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A NOTE ON LOCAL NOETHER LATTICES

BY

JOHNNY A. JOHNSON (HOUSTON, TEXAS)

1. Introduction. In a recent paper [5] Satyanarayana introduced a class of rings which he called generalized primary rings. A *generalized primary ring* is a commutative ring with an identity in which all ideals are primary. Among other things he showed that a generalized primary ring has a unique maximal ideal, and in Theorem 4.12 he proved the following result for Noetherian generalized primary rings (see Section 3 for the definition of a discrete valuation ring used below):

(1.1) *Let R be a Noetherian generalized primary ring with the unique maximal ideal M. Then R is a discrete valuation ring if and only if M is a principal ideal.*

In this paper we show that this result holds for a much larger class of Noetherian rings and are able to obtain his result as a corollary (Corollary 3.2).

2. The lattice case. In order to establish the result of this section we will require the following definitions. By a *multiplicative lattice* we mean a complete lattice provided with a commutative, associative, join-distributive multiplication for which the unit element is a multiplicative identity. We shall use juxtaposition to denote multiplication, $\vee$ and $\wedge$ to denote the lattice operations, and $\leqslant$ to denote the partial order of the lattice. If A and B are elements of the multiplicative lattice L, then $A:B$ is defined to be the join of all elements C in L such that $CB \leqslant A$. An element E of L is said to be *meet principal* in case

$$(2.1) \qquad \big(A\wedge(B:E)\big)E = AE\wedge B \quad \text{for all } A, B \text{ in } L.$$

E is said to be *join principal* in case

$$(2.2) \qquad (A\vee BE):E = (A:E)\vee B \quad \text{for all } A, B \text{ in } L.$$

If E is both meet principal and join principal, E is said to be *principal*. A multiplicative lattice L is said to be a *Noether lattice* if L satisfies the ascending chain condition, is modular, and every element of L is a join of principal elements. A Noether lattice is said to be *local* if it has precisely one maximal element. For further properties and definitions concerning Noether lattices the reader is referred to [1]-[4].

We can now establish the following result:

THEOREM 2.1. *Let L be a local Noether lattice with the unique maximal element M. Then the following three statements are equivalent*:

(2.3) *Every element of L is principal.*

(2.4) *M is meet principal.*

(2.5) *For every element $A \neq 0$ of L, there exists a non-negative integer n such that $A = M^n$.*

Proof. Clearly, (2.3) implies (2.4). To show that (2.4) implies (2.5), assume that M is meet principal and let $A \neq 0$ be an element of L. Since $A \neq 0$, there exists, by the Intersection Theorem for Noether lattices [1], a positive integer n such that $A \not\leqslant M^n$. Let k be the least such a positive integer. Thus $A \not\leqslant M^k$ and $A \leqslant M^{k-1}$. Since M is meet principal, it follows that M^{k-1} is meet principal. Thus, we have

$$A = A \wedge M^{k-1} = (A:M^{k-1})M^{k-1}.$$

If $A:M^{k-1} \neq I$, then $A:M^{k-1} \leqslant M$, so that we would have

$$A = A \wedge M^{k-1} = (A:M^{k-1})M^{k-1} \leqslant MM^{k-1} = M^k,$$

which contradicts the choice of k. Therefore, $A:M^{k-1} = I$ and, consequently, $A = M^{k-1}$. Thus (2.4) implies (2.5).

Assume now that for every element $A \neq 0$ of L there exists a non-negative integer n such that $A = M^n$. Let B be an element of L and suppose $B = B_1 \vee \ldots \vee B_k$ is a minimal representation of B as the join of principal elements of L. Suppose $k > 1$. Let n_1 and n_2 be such that $B_1 = M^{n_1}$ and $B_2 = M^{n_2}$. We can assume without loss of generality that $n_2 \leqslant n_1$. Thus $B_1 = M^{n_1} \leqslant M^{n_2} = B_2$, and so $B = B_2 \vee \ldots \vee B_k$ which contradicts the fact that $B = B_1 \vee \ldots \vee B_k$ was a minimal representation. It follows that (2.5) implies (2.3).

3. The ring case. In this section we shall apply Theorem 2.1 of the previous section to the case of the lattice of ideals of a Noetherian ring to obtain our final result. As in [5], we define a *discrete valuation ring* (not necessarily an integral domain) to be a commutative ring with an identity in which every ideal is principal and all ideals form a chain under set inclusion.

We now prove the following theorem:

THEOREM 3.1. *Let R be a local Noetherian ring with the unique maximal ideal M. Then R is a discrete valuation ring if and only if M is a principal ideal.*

Proof. It is sufficient to show that R is a discrete valuation ring if M is a principal ideal. Thus, assume that M is a principal ideal.

Let $L(R)$ denote the lattice of ideals of R with the usual ideal operations. It is well known that $L(R)$ is a Noether lattice. Furthermore, in this case $L(R)$ is a local Noether lattice. For local Noether lattices, it is known that the principal elements of $L(R)$ are precisely the principal ideals of R (see [2], Corollary 1.6). Thus, since M is a principal ideal, it follows that M is a principal element of the Noether lattice $L(R)$, and is, therefore, meet principal. Consequently, by (2.4) of Theorem 2.1, every element of $L(R)$ is principal, and hence every ideal of R is principal. In addition, it follows from (2.5) of Theorem 2.1 that all of the ideals of R form a chain under set inclusion. Hence R is a discrete valuation ring.

As a corollary to the above-mentioned result we obtain Theorem 4.12 of [5]:

COROLLARY 3.2. *Let R be a Noetherian generalized primary ring with the unique maximal ideal M. Then R is a discrete valuation ring if and only if M is a principal ideal.*

Proof. Since Noetherian generalized primary rings are local Noetherian rings (see [5], Corollary 2.3), the result follows immediately.

REFERENCES

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CALIFORNIA INSTITUTE OF TECHNOLOGY
JET PROPULSION LABORATORY
PASADENA, CALIFORNIA
UNIVERSITY OF HOUSTON
HOUSTON, TEXAS

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