2p} of the point (X,Y) ranging in $\overline{D}\times\overline{D}$, except at the diagonal, 2° u as the function of X as well as the function of Y satisfies in D equation (1), 3° for each function W of class C^{2p} in D and of class C^{2p-1} in $\overline{D}$, satisfying equation (1) in D one has

$$W(X)=a_3\sum_{i=0}^{p-1}\iint\limits_{\partial D}\Big[\Delta^i u(r)\frac{d\Delta^{p-i-1}W(Y)}{dn}-\Delta^i W(Y)\frac{d\Delta^{p-i-1}u(r)}{dn}\Big]dS_Y\,, \tag{2}$$

for every $X\in D$, with $a_3=-(4\pi)^{-1}$, where d/dn denotes a normal derivative in the internal direction.

Given a function U of class C^{2p} of the real variable r defined for $r>0$ we have

$$\Delta^j U(r)=U^{(2j)}(r)+2jr^{-1}U^{(2j-1)}(r)\qquad (j=0,1,2,\ldots)\,, \tag{3}$$

whence the transformation

$$U(r) = r^{-1}V(r)$$

converts $\Delta^j U(r)$ into $r^{-1}V^{(2j)}(r)$.

Therefore equation (1) takes on the form

$$V^{(2p)}(r) + k(r)V(r) = 0 \,. \tag{4}$$

Let us consider this equation in the complex domain with initial data

$$V(0) = V'(0) = \ldots = V^{(2p-3)}(0) = 0 \,,$$
$$V^{(2p-2)}(0) = 1 \,, \qquad V^{(2p-1)}(0) = c \,, \qquad c \neq 0 \,.$$

From well-known theorems about the solution of linear equations in complex domains ([2], p. 370) it easily follows that equation (4) has one and only one entire solution. It may be obtained by formal computations. Let

$$k(r) = \sum_{j=0}^{\infty} b_j r^j$$

be the development of k into the Maclaurin series. Upon setting

$$V(r) = r^{2p-2} + \sum_{j=0}^{\infty} d_{2p+j} r^j$$

we easily obtain the system of equations

$$\sum_{i=2p-2}^{j} d_i b_{j-i} + \frac{(2p+j)!}{j!} d_{2p+j} = 0 \qquad (j = 2p-2, 2p-1, \ldots)$$

which, together with conditions $d_{2p-2} = 1$, $d_{2p-1} = C_1$ determines the coefficients uniquely.

It follows that all the coefficients d_j are real, whence the function k is real.

The function $U(r) = rV(r)$ is obviously of form $r^{2p-3} + O(r^{2p-2})$.

Theorem. *The above constructed function is a fundamental solution of equation* (1).

Proof. Let us check conditions 3°, the remaing ones being obvious. Since

$$U(r) = r^{2p-3} + O(r^{2p-2}) \,,$$

we infer by (3) that

$$\frac{d}{dr}\big(\Delta^j U(r)\big) = \frac{d}{dr}\Big(U^{(2j)}(r) + \frac{2j}{r} U^{(2j-1)}(r)\Big) = O(1) \tag{5}$$

for $j = 0, 1, \dots, p-2$, whence

$$\frac{d}{dr}\big(\Delta^{p-1}U(r)\big) = \frac{d}{dr}\left(U^{(2p-2)}(r)+\frac{2p-2}{r}U^{(2p-3)}(r)\right)$$
$$= \frac{d}{dr}\left(\frac{C_2}{r}+O(r)\right) = -\frac{C_2}{r^2}+O(1)\,,$$

$C_2 \neq 0$, being a convenient constant. Let the functions u and W by of class C^{2p} in a bounded domain D, and of class C^{2p-1} in $\bar{D}$, let the boundary ∂D of D be of class C^1_σ, then (see [4], p. 18)

$$\text{(6)}\qquad \iiint_D (u\Delta^p W - W\Delta^p u)\,dY = -\sum_{i=0}^{p-1}\iint_{\partial D}\left(\Delta^i u\frac{d\Delta^{p-i-1}W}{dn} - \Delta^i W\frac{d\Delta^{p-i-1}u}{dn}\right)dS_Y\,.$$

We assume that W is a C^{2p} regular solution of (1) in D and that W is C^{2p-1} in $\bar{D}$. We substitute the solution of (1), $U(r)$, into the place of u in formula (6). Thus we can apply identity (6) to the domain $D\setminus K_R$, K_R being a ball with centre X and radius R. This yields

$$\text{(7)}\qquad \iiint_{D\setminus K_R} (U\Delta^p W - W\Delta^p U)\,dY = -\sum_{i=0}^{p-1}\iint_{\partial(D\setminus K_R)}\left(\Delta^i U\frac{d\Delta^{p-i-1}W}{dn} - \Delta^i W\frac{d\Delta^{p-i-1}U}{dn}\right)dS_Y\,.$$

The left-hand side in (7) equals zero, whence

$$\text{(8)}\qquad \sum_{i=0}^{p-1}\iint_{\partial D}\left(\Delta^i U\frac{d\Delta^{p-i-1}W}{dn} - \Delta^i W\frac{d\Delta^{p-i-1}U}{dn}\right)dS_Y = \sum_{i=0}^{p-1}\iint_{\partial K_R}\left(\Delta^i W\frac{d\Delta^{p-i-1}U}{dr} - \Delta^i U\frac{d\Delta^{p-i-1}W}{dr}\right)dS_Y\,.$$

We write the sum of integrals in the right-hand side of (8) in a brief form $\sum\limits_{i=1}^{p-1}(J_i - K_i)$. Thus we show that

$$\text{(9)}\qquad \begin{gathered}\lim_{R\to 0} J_0 = -4\pi C_2 W(X)\,,\qquad \lim_{R\to 0} J_1 = \dots = \lim_{R\to 0} J_{p-1} = 0\,,\\ \lim_{R\to 0} K_0 = \dots = \lim_{R\to 0} K_{p-1} = 0\,.\end{gathered}$$

In view of the mean value theorem we have points $Q_1, \ldots, Q_{p-1}$ all $\in \partial K_R$, such that there hold equalities

$$J_0 = 4\pi R^2\big(-C_2 R^{-2} + O(1)\big) W(Q_1) \xrightarrow[R\to 0]{} -4\pi C_2 W(X)\,,$$

$$J_i = 4\pi R^2 \Delta^i W(Q_i) O(1) \xrightarrow[R\to 0]{} 0$$

for $i = 1, \ldots, p-1$. An analogical estimation we have for the integrals K_i, which implies (9). Thus by (8) we obtain (2).

References

[1] M. Filar, *Konstrukcja rozwiązania podstawowego dla równania* $\Delta^p u(x_1, x_2, x_3) + ku(x_1, x_2, x_3) = 0$, Prace Mat. 9 (1965), pp. 207-212.

[2] E. Goursat, *Cours d'analyse mathématique*, t. 2, Paris 1926.

[3] J. Musiałek, *Construction of the fundamental solution for the equation* $\Delta^2 u(X) + ku(X) = 0$, Prace Mat. 9 (1965), pp. 213-236.

[4] M. Nicolescu, *Les fonctions polyharmoniques*, Paris 1936.

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