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APPLICATIONES MATHEMATICAE
20, 3 (1990), pp. 465–482

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STRONG SOLUTIONS OF NONLINEAR EQUATIONS OF VIBRATIONS OF SHELLS

1. Introduction. The existence and uniqueness of weak solutions of general boundary value problems of the nonlinear theory of vibrations of plates were considered in [7]. Introducing the curvature terms (see [13], pp. 19–27) into equations (5)–(8) of [7] we obtain the equations of vibrations of shells

(1.1) $$ghw''-I\Delta w''+D\Delta^2 w-k_{ij}s_{ij}-(s_{ij}w_{,i})_{,j}=f_3 \quad \text{in } G\times(0, T),$$

(1.2) $$ghu_i''-s_{ij,j}=f_i$$

(1.3) $$s_{ij}=c_{ijkl}e_{kl}$$ (the summation convention),

(1.4) $$e_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i}+w_{,i}w_{,j})-k_{ij}w,$$

where $i, j, k, l = 1, 2$. Here G is a given bounded domain of R^2, $(0, T)$ an open interval of R, and the real functions

$$w=w(x_1, x_2, t), \quad u_i=u_i(x_1, x_2, t), \quad i=1, 2, \ (x_1, x_2)\in G, \ t\in(0, T),$$

are sought for. Physically, the functions w, u_i denote the displacements in the normal and tangential directions with respect to the middle surface of the shell. The functions

$$f_m=f_m(x_1, x_2, t), \quad m=1, 2, 3,$$

are assumed to be given and represent the external forces. The symbol $(\cdot)''$ or, generally, $(\cdot)^{(p)}$, $p=1, 2, \ldots$, denotes the derivative with respect to t, $(\cdot)_{,i}$, $i=1, 2$, the derivative in the x_i-direction, and

$$\Delta(\cdot)=(\cdot)_{,ii}=(\cdot)_{,11}+(\cdot)_{,22}.$$

The constants g, h, I, D, c_{ijkl} $(i, j, k, l=1, 2)$ are given and denote the mass density, the tickness, the inertia coefficient, the flexural rigidity and material coefficients of the shell, respectively. Finally, $k_{ii}=k_{ii}(x_1, x_2, t)$ (no summation) are given functions denoting the curvatures of the shell and $k_{ij}=0$ for $i\neq j$.

The system (1.1)–(1.4) differs from that considered in [7] by the terms with derivatives of lower order and it is not hard to check that all the statements of [7] are true for (1.1)–(1.4).

In the present paper we shall show that in the case of clamped shell, i.e., if the boundary conditions are of the form

$$\begin{aligned} w = w_{,n} &= 0 \\ u = (u_1, u_2) &= 0 \end{aligned} \quad \text{on } \partial G \times (0, T) \tag{1.5}$$

(∂G is the boundary of G and $(\cdot)_{,n}$ denotes the normal derivative with respect to ∂G), the weak solutions of (1.1)–(1.5) with initial conditions

$$\begin{aligned} w(x, 0) &= w^0(x), \quad & w(x, 0) &= w^1(x) \\ u(x, 0) &= u^0(x), \quad & u(x, 0) &= u^1(x) \end{aligned} \quad \text{for } x = (x_1, x_2) \in G \tag{1.6}$$

are also the strong solutions, i.e., the solutions for which all the derivatives which appear in (1.1)–(1.2) are square integrable functions. Let us note that the only previous results regarding the regularity questions for similar initial-boundary value problems are concerned with the square integrability of derivatives of lower order (see [4], [5]). The more precise formulation of our results is given in Theorem 1 of the next section.

2. The main theorem. Let $W^{m,p}(G)$, $1 \leqslant p \leqslant \infty$, $m = 0, \pm 1, \pm 2, \ldots$, be the usual Sobolev space with the norm $\|\cdot\|_{m,p}$ (see [6], Chapitre I, Section 4) ([1]). In the case $p = 2$ we shall write $W^{m,p}(G) = H^m(G)$ and $\|\cdot\|_{m,p} = \|\cdot\|_m$. We shall use the symbol $\|\cdot\|_m$ also to denote the norm of the space $\{H^m(G)\}^2$. Let $H_0^m(G)$ be the closure of $C_0^\infty(G)$ in $H^m(G)$. The symbol $(\cdot, \cdot)$ will denote the duality between $H_0^m(G)$ and their dual $H^{-m}(G)$. In the special case $m = 0$, $(\cdot, \cdot)$ reduces to the inner product of the space $H^0(G) = L^2(G)$ of square integrable functions. For a given Banach space X the symbols $L^\infty(0, T, X)$ and $C(0, T, X)$ denote the spaces of all measurable, almost everywhere bounded and, respectively, continuous mappings $[0, T] \to X$. We shall frequently use the abbreviated notation

$$X^m = L^\infty(0, T, H^m(G)), \qquad X_0^m = L^\infty(0, T, H_0^m(G)).$$

Now we can formulate precise definitions of weak and strong solutions of our problem (1.1)–(1.6).

DEFINITION 1. A system of functions w, $u = (u_1, u_2)$ such that

$$w \in X_0^2, \quad w' \in X_0^1, \quad w'' \in X^{-2}, \quad u \in \{X_0^1\}^2, \quad u' \in \{X^0\}^2, \quad u'' \in \{X^{-1}\}^2$$

is said to be a *weak solution of the problem* (1.1)–(1.6) if for arbitrary functions

$$\phi \in H_0^2(G), \qquad \psi = (\psi_1, \psi_2) \in \{H_0^1(G)\}^2$$

([1]) $\|\phi\|_{m,p} = \left(\int_G \sum_{\alpha_1 + \alpha_2 \leqslant m} \left| \frac{\partial^{\alpha_1 + \alpha_2} \phi(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right|^p dx \right)^{1/p}$, $\|\phi\|_{m,\infty} = \sum_{\alpha_1 + \alpha_2 \leqslant m} \operatorname{ess\,sup} \left| \frac{\partial^{\alpha_1 + \alpha_2} \phi(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right|$.

and for almost every $t \in (0, T)$ the integral identities

$$(2.1)\quad (ghw''(t), \phi)+(I\nabla w''(t), \nabla\phi)+(D\Delta w(t), \Delta\phi) -(k_{ij}(t)s_{ij}(t), \phi)+(s_{ij}(t)w_{,i}(t), \phi_{,j}) = (f_3(t), \phi),$$

$$(2.2)\quad (ghu_i''(t), \psi_i)+(s_{ij}(t), \psi_{i,j}) = (f_i(t), \psi_i)$$

and the initial conditions

$$(2.3)\quad w^{(k)}(0) = w^k, \quad u^{(k)}(0) = u^k, \quad k = 0, 1,$$

are satisfied.

Remark. As is well known (see [6], Chapitre I, Section 4.3), our assumptions concerning the regularity of functions w, u imply that these functions can be identified with functions having the following properties:

$$w \in C(0, T, H^1(G)), \quad w' \in C(0, T, H^{-2}(G)),$$
$$u_i \in C(0, T, H^0(G)), \quad u_i' \in C(0, T, H^{-1}(G)), \quad i = 1, 2.$$

Thus, elements

$$w(0) \in H^1(G), \quad w'(0) \in H^{-2}(G), \quad u(0) \in \{H^0(G)\}^2, \quad u'(0) \in \{H^{-1}(G)\}^2$$

are well defined.

DEFINITION 2. A system of functions w, $u = (u_1, u_2)$ such that

$$w \in X^4, \quad w^{(m)} \in X_0^2, \quad w^{(3)} \in X_0^1,$$
$$u_i \in X^3, \quad u_i^{(m)} \in X_0^1, \quad u_i^{(3)} \in X^0, \quad i = 1, 2, \ m = 0, 1, 2,$$

is said to be a *strong solution of the problem* (1.1)–(1.6) if the functions w, u satisfy equations (1.1), (1.2) almost everywhere on $G \times (0, T)$, condition (1.6) almost everywhere on G, and condition (1.5) in the classical sense.

Using these definitions we can state our main result.

THEOREM 1. *If the conditions*

$$(2.4)\quad \partial G \in C^4, \quad k_{ii}^{(p)} \in L^\infty(G \times (0, T)), \quad k_{ii}^{(m)} \in W^{2,\infty}(G \times (0, T)),$$
$$i = 1, 2, \ p = 0, 1, 2, 3, \ m = 0, 1,$$

$$(2.5)\quad c_{ijkl} = c_{jikl} = c_{klij}, \quad c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geqslant c\varepsilon_{ij}\varepsilon_{ij}$$

for some positive constant c *and for arbitrary* ε_{ij}, ε_{kl}, $i, j, k, l = 1, 2$,

$$(2.6)\quad \begin{gathered} w^m \in H^{4-m}(G) \cap H_0^2(G), \quad u^m \in H^{3-m}(G) \cap H_0^1(G), \quad m = 0, 1, \\ -\tfrac{1}{2}c_{ijkl}(u_{l,k}^0 + u_{k,l}^0)_{,j} \in H_0^1(G), \quad i = 1, 2, \end{gathered}$$

$$(2.7)\quad f_r^{(p)} \in X^0, \ p = 0, 1, 2, \ r = 1, 2, 3, \quad f_i^{(m)} \in X^1, \ i = 1, 2, \ m = 0, 1,$$

are satisfied, then there exists a weak solution of the problem (1.1)–(1.6) *which is also a strong one.*

The proof of this theorem is realized in several steps.

3. Approximate solutions. Let us consider the following two eigenvalue problems:

$$\Delta^2 w = \lambda w \text{ in } G, \quad w = w_{,n} = 0 \text{ on } \partial G, \tag{3.1}$$

$$Tu = \mu u \text{ in } G, \quad u = 0 \text{ on } \partial G, \tag{3.2}$$

where

$$\begin{gathered} Tu = \big((Tu)_1, (Tu)_2\big), \quad (Tu)_i = -\bar{s}_{ij,j}, \; i = 1, 2, \\ \bar{s}_{ij} = c_{ijkl}\bar{e}_{kl}, \quad \bar{e}_{kl} = \tfrac{1}{2}(u_{k,l}+u_{l,k}). \end{gathered} \tag{3.3}$$

Repeating the argument of Section 17, Chapter III of [10] (see also Chapter II of [9]) we can prove that the problems (3.1), (3.2) have sequences of eigenvalues $\{\lambda_r\}_{r=1}^{\infty}$, $\{\mu_r\}_{r=1}^{\infty}$ such that the corresponding eigenfunctions $\{z_r\}_{r=1}^{\infty}$, $\{v_r\}_{r=1}^{\infty}$ form bases of

$$H^4(G)\cap H_0^2(G) \quad \text{and} \quad \{u\in\{H^3(G)\cap H_0^1(G)\}^2\colon\ Tu\in\{H_0^1(G)\}^2\},$$

respectively.

Let us define the Galerkin approximate solutions of (1.1)–(1.6) as functions of the form

$$\begin{gathered} w_n(x, t) = \sum_{r=1}^{n} g_{rn}(t)z_r(x), \quad t\in[0, T],\ x\in G, \\ \qquad\qquad\qquad\qquad n = 1, 2, \ldots, \\ u_n(x, t) = \sum_{r=1}^{n} h_{rn}(t)v_r(x), \quad u_n = (u_{n1}, u_{n2}),\ v_r = (v_{r1}, v_{r2}), \end{gathered} \tag{3.4}$$

satisfying the following system of ordinary differential equations:

$$\begin{gathered} (ghw_n'', z_r)+(I\nabla w_n'', \nabla z_r)+(D\Delta w_n, \Delta z_r)-(k_{ij}s_{nij}, z_r) \\ +(s_{nij}w_{n,i}, z_{r,j}) = (f_3, z_r), \quad r = 1, 2, \ldots, n, \\ (ghu_{ni}'', v_{ri})+(s_{nij}, v_{ri,j}) = (f_i, v_{ri}), \quad i = 1, 2, \end{gathered} \tag{3.5}$$

where

$$s_{nij} = c_{ijkl}\cdot\tfrac{1}{2}(u_{ni,j}+u_{nj,i}+w_{n,i}w_{n,j})-k_{ij}w_n, \quad i, j = 1, 2.$$

The system (3.5) is supplemented by the initial conditions

$$\begin{gathered} w_n^{(m)}(0) = w_n^m \to w^m \text{ as } n\to\infty \quad \text{in } H^{4-m}(G)\cap H_0^2(G), \\ \qquad\qquad\qquad\qquad m = 0, 1, \\ u_n^{(m)}(0) = u_n^m \to u^m \text{ as } n\to\infty \quad \text{in } H^{3-m}(G)\cap H_0^1(G), \end{gathered} \tag{3.6}$$

where w_n^m, u_n^m are projections of w^m, u^m on the spaces spanned by $\{z_1, \ldots, z_n\}$ and $\{v_1, \ldots, v_n\}$, respectively.

By the theory of ordinary differential equations there exists a solution of the problem (3.5), (3.6) defined on some interval $[0, T_n] \subset [0, T]$. The estimate (4.1) of the next section implies the existence of a solution defined on $[0, T]$.

4. A priori estimate. In [7] the following estimates were proved:

$$(4.1)\qquad \begin{matrix} \|w_n^{(m)}(t)\|_2,\ \|w_n^{(2)}(t)\|_1 \leqslant \text{Const}, \\ \|u_n^{(m)}(t)\|_1,\ \|u_n^{(2)}(t)\|_0 \leqslant \text{Const}, \end{matrix} \qquad t\in[0, T],\ m = 0, 1,$$

under the additional assumption that $k_{ij} = 0$, $i, j = 1, 2$ (see estimates (81) of [7]). The modifications needed in the case of arbitrary k_{ij} are not essential and will be omitted. Instead, we shall give a detailed proof of stronger estimates

$$(4.2)\qquad \|w_n^{(p)}(t)\|_{4-p},\ \|u_n^{(p)}(t)\|_{3-p} \leqslant \text{Const}, \quad p = 2, 3,\ t\in[0, T],$$

which play a fundamental role in our considerations.

The proof of (4.2) is based on the a priori equality of the form

$$(4.3)\qquad E(t) = E(0)+F(t)+G(t)+H(t), \quad t\in[0, T],$$

where (we drop the fixed index n in the symbols w_n, u_n)

$$E(t) = E_1(t)+E_2(t),$$

$$(4.4)\qquad E_1(t) = gh\big(\|w'''(t)\|_0^2+\|u'''(t)\|_0^2\big)+I\,\|\nabla w'''(t)\|_0^2+D\,\|\Delta w''(t)\|_0^2,$$

$$(4.5)\qquad E_2(t) = \big(s_{ij}''(t),\ e_{ij}''(t)\big),$$

$$(4.6)\qquad F(t) = -\big(s_{ij}(s),\ w_{,i}''(s)w_{,j}''(s)\big)\big|_0^t-4\big(s_{ij}'(s),\ w_{,i}'(s)w_{,j}''(s)\big|_0^t$$
$$+10\int_0^t\big(s_{ij}''(s),\ w_{,i}'(s)w_{,j}''(s)\big)ds+5\int_0^t\big(s_{ij}'(s),\ w_{,i}''(s)w_{,j}''(s)\big)ds,$$

$$(4.7)\qquad G(t) = 2\int_0^t\Big\{\big(k_{ij}''(s)s_{ij}(s),\ w'''(s)\big)+2\big(k_{ij}'(s)s_{ij}'(s),\ w'''(s)\big)$$
$$-\big(k_{ij}'''(s)s_{ij}''(s),\ w(s)\big)-3\big(k_{ij}''(s)s_{ij}''(s),\ w'(s)\big)$$
$$-3\big(k_{ij}'(s)s_{ij}''(s),\ w''(s)\big)\Big\}ds,$$

$$(4.8)\qquad H(t) = 2\int_0^t\Big\{\big(f_3''(s),\ w'''(s)\big)+\big(f_i''(s),\ u_i'''(s)\big)\Big\}ds.$$

Remark. Assumptions (2.7) imply that the system (3.5), (3.6) admits a local solution with absolutely continuous third derivative with respect to t and with fourth derivative existing almost everywhere.

5. The proof of a priori equality (4.3). To check that equality (4.3) holds true, let us differentiate equalities (3.5) twice with respect to t, multiply them by $g_{rn}'''(t)$ and $h_{rn}'''(t)$, respectively, and take the sums over $r = 1, \ldots, n$. We obtain

$$(5.1)\quad (ghw^{(4)}, w^{(3)})+(I\nabla w^{(4)}, \nabla w^{(3)})+(D\Delta w^{(2)}, \Delta w^{(3)})$$

$$-\big((k_{ij}s_{ij})^{(2)}, w^{(3)}\big)+\big((s_{ij}w_{,i})^{(2)}, w_{,j}^{(3)}\big) = (f_3^{(2)}, w^{(3)}),$$

$$(5.2)\quad (ghu_i^{(4)}, u_i^{(3)})+(s_{ij}^{(2)}, u_{i,j}^{(3)}) = (f_i^{(2)}, u_i^{(3)}).$$

Adding these equalities side by side, integrating over $[0, t]$ and using the elementary formulas

$$\int_0^t \{gh[(w^{(4)}, w^{(3)})+(u^{(4)}, u^{(3)})]+(I\nabla w^{(4)}, \nabla w^{(3)})+D(\Delta w^{(2)}, \Delta w^{(3)})\}\,dt$$

$$= \tfrac{1}{2}\big(E_1(t)-E_1(0)\big),$$

$$\begin{aligned} -\big((k_{ij}s_{ij})'', w'''\big) &= -\big((k_{ij}''s_{ij}+2k_{ij}'s_{ij}'+k_{ij}s_{ij}''), w'''\big)\\ &= -(s_{ij}'', k_{ij}'''w+3k_{ij}''w'+3k_{ij}'w''+k_{ij}w''')-(k_{ij}''s_{ij}, w''')\\ &\quad -2(k_{ij}'s_{ij}', w''')+(k_{ij}'''s_{ij}'', w)+3(k_{ij}''s_{ij}'', w')+3(k_{ij}'s_{ij}'', w'')\\ &= -\big(s_{ij}'', (k_{ij}w)'''\big)-(k_{ij}''s_{ij}, w''')-2(k_{ij}'s_{ij}', w''')+(k_{ij}'''s_{ij}'', w)\\ &\quad +3(k_{ij}''s_{ij}'', w')+3(k_{ij}'s_{ij}'', w''), \end{aligned}$$

i.e.,

$$-\int_0^t \big((k_{ij}s_{ij})'', w'''\big)ds = -\int_0^t \big(s_{ij}'', (k_{ij}w)'''\big)ds-\tfrac{1}{2}G(t)$$

and

$$\begin{aligned} &(s_{ij}'', u_{i,j}''')-\big(s_{ij}'', (k_{ij}w)'''\big)+\big((s_{ij}w_{,i})'', w_{,j}'''\big)\\ &= (s_{ij}'', u_{i,j}''')-\big(s_{ij}'', (k_{ij}w)'''\big)+(s_{ij}'', w_{,i}w_{,j}''')+2(s_{ij}', w_{,i}'w_{,j}''')+(s_{ij}, w_{,i}''w_{,j}''')\\ &= (s_{ij}'', u_{i,j}''')-\big(s_{ij}'', (k_{ij}w)'''\big)+\tfrac{1}{2}(s_{ij}'', w_{,i}'''w_{,j}+3w_{,i}''w_{,j}'+3w_{,i}'w_{,j}''+w_{,i}w_{,j}''')\\ &\quad -3(s_{ij}'', w_{,i}'w_{,j}'')+2(s_{ij}', w_{,i}'w_{,j}''')+(s_{ij}, w_{,i}''w_{,j}''')\\ &= (s_{ij}'', e_{ij}''')+\tfrac{1}{2}\big((s_{ij}', w_{,i}''w_{,j}'')+2(s_{ij}, w_{,i}''w_{,j}''')\big)\\ &\quad -\tfrac{1}{2}(s_{ij}', w_{,i}''w_{,j}'')+2(s_{ij}', w_{,i}'w_{,j}''')-3(s_{ij}'', w_{,i}'w_{,j}'')\\ &= (s_{ij}'', e_{ij}''')+\tfrac{1}{2}(s_{ij}, w_{,i}''w_{,j}'')'+2[(s_{ij}', w_{,i}'w_{,j}''')+(s_{ij}', w_{,i}''w_{,j}'')+(s_{ij}'', w_{,i}'w_{,j}'')]\\ &\quad -\tfrac{5}{2}(s_{ij}', w_{,i}''w_{,j}'')-5(s_{ij}'', w_{,i}'w_{,j}'')\\ &= \tfrac{1}{2}(s_{ij}'', e_{ij}'')'+\tfrac{1}{2}(s_{ij}, w_{,i}''w_{,j}'')'+2(s_{ij}', w_{,i}'w_{,j}'')'-5(s_{ij}'', w_{,i}'w_{,j}'')-\tfrac{5}{2}(s_{ij}', w_{,i}''w_{,j}''), \end{aligned}$$

which implies

$$\int_0^t \big[(s_{ij}'', u_{i,j}''')-\big(s_{ij}'', (k_{ij}w)'''\big)+\big((s_{ij}w_{,i})'', w_{,j}'''\big)\big]\,dt = \tfrac{1}{2}(s_{ij}'', e_{ij}'')\big|_0^t-\tfrac{1}{2}F(t),$$

we arrive at equality (4.3).

6. Estimates for $E(0)$. An obvious modification of calculations leading to formulas (5.1), (5.2) gives

$$(6.1)\quad (ghw'''(0), w'''(0))+(I\nabla w'''(0), \nabla w'''(0))-(D\nabla\Delta w'(0), \nabla w'''(0)) -((k_{ij}s_{ij})'(0), w'''(0))+((s_{ij}w_{,i})'(0), w_{,j}'''(0)) = (f_3'(0), w'''(0)),$$

$$(6.2)\quad (ghu_i'''(0), u_i'''(0))+(s_{ij}'(0), u_{i,j}'''(0)) = (f_i'(0), u_i'''(0)).$$

Using (6.1) and the equivalence of the norms $gh\|\cdot\|_0+I\|\nabla(\cdot)\|_0$ and $\|\cdot\|_1$ of the space $H^1(G)$, we obtain

$$(6.3)\quad c_1\|w'''(0)\|_1^2 \leqslant gh\|w'''(0)\|_0^2+I\|\nabla w'''(0)\|_0^2 = D(\nabla\Delta w'(0), \nabla w'''(0)) +((k_{ij}s_{ij})'(0), w'''(0))-((s_{ij}w_{,i})'(0), w_{,j}'''(0))+(f_3'(0), w'''(0)).$$

Here and in the sequel c_k, $k = 1, 2, \ldots$, denote positive constants not depending on w_n, u_n. Relation (3.6) enables us to write

$$(6.4)\quad \|\nabla\Delta w'(0)\|_0 \leqslant c_2\|w'(0)\|_3 \leqslant \text{Const}.$$

Now, let us note that from the continuity of the imbedding $H^1(G) \subset W^{0,4}(G)$ and from (4.1) it follows that

$$\|w_{,i}(s)w_{,j}^{(m)}(s)\|_0 \leqslant c_2\|w_{,i}(s)\|_{0,4}\|w_{,j}^{(m)}(s)\|_{0,4}$$
$$\leqslant c_3\|w_{,i}(s)\|_1\|w_{,j}^{(m)}(s)\|_1 \leqslant c_4\|w(s)\|_2\|w^{(m)}(s)\|_2 \leqslant \text{Const}, \quad s\in[0, T], m = 0, 1,$$

and, consequently,

$$(6.5)\quad \|s_{ij}^{(m)}(s)\|_0 \leqslant c_5\Big[\|u^{(m)}(s)\|_1+\sum_{p=0}^{m}\big(\|w^{(p)}(s)\|_2\|w^{(m-p)}(s)\|_2 +\|k_{ij}^{(p)}(s)\|_{0,\infty}\|w^{(m-p)}(s)\|_0\big)\Big] \leqslant \text{Const}, \quad m = 0, 1,\ s\in[0, T].$$

Inequality (6.5) and the continuity of the imbedding $H^2(G) \subset L^\infty(G)$ allow us to write

$$\|(k_{ij}s_{ij})^{(m)}(s)\|_0 \leqslant \sum_{p=0}^{m}\|k_{ij}^{(p)}(s)\|_{0,\infty}\|s_{ij}^{(m-p)}(s)\|_0 \leqslant \text{Const},$$

$$(6.6)\quad \|(s_{ij}w_{,i})^{(m)}(s)\|_0 \leqslant \sum_{p=0}^{m}\|s_{ij}^{(p)}(s)\|_0\|w_{,i}^{(m-p)}(s)\|_{0,\infty} \leqslant c_6\sum_{p=0}^{m}\|w_{,i}^{(m-p)}(s)\|_2 \leqslant c_7\sum_{p=0}^{m}\|w^{(m-p)}(s)\|_3, \quad m = 0, 1,\ s\in[0, T].$$

Let us remark that due to (3.6) we have $\|w^{(m-p)}(0)\|_3 \leqslant \text{Const}$, and therefore the expressions $\|(k_{ij}s_{ij})'(0)\|_0$ and $\|(s_{ij}w_{,i})'(0)\|_0$ are bounded by a constant. We have also $\|f_3'(0)\|_0 \leqslant$ Const. In consequence, we get the inequality

$$c_1\|w'''(0)\|_1^2 \leqslant c_8\|w'''(0)\|_1,$$

i.e.,

$$\|w'''(0)\|_1 \leqslant \text{Const.} \tag{6.7}$$

Similar considerations concerning equality (6.2) give

$$gh\|u'''(0)\|_0^2 = \big(s'_{ij,j}(0)+f_i'(0),\ u_i'''(0)\big) \leqslant \|s'_{ij,j}(0)+f_i'(0)\|_0\,\|u_i'''(0)\|_0.$$

Using the continuity of the imbedding $H^3(G) \subset W^{1,\infty}(G)$ we obtain, for $m = 0, 1$,

$$\begin{aligned}
&\|(w_{,i}w_{,j})^{(m)}_{,j}(0)\|_0\\
&\leqslant \sum_{p=0}^{m}\big(\|(w^{(p)}_{,ij}w^{(m-p)}_{,j})(0)\|_0+\|(w^{(p)}_{,i}w^{(m-p)}_{,jj})(0)\|_0\big)\\
&\leqslant c_9\sum_{p=0}^{m}\big(\|w^{(p)}_{,ij}(0)\|_0\,\|w^{(m-p)}_{,j}(0)\|_{0,\infty}+\|w^{(p)}_{,i}(0)\|_{0,\infty}\,\|w^{(m-p)}_{,jj}(0)\|_0\big)\\
&\leqslant c_{10}\big(\|w^{(m)}(0)\|_2\,\|w(0)\|_{1,\infty}+\|w(0)\|_2\,\|w^{(m)}(0)\|_{1,\infty}\big)\\
&\leqslant c_{11}\big(\|w^{(m)}(0)\|_2\,\|w(0)\|_3+\|w(0)\|_2\,\|w^{(m)}(0)\|_3\big) \leqslant \text{Const},
\end{aligned}$$

(6.8)

$$\begin{aligned}
\|(k_{ij}w)^{(m)}_{,j}(0)\|_0 &\leqslant \sum_{p=0}^{m}\big(\|(k^{(p)}_{ij,j}w^{(m-p)})(0)\|_0+\|(k^{(p)}_{ij}w^{(m-p)}_{,j})(0)\|_0\big)\\
&\leqslant \sum_{p=0}^{m}\big(\|k^{(p)}_{ij,j}(0)\|_{0,\infty}\,\|w^{(m-p)}(0)\|_0+\|k^{(p)}_{ij}(0)\|_{0,\infty}\,\|w^{(m-p)}_{,j}(0)\|_0\big)\\
&\leqslant c_{12}\big(\|w(0)\|_1+\|w^{(m)}(0)\|_1\big) \leqslant \text{Const}
\end{aligned}$$

and, consequently,

$$\begin{aligned}
\|s^{(m)}_{ij,j}(0)\|_0 \leqslant c_{13}\big(\|u^{(m)}(0)\|_2&+\|(w_{,i}w_{,j})^{(m)}_{,j}(0)\|_0\\
&+\|(k_{ij}w)^{(m)}_{,j}(0)\|_0\big) \leqslant \text{Const}, \qquad m = 0, 1.
\end{aligned} \tag{6.9}$$

If we take into account that $\|f_i'(0)\|_0 \leqslant$ Const, $i = 1, 2$, we get

$$gh\|u'''(0)\|_0^2 \leqslant c_{14}\|u'''(0)\|_0,$$

i.e.,

$$\|u'''(0)\|_0 \leqslant \text{Const.} \tag{6.10}$$

It remains to obtain the estimates for $\|w''(0)\|_2$ and $\|u''(0)\|_1$. To this purpose let us multiply the r-th equation of (3.5) by $\lambda_r^{1/2}g''_{rn}(0)$ and $\mu_r h''_{rn}(0)$, respectively, use (3.1), (3.2) and sum over $r = 1, \ldots, n$. We arrive at the equalities

$$(6.11)\qquad \big(ghw''(0)-I\Delta w''(0)+D\Delta^2 w(0)-(k_{ij}s_{ij})(0)-(s_{ij}w_{,i})_{,j}(0),\ (\Delta^2)^{1/2}w''(0)\big) = \big(f_3(0),\ (\Delta^2)^{1/2}w''(0)\big),$$

$$(6.12)\qquad \big(ghu_i''(0)-s_{ij,j}(0),\ \bar{s}_{ij,j}''(0)\big) = \big(f_i(0),\ \bar{s}_{ij,j}''(0)\big),\qquad i=1,2$$

(see formula (3.3)). Here $(\Delta^2)^{1/2}$ is the square root of the self-adjoint positive definite operator Δ^2 considered as the unbounded operator in $L^2(G)$ with the domain of definition

$$D(\Delta^2) = H^4(G)\cap H_0^2(G).$$

Relation (6.11) and inequality (A.1) (see Appendix) give

$$(6.13)\qquad c_{15}\|w''(0)\|_2^2 \leqslant I\big(\Delta w''(0),\ (\Delta^2)^{1/2}w''(0)\big)$$
$$= \big(ghw''(0)+D\Delta^2 w(0)-k_{ij}(0)s_{ij}(0)-(s_{ij}w_{,i})_{,j}(0)-f_3(0),\ (\Delta^2)^{1/2}w''(0)\big).$$

Let us note that from (4.1) it follows that

$$\|ghw''(0)\|_0 \leqslant c_{16}\|w''(0)\|_1 \leqslant \mathrm{Const},$$

and (3.6) gives

$$\|D\Delta^2 w(0)-f_3(0)\|_0 \leqslant c_{17}\big(\|w(0)\|_4+\|f_3(0)\|_0\big) \leqslant \mathrm{Const}.$$

Furthermore, from (6.6) we get

$$\|(k_{ij}s_{ij})(0)\|_0 \leqslant \mathrm{Const}$$

and from (6.9) and the continuity of the imbeddings

$$H^4(G) \subset W^{2,\infty}(G) \subset W^{1,\infty}(G)$$

we have

$$\|(s_{ij}w_{,i})_{,j}(0)\|_0 \leqslant \|s_{ij,j}(0)w_{,i}(0)\|_0+\|s_{ij}(0)w_{,ij}(0)\|_0$$
$$\leqslant \|s_{ij,j}(0)\|_0\|w_{,i}(0)\|_{0,\infty}+\|s_{ij}(0)\|_0\|w_{,ij}(0)\|_{0,\infty}$$
$$\leqslant c_{18}\big(\|w(0)\|_{1,\infty}+\|w(0)\|_{2,\infty}\big) \leqslant c_{19}\|w(0)\|_4 \leqslant \mathrm{Const}.$$

Thus relations (6.13) and (A.3) imply

$$\|w''(0)\|_2^2 \leqslant c_{20}\|w''(0)\|_2,$$

i.e.,

$$(6.14)\qquad \|w''(0)\|_2 \leqslant \mathrm{Const}.$$

Relations (6.7), (6.10), (6.14) give $E_1(0) \leqslant \mathrm{Const}$.

Now, using relation (6.12), Korn's inequality

$$(6.15)\qquad c_{21}\|u\|_1^2 \leqslant (\bar{e}_{ij},\ \bar{e}_{ij}) \leqslant c_{22}(\bar{s}_{ij},\ \bar{e}_{ij})$$

(see, e.g., [11], p. 148) and integrating by parts we obtain

(6.16) $$c_{23}\|u''(0)\|_1^2 \leqslant gh(\bar{e}''_{ij}(0), \bar{s}''_{ij}(0)) = gh(u''_{i,j}(0), \bar{s}''_{ij}(0))$$

$$= -(ghu''_i(0), \bar{s}''_{ij,j}(0)) = -(s_{ik,k}(0)+f_i(0), \bar{s}''_{ij,j}(0)) = (s_{ik,kj}(0)+f_{i,j}(0), \bar{s}''_{ij}(0)).$$

The estimate $\|s_{ik,kj}(0)\|_0 \leqslant \text{Const}$ is a consequence of the inequalities

$$\|(u_{i,k}+u_{k,i})_{,kj}(0)\|_0 \leqslant c_{24}\|u(0)\|_3 \leqslant \text{Const},$$

$$\|(w_{,i}w_{,k})_{,kj}(0)\|_0 \leqslant c_{25}(\|w(0)\|_{1,\infty}\|w(0)\|_3+\|w(0)\|_{2,4}\|w(0)\|_{2,4})$$
$$\leqslant c_{26}\|w(0)\|_3^2 \leqslant \text{Const},$$

$$\|(k_{ik}w)_{,kj}(0)\|_0 \leqslant \|k_{ik}(0)\|_{0,\infty}\|w_{,kj}(0)\|_0+\|w_{,k}(0)\|_0\|k_{ik,j}(0)\|_{0,\infty}$$
$$+(\|k_{ik,k}(0)\|_{0,\infty}+\|k_{ik,kj}(0)\|_{0,\infty})\|w(0)\|_1 \leqslant c_{27}\|w(0)\|_2 \leqslant \text{Const},$$

which follow from the continuity of the imbeddings

$$H^3(G) \subset W^{1,\infty}(G), \quad H^3(G) \subset W^{2,4}(G), \quad H^2(G) \subset H^1(G),$$

relation (3.6) and assumptions (2.4). If we take into account that, by assumption, $\|f_{i,j}(0)\|_0 \leqslant \text{Const}$ and that

$$\|\bar{s}''_{ij}(0)\|_0 \leqslant c_{28}\|u''(0)\|_1$$

(see (3.3)), from (6.16) we get

$$c_{23}\|u''(0)\|_1^2 \leqslant c_{29}\|u''(0)\|_1,$$

i.e.,

(6.17) $$\|u''(0)\|_1 \leqslant \text{Const}.$$

The last inequality and (6.14) imply that $E_2(0) \leqslant \text{Const}$. In fact, we have

$$(s''_{ij}(0), e''_{ij}(0)) \leqslant c_{30}(e''_{ij}(0), e''_{ij}(0))$$
$$\leqslant c_{31}\left(\|u''(0)\|_1^2+\sum_{p=0}^{2}\sum_{i,j=1}^{2}\binom{2}{p}\|w_{,i}^{(p)}(0)\|_{0,4}^2\|w_{,j}^{(2-p)}(0)\|_{0,4}^2+\|w''(0)\|_0^2\right)$$
$$\leqslant c_{32}\left(c_{33}+\sum_{p=0}^{2}\|w^{(p)}(0)\|_2^2\|w^{(2-p)}(0)\|_2^2+c_{34}\right) \leqslant \text{Const},$$

where the first inequality is a consequence of (2.5). Thus relations (6.7), (6.10), (6.14), (6.17) imply

(6.18) $$E(0) \leqslant \text{Const}.$$

7. Estimates for $F(t)$, $G(t)$, $H(t)$, $t \in [0, T]$. We begin with the estimate for $F(t)$. To this purpose let us note that from Hölder's inequality we obtain

(7.1) $$(s_{ij}^{(m)}(s), w'_{,i}(s)w'_{,j}(s)) \leqslant \|s_{ij}^{(m)}(s)\|_0\|w'_{,i}(s)\|_{0,4}\|w'_{,j}(s)\|_{0,4}$$

for $m = 0, 1, s\in[0, T]$. The first term on the right-hand side was estimated by a constant in formula (6.5). To estimate the remaining ones we use the inequality

$$\|v\|_{0,4}^2 \leqslant c_{35}\|v\|_1\|v\|_0 \quad \text{for } v\in H^1(G) \tag{7.2}$$

(see Remarque 3.1, Chapitre 6 of [6]) and the elementary inequality

$$ab \leqslant \frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2 \quad \text{for } a, b\in R,\ \varepsilon>0.$$

In consequence we obtain

$$\begin{aligned}(s_{ij}^{(m)}(s), w''_{,i}(s)w''_{,j}(s)) &\leqslant c_{36}\|w''(s)\|_1\|w''(s)\|_2 \\ &\leqslant \frac{c_{36}\varepsilon_m}{2}\|w''(s)\|_2^2+\frac{c_{36}}{2\varepsilon_m}\|w''(s)\|_1^2 \\ &\leqslant \frac{c_{36}\varepsilon_m}{2}\|w''(s)\|_2^2+c_{37}, \qquad m=0,1,\ s\in[0,T],\end{aligned} \tag{7.3}$$

where ε_0 is an arbitrary positive number and $\varepsilon_1 = 1$.

Using the Sobolev imbedding theorems instead of (7.2) we get, for $s\in[0, T]$,

$$\begin{aligned}(s'_{ij}(s), w'_{,i}(s)_{,i}w''_{,j}(s)) &\leqslant \|s'_{ij}(s)\|_0\|w'_{,i}(s)\|_{0,4}\|w''_{,j}(s)\|_{0,4} \\ &\leqslant c_{38}\|w'(s)\|_2\|w''(s)\|_2 \leqslant \frac{c_{38}\varepsilon_0}{2}\|w''(s)\|_2^2+\frac{c_{39}}{2\varepsilon_0}\end{aligned} \tag{7.4}$$

and

$$\begin{aligned}(s''_{ij}(s), w'_{,i}(s)w''_{,j}(s)) &\leqslant \|s''_{ij}(s)\|_0\|w'_{,i}(s)\|_{0,4}\|w''_{,j}(s)\|_{0,4} \\ &\leqslant c_{40}\|s''_{ij}(s)\|_0\|w''_{,j}(s)\|_{0,4} \leqslant c_{41}\big((s''_{ij}(s), s''_{ij}(s))+\|w''(s)\|_2^2\big).\end{aligned}$$

To complete the estimates for $F(t)$ it is sufficient to note that (2.5) implies

$$(s''_{ij}, s''_{ij}) \leqslant c_{42}(e''_{ij}, e''_{ij}) \leqslant c_{43}(s''_{ij}, e''_{ij}), \tag{7.5}$$

and therefore we have

$$\begin{aligned}F(t) &\leqslant \frac{c_{44}\varepsilon_0}{2}\|w''(t)\|_2^2+c_{45} \\ &\quad +c_{46}\int_0^t\big(\|w''(s)\|_2^2+(s''_{ij}(s), e''_{ij}(s))\big)ds, \qquad t\in[0,T].\end{aligned} \tag{7.6}$$

The estimate for $G(t)$ follows from the inequalities

$$\begin{aligned}(k_{ij}^{(m)}s_{ij}^{(p)}, w^{(q)}) &\leqslant \|k_{ij}^{(m)}\|_{0,\infty}\|s_{ij}^{(p)}\|_0\|w^{(q)}\|_0 \leqslant c_{47}\|s_{ij}^{(p)}\|_0\|w^{(q)}\|_0 \\ &\leqslant c_{48}(\|s_{ij}^{(p)}\|_0^2+\|w^{(q)}\|_0^2) \leqslant c_{49}\big(c_{50}+(s''_{ij}, e''_{ij})+\|w'''\|_1^2\big)\end{aligned}$$

for $m = 1, 2, 3$, $p = 0, 1, 2$, $q = 0, 1, 2, 3$, in which the estimates

$$\|s_{ij}^{(r)}\|_0 \leqslant \text{Const}, \qquad r = 0, 1$$

(see (6.5)),

$$\|w^{(q)}\|_0 \leqslant c_{51} \|w^{(q)}\|_1, \qquad q = 0, 1, 2, 3,$$

and

$$\|w^{(p)}\|_1 \leqslant \text{Const}, \qquad p = 0, 1, 2$$

(see (4.1) and (7.5)) were used. In consequence we have, for $t \in [0, T]$,

$$G(t) \leqslant c_{52} + c_{53} \int_0^t \left(\|w'''(s)\|_1^2 + \left(s_{ij}''(s), e_{ij}''(s)\right)\right) ds. \tag{7.7}$$

Finally, assumptions (2.7) allow us to write the estimate for the expression $H(t)$:

$$\begin{aligned} H(t) &\leqslant \int_0^t \left(\|f_3''(s)\|_{-1}^2 + \|w'''(s)\|_1^2 + \|f_1''(s)\|_0^2 + \|f_2''(s)\|_0^2 + \|u'''(s)\|_0^2\right) ds \\ &\leqslant c_{54} + c_{55} \int_0^t \left(\|w'''(s)\|_1^2 + \|u'''(s)\|_0^2\right) ds, \qquad t \in [0, T]. \end{aligned} \tag{7.8}$$

8. Completion of the proof of estimates (4.2). Taking into account the a priori equality (4.3) and inequalities (6.18), (7.6)–(7.8), for sufficiently small ε_0 we obtain

$$E(t) \leqslant c_{56} + c_{57} \int_0^t E(s)\, ds, \qquad t \in [0, T].$$

Gronwall's lemma gives

$$E(t) \leqslant \text{Const}, \qquad t \in [0, T], \tag{8.1}$$

and, in consequence,

$$\left(e_{ij}''(t), e_{ij}''(t)\right) \leqslant c_{58} E_1(t) \leqslant \text{Const}, \qquad t \in [0, T].$$

Since the terms $\|w^{(p)}(t)\|_2$, $p = 0, 1, 2$, $t \in [0, T]$, are estimated by a constant (see (4.1) and (8.1)), we have

$$\|\bar{e}_{ij}''(t)\|_0 \leqslant \text{Const}, \qquad t \in [0, T].$$

By Korn's inequality (6.15) we have also

$$\|u''(t)\|_1 \leqslant \text{Const}, \qquad t \in [0, T].$$

Thus estimates (4.2) are proved.

Let us recall that, in a general schema of Galerkin method of proving the existence of weak solutions, the last ones are obtained as the weak limits (in appropriate spaces) of subsequences of Galerkin approximate solutions which in our case are defined in (3.4)–(3.6). Estimates (4.1), (4.2) imply that the sequences w_n, u_n have subsequences with the properties

$$w_\nu^{(m)} \to w^{(m)} \quad \text{as } \nu \to \infty \text{ weak* in } X_0^2,$$

$$w_\nu''' \to w''' \quad \text{as } \nu \to \infty \text{ weak* in } X_0^1,$$

$$u_\nu^{(m)} \to u^{(m)} \quad \text{as } \nu \to \infty \text{ weak* in } \{X_0^1\}^2, \ m = 0, 1, 2,$$

$$u_\nu''' \to u''' \quad \text{as } \nu \to \infty \text{ weak* in } \{X^0\}^2,$$

where the weak* convergence has the following meaning: if X is a Banach space and X its dual, then $f_n \to f$ as $n \to \infty$ weak* in $L^\infty(0, T, X)$ if

$$\int_0^T (f_n(t), \phi(t))dt \to \int_0^T (f(t), \phi(t))dt \quad \text{as } n \to \infty$$

for arbitrary $\phi \in L^1(0, T, X)$. For checking that the weak* limits w, u satisfy the requirements of Definition 1, estimates (4.1) are sufficient. The corresponding argument is given in [7]. In our case, however, the weak solution w, u is more regular. We have namely

(8.2) $w^{(m)} \in X_0^2, \quad u^{(m)} \in \{X_0^1\}^2, \quad m = 0, 1, 2, \quad w''' \in X_0^1, \quad u''' \in \{X^0\}^2.$

Relations (8.2) play a crucial role in the next section.

9. Application of the elliptic regularity theory. Let us write the system (1.1), (1.2) in the form

(9.1) $$D\Delta^2 w = -ghw'' + I\Delta w'' + k_{ij}s_{ij} + (s_{ij}w_{,i})_{,j} + f_3 \equiv F_3,$$

(9.2) $$Tu = F, \quad F = (F_1, F_2),$$

where Tu is defined in (3.3) and

(9.3) $$F_i = -ghu_i'' + c_{ijkl}(\tfrac{1}{2}w_{,k}w_{,l} - k_{kl}w)_{,j} + f_i, \quad i = 1, 2.$$

With the use of (8.2), Hölder's inequality and the continuity of the imbedding $H^2(G) \subset L^{2r/(2-r)}(G)$, $1 < r < 2$, we can show that almost everywhere on $[0, T]$

$$\int_G |s_{ij}|^r |w_{,i}|^r dx \leqslant \Big(\int_G |s_{ij}|^2 dx\Big)^{r/2} \Big(\int_G |w_{,i}|^{2r/(2-r)} dx\Big)^{(2-r)/2}$$

$$\leqslant \|s_{ij}\|_0^r \, \|w_{,i}\|_{0,2r/(2-r)}^r \leqslant C_1 \|s_{ij}\|_0^r \, \|w\|_2^r$$

(in this section the positive constants are denoted by C_i, $i = 1, 2, \ldots$). The last inequality means that almost everywhere on $[0, T]$

$$\|s_{ij}(t) w_{,i}(t)\|_{0,r} \leqslant C_2 \|s_{ij}(t)\|_0 \, \|w(t)\|_2 \leqslant \text{Const.}$$

On the other hand, since $t \to s_{ij}(t)$, $t \to w_{,i}(t)$ are measurable mappings $[0, T] \to L^2(G)$ and $[0, T] \to H^2(G)$, respectively, the mapping $t \to s_{ij}(t)w_{,i}(t)$ is measurable from $[0, T]$ to $L^r(G)$ (see [12], Corollaire 3, Théorème 26, Chapitre IV.3). Thus

$$s_{ij}w_{,i} \in L^\infty(0, T, L^r(G)).$$

The continuity of the operator

$$\frac{\partial}{\partial x_i}: L^r(G) \to W^{-1,r}(G)$$

gives

$$(s_{ij}w_{,i})_{,j} \in L^\infty(0, T, W^{-1,r}(G)).$$

From (8.2), assumptions (2.4) and (2.7) it follows that the remaining terms of the expression F_3 belong to $L^\infty(0, T, W^{-1,r}(G))$, i.e.,

$$F_3 \in L^\infty(0, T, W^{-1,r}(G)).$$

Using the definition of a weak solution and integrating by parts we obtain for almost every $t \in [0, T]$ and for arbitrary $\phi \in H_0^2(G)$

$$(D\Delta w(t), \Delta\phi) = (F_3(t), \phi),$$

i.e., $w(t)$ is a weak solution of the elliptic boundary value problem $D\Delta^2 w(t) = F_3(t)$ with homogeneous Dirichlet boundary conditions. Regularity theory for such problems (see [1]) gives

$$w(t) \in W^{3,r}(G) \cap H_0^2(G) \quad \text{and} \quad \|w(t)\|_{3,r} \leqslant C_3 \|F_3\|_{-1,r} \leqslant \text{Const}$$

for almost every $t \in [0, T]$. Since the operator $F_3 \to w$ is continuous, $t \to w(t)$ is measurable from $[0, T]$ to $W^{3,r}(G) \cap H_0^2(G)$, and therefore

(9.4) $$w \in L^\infty(0, T, W^{3,r}(G) \cap H_0^2(G)).$$

Inclusion (9.4) implies that, for $k, l, m = 1, 2$,

$$(w_{,k}w_{,l})_{,m} = w_{,km}w_{,l} + w_{,k}w_{,lm} \in L^\infty(0, T, L^2(G)).$$

In fact, almost everywhere on $[0, T]$ we have

$$\begin{aligned}
\|w_{,km}w_{,l}\|_0^2 &= \int_G |w_{,km}|^2 |w_{,l}|^2 dx \\
&\leqslant \Big(\int_G |w_{,km}|^{2r/(2-r)} dx\Big)^{(2-r)/r} \Big(\int_G |w_{,l}|^{2r/(2r-2)} dx\Big)^{(2r-2)/r} \\
&= \|w_{,km}\|_{0,2r/(2-r)}^2 \|w_{,l}\|_{0,2r/(2r-2)}^2 \\
&\leqslant C_4 \|w_{,km}\|_{1,r}^2 \|w_{,l}\|_{2,r}^2 \leqslant C_5 \|w\|_{3,r}^4 \leqslant \text{Const}.
\end{aligned}$$

Here we have used Hölder's inequality, the continuity of the imbeddings

$$W^{1,r}(G) \subset L^{2r/(2-r)}(G), \quad W^{2,r}(G) \subset L^{2r/(2r-2)}(G), \quad 1 < r < 2,$$

and the continuity of differential operators

$$\frac{\partial^2}{\partial x_k \partial x_m}\colon W^{3,r}(G)\to W^{1,r}(G), \qquad \frac{\partial}{\partial x_l}\colon W^{3,r}(G)\to W^{2,r}(G).$$

Since the $L^2(G)$-norm of the remaining terms of F_i ($i = 1, 2$) can be estimated by a constant without difficulties, we have $\|F_i(t)\|_0 \leqslant \text{Const}$ for almost every $t\in[0, T]$, $i = 1, 2$. The measurability of $t\to F_i(t)$, $i = 1, 2$, can be proved similarly as before. In consequence,

$$F\in L^\infty(0, T, \{L^2(G)\}^2)$$

and as before u is a weak solution of the elliptic boundary value problem for the system (9.2) with the homogeneous Dirichlet boundary conditions. Due to the regularity theory for elliptic systems (see [8], Section 6) we have

$$\|u(t)\|_2 \leqslant C_6\|F(t)\|_0 \leqslant \text{Const}$$

and

(9.5)
$$u\in L^\infty(0, T, \{H^2(G)\cap H_0^1(G)\}^2).$$

Here and in the sequel we omit the simple argument concerning measurability.

Now we can repeat the procedure leading to (9.4), (9.5). We have namely, for almost every $t\in[0, T]$,

$$\|s_{ij,j}\|_0 \leqslant C_7\big(\|(u_{i,j}+u_{j,i})_{,j}\|_0+\|(w_{,i}w_{,j})_{,j}\|_0+\|w\|_0\big)$$

and relations (8.2), (9.4), (9.5) imply that $\|s_{ij,j}(t)\|_0 \leqslant \text{Const}$ for almost every $t\in[0, T]$. The continuity of the imbeddings

$$W^{2,r}(G)\subset L^\infty(G), \qquad W^{1,2}(G)\subset L^{2r/(2r-2)}(G), \qquad W^{1,r}(G)\subset L^{2r/(2-r)}(G)$$

and Hölder's inequality yield

$$\begin{aligned}\|s_{ij,j}w_{,i}\|_0 &\leqslant C_8\|s_{ij,j}\|_0\|w_{,i}\|_{0,\infty}\\ &\leqslant C_9\|w_{,i}\|_{2,r}\leqslant C_{10}\|w\|_{3,r}\leqslant \text{Const},\\ \|s_{ij}w_{,ij}\|_0 &\leqslant C_{11}\|s_{ij}\|_{0,2r/(2r-2)}\|w_{,ij}\|_{0,2r/(2-r)}\leqslant C_{12}\|s_{ij}\|_1\|w_{,ij}\|_{1,r}\\ &\leqslant C_{13}\|s_{ij}\|_1\|w\|_{3,r}\leqslant \text{Const}.\end{aligned}$$

The last two relations give

$$\|(s_{ij}w_{,i})_{,j}(t)\|_0 \leqslant \text{Const}$$

almost everywhere on $[0, T]$, i.e.,

$$(s_{ij}w_{,i})_{,j}\in L^\infty(0, T, L^2(G)).$$

It is not difficult to see that due to (8.3) the remaining terms of F_3 also satisfy this inclusion and we have

$$F_3 \in L^\infty(0, T, L^2(G)).$$

By using the regularity theory we obtain

$$\|w(t)\|_4 \leqslant C_{14}\|F_3(t)\|_0 \leqslant \text{Const}$$

for almost every $t \in [0, T]$ and

$$w \in L^\infty(0, T, H^4(G) \cap H_0^2(G)). \tag{9.6}$$

Relation (9.6) and the continuity of the imbeddings

$$W^{4,2}(G) \subset W^{3,4}(G) \subset W^{2,4}(G) \subset W^{1,4}(G)$$

show that for almost every $t \in [0, T]$

$$\|(w_{,i}w_{,j})_{,kl}\|_0 \leqslant \|w_{,ikm}w_{,j}\|_0 + \|w_{,ik}w_{,jl}\|_0 + \|w_{,il}w_{,jk}\|_0 + \|w_{,i}w_{,jkl}\|_0$$
$$\leqslant C_{15}(\|w\|_{3,4}\|w\|_{1,4} + 2\|w\|_{2,4}^2) \leqslant C_{16}\|w\|_4^2 \leqslant \text{Const}.$$

As a corollary we obtain $(w_{,i}w_{,j})_{,l} \in L^\infty(0, T, H^1(G))$. The remaining terms of F_i, $i = 1, 2$, also have this property and we get

$$F \in L^\infty(0, T, \{H^1(G)\}^2).$$

Using the regularity theory once more we obtain

$$u \in L^\infty(0, T, \{H^3(G) \cap H_0^1(G)\}^2). \tag{9.7}$$

10. Completion of the proof. The regularity results (8.2), (9.6), (9.7) allow us to integrate by parts in formulas (2.1), (2.2) defining weak solutions. In consequence we obtain

$$(D\Delta^2 w - F_3, \phi) = 0 \quad \text{for arbitrary } \phi \in C_0^\infty(G),$$
$$(Tu - F, \psi) = 0 \quad \text{for arbitrary } \psi \in C_0^\infty(G).$$

Thus, the expressions $\Delta^2 w - F_3$, $Tu - F$ vanish in the distributional sense and are square integrable. It is well known that in this case they must vanish almost everywhere on $G \times (0, T)$, i.e., equations (1.1), (1.2) are satisfied almost everywhere.

On the other hand, the imbeddings

$$H^4(G) \subset C^2(\bar{G}), \quad H^3(G) \subset C^1(\bar{G})$$

imply that, for almost every $t \in [0, T]$,

$$w(t) \in C^2(\bar{G}) \cap H_0^2(G), \quad u(t) \in \{C^1(\bar{G}) \cap H_0^1(G)\}^2,$$

and this implies that the boundary conditions are satisfied in the classical sense (see [2], Lemma 9.1).

Finally, the obtained regularity of solutions implies

$$w,\ w' \in C\big(0,\ T,\ H_0^2(G)\big), \qquad u_i,\ u_i' \in C\big(0,\ T,\ H_0^1(G)\big)$$

(see [6], Chapitre I, Section 4.3) and the initial conditions (2.3) are satisfied in the sense of the equality of two functions from the spaces $H_0^2(G)$ and $H_0^1(G)$, respectively, i.e., almost everywhere on G.

Appendix. In this appendix we prove the estimate

(A.1) $$C\,\|w_n''(0)\|_2^2 \leqslant I\big(\Delta w_n''(0),\ (\Delta^2)^{1/2} w_n''(0)\big),$$

where w_n is defined in (3.4) and C denotes a positive constant. To this purpose let us suppose that the system $\{z_r\}_{r=1}^{\infty}$ is orthogonal with respect to the scalar product

$$[\phi,\ \psi] = (\Delta^{1/2}\phi,\ \Delta^{1/2}\psi),$$

where $\Delta^{1/2}$ is a square root of the Laplace operator considered as an unbounded operator in $L^2(G)$ with the domain of definition

$$D(\Delta) = H^2(G) \cap H_0^1(G).$$

This assumption implies also the orthogonality with respect to the scalar product

$$\{\phi,\ \psi\} = \big(\Delta^{1/2}(\Delta^2)^{1/2}\phi,\ \Delta^{1/2}(\Delta^2)^{1/2}\psi\big),$$

where $(\Delta^2)^{1/2}$ is the square root of Δ^2 considered as an unbounded operator in $L^2(G)$ with the domain of definition

$$D(\Delta^2) = H^4(G) \cap H_0^2(G).$$

In consequence the following equalities are true:

$$\big(\Delta w_n''(0),\ (\Delta^2)^{1/2} w_n''(0)\big) = \Big(\Delta \sum_{r=1}^{n} g_{rn}''(0) z_r,\ (\Delta^2)^{1/2} \sum_{r=1}^{n} g_{rn}''(0) z_r\Big)$$

$$= \sum_{r,q=1}^{n} \lambda^{-1/2} g_{rn}''(0) g_{qn}''(0) \big(\Delta (\Delta^2)^{1/2} z_r,\ (\Delta^2)^{1/2} z_q\big)$$

$$= \sum_{r,q=1}^{n} \lambda^{-1/2} g_{rn}''(0) g_{qn}''(0) \big(\Delta^{1/2} (\Delta^2)^{1/2} z_r,\ \Delta^{1/2}(\Delta^2)^{1/2} z_q\big)$$

$$= \sum_{r=1}^{n} \lambda^{-1/2} \big(g_{rn}''(0)\big)^2 \|\Delta^{1/2}(\Delta^2)^{1/2} z_r\|_0^2 = \sum_{r=1}^{n} \big(g_{rn}''(0)\big)^2 \|\Delta^{1/2}(\Delta^2)^{1/4} z_r\|_0^2$$

$$= \|\Delta^{1/2}(\Delta^2)^{1/4} w_n''(0)\|_0^2,$$

where $(\Delta^2)^{1/4}$ is the square root of the operator $(\Delta^2)^{1/2}$ with

$$D\big((\Delta^2)^{1/2}\big) = H_0^2(G).$$

It is known (see [3]) that

$$C_0\|\phi\|_1 \leqslant \|\Delta^{1/2}\phi\|_0 \leqslant C_1\|\phi\|_1, \qquad \phi\in D(\Delta^{1/2}) = H_0^1(G),$$

(A.2) $$C_{2m}\|\psi_m\|_m \leqslant \|(\Delta^2)^{m/4}\psi_m\|_0 \leqslant C_{2m+1}\|\psi_m\|_m, \qquad m = 1, 2, \; \psi_m\in D\big((\Delta^2)^{m/4}\big),$$

$$H_0^1(G) \subset D\big((\Delta^2)^{1/4}\big) \subset H^1(G), \qquad D\big((\Delta^2)^{1/2}\big) = H_0^2(G),$$

where $C_1, \ldots, C_5$ denote positive constants. Using (A.2) we obtain

(A.3) $$C_4\|w_n''(0)\|_2 \leqslant \|(\Delta^2)^{1/2}w_n''(0)\|_0 \leqslant C_3\|(\Delta^2)^{1/4}w_n''(0)\|_1$$
$$\leqslant \frac{C_3}{C_0}\|\Delta^{1/2}(\Delta^2)^{1/4}w_n''(0)\|_0 \leqslant \frac{C_3C_1}{C_0}\|(\Delta^2)^{1/4}w_n''(0)\|_1$$
$$\leqslant \frac{C_3C_1}{C_2C_0}\|(\Delta^2)^{1/2}w_n''(0)\|_0 \leqslant \frac{C_5C_3C_1}{C_2C_0}\|w_n''(0)\|_2$$

and (A.1) is proved.

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Received on 1987.10.06;
revised version on 1988.02.09