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Zdzisław Opial – a mathematician 1930–1974

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Zdzisław Opial studied mathematics at the Jagiellonian University in Kraków from 1949 till 1954. He was not a typical student. He was sharp, critical and hard working. He knew much more than any of his fellow students. He was self-educated to a large extent and when entering the university, his knowledge was far above that of the best graduates of high schools. His interests were wide-ranging and not restricted to mathematics. Philosophy was another subject he studied intensively. He learned French fluently in high school. He read much and collected books with a great passion. He was interested in arts, films, the theater. As a student he played soccer for one or two seasons in one of the teams of the local league.

At the second year of his university studies, Opial joined Ważewski's seminar on differential equations and soon became one of its active members. This seminar and the personal influence of Tadeusz Ważewski were crucial in forming Opial's research interest. That was ordinary differential equations, the area in which he quickly became an expert and acquired an outstanding internationally recognized position.

Before presenting some of his mathematical results let us recall the main steps of his academic career. During all of his professional life Opial was associated with the Jagiellonian University. In 1951 while still a student, he was appointed to the position of assistant. He received a doctor's degree in 1957 and his second degree – habilitation – in 1960. He was promoted to the position of docent in 1962 and became a professor in 1967. He served as one of the prorectors of the University from 1969 till 1972. Opial received a fellowship of the Polish Academy of Sciences to study abroad and he used it to spend the academic year 1959/1960 in Paris. He spent the year 1966/1967 as a visiting professor at the Lefschetz Center for Dynamical Systems of Brown University in Providence, Rhode Island.

The first period of his mathematical life up to his trip to Paris was marked with a very intense and fruitful research concentrated mainly on ordinary differential equations. It is enough to mention that more than half of all his research papers are from that comparatively short period. It would be difficult to describe all of his results from that time since they are contained in 44 publications. If they were put together they would form a sizable volume of almost four hundred pages.

In his graduation thesis [2], Opial obtained an important and original result. Namely, he proved that if the functions $f_i(x, y_1, \ldots, y_n)$, $i = 1, 2, \ldots, n$, are non-decreasing with respect to the y variables and if the continuous functions $\Phi_i(x)$ satisfy the following system of integral inequalities

$$\Phi_i(x) \leqslant \beta_i + \int_a^x f_i(x, \Phi_1(x), \ldots, \Phi_i(x))dx, \qquad i = 1, \ldots, n,$$

then

$$\Phi_i(x) \leqslant \mu_i(x) \qquad \text{for } a \leqslant x \leqslant a+\alpha,$$

where μ_i is the maximal solution of the corresponding differential equation. At the same time he proved, by constructing an example, that the monotonicity assumption in this theorem is essential. This result, now referred to as Opial's theorem on integral inequalities, was known before for $n = 1$ and is a generalization of the popular Gronwall integral inequality.

A comparatively large part of Opial's papers deal with various asymptotic properties of solutions of both linear and non-linear second order differential equations. Ten of these papers, i.e., among others, [4], [6], [9], [10], [15], [17], [18], [29] are concerned with the linear equation

$$x'' + a(t)x = 0 \tag{1}$$

or with its more general form

$$(p(t)x')' + g(t)x = 0. \tag{2}$$

Hille proved that if $\int_0^\infty a(t)dt$ is finite and $t\int_0^\infty a(s)ds \geqslant \frac{1}{4}+\varepsilon$, then each solution of (1) is oscillatory. A far reaching generalization of this result was obtained by Opial in [6]. In [14] he considers the same problem for equations (2) and he shows that a sufficient condition for oscillation of solutions of (2) is the existence of a function $w(t)$ of class C^1 such that

$$\lim_{t\to\infty} \int_0^t w(s)\big(g(s) - \tfrac{1}{4}p(s)[w'(s)/w(s)]\big)ds = +\infty.$$

The above result generalizes the fact that equation (1) for $a(t) = a/t^2$ is oscillatory if and only if $a > \frac{1}{4}$.

If the coefficient in (1) is continuous and non-decreasing, then

$$a(t)x'^2 + x^2$$

is a Lyapunov function for equation (1). This implies stability and an additional property: if $a(t)$ tends to infinity when $t \to \infty$, then there exists a solution of (1) which tends to zero. Two papers of Opial ([9], [15]) are devoted to this effect (discovered by Biernacki, Milloux, and others). Among others the question how big the perturbation $g(t)$ can be in order that this effect is preserved for the equation $x''+sa(t)+g(t)cx = 0$ is answered in these papers.

If $a(t)$ is non-negative and periodic, stability of equation (1) takes place (Lyapunov) provided that

$$\int_0^\pi a(t)dt \leqslant 4/\pi.$$

A similar result was also given by Borg. Opial [10] obtained a beautiful and "final" result along these lines.

Finally, the subject of two other papers from this group ([18], [29]) is the asymptotic distribution of zeros of characteristic functions in Sturm's problem for equation (2) and the existence of square integrable solutions of (1).

The linear equation

$$x''+a(t)x'+b(t)x = 0 \tag{3}$$

with $b(t)$ bounded from above and from below by positive constants and with sufficiently strong dumping $a(t)$ behaves similarly to an equation with constant positive coefficients, in particular solutions of (3) are bounded for positive time t. If the dumping $a(t)$ tends to infinity quickly enough, then there exist solutions which have limits different from zero as t approaches infinity. This is the case, as was shown by Opial in [5], if and only if

$$\int_0^\infty \left[\exp\left(\int_0^s -a(t)dt\right) \int_0^s \left(\exp \int_0^t a(u)du\right)dt \right] ds < +\infty .$$

Applying Ważewski's topological method, Opial proved in [7] that the equation

$$u'' = f(t, u', u) \tag{4}$$

has a bounded solution if f is increasing with respect to u. He used this result ([21]) to show that (4) has a periodic solution if f is periodic in t. The case of almost periodic solutions was treated in [24].

In another paper [16] dealing with equation (3), Opial gives estimates of the distance of two consecutive zeros of a non-trivial solution. Namely, if a solution $x(t)$ of (3) equals zero for $t = 0$ and $t = h > 0$ and is different from zero in between them, as was proved by Opial, we have the estimate

$$\pi^2 \leqslant 4mh + kh^2,$$

where

$$2m = \max\{|a(t|\colon\ 0 \leqslant t \leqslant h\} \quad \text{and} \quad k = \max\{|b(t)|\colon\ 0 \leqslant t \leqslant h\}.$$

This considerably extends an analogous result due to de la Vallée Poussin. We mentioned this result also for the reason that in its proof Opial used an integral inequality of the following type

$$\int_0^h |x(t)x'(t)|dt \leqslant \alpha^2 \int_0^h x'^2(t)dt. \tag{5}$$

The existence of a constant α such that (5) holds true follows from the Schwarz inequality. Opial found the best possible constant. In [33] he proved that for each $x(t)$ of class C^1 on $[0, h]$ and equal to zero on both ends of the interval, inequality (5) holds with $\alpha = h/4$, while we have equality if $x(t) = ct$ if $0 \leqslant t \leqslant h/2$ and $x(t) = ch - ct$ if $h/2 \leqslant t \leqslant h$, where c is an arbitrary constant. This inequality made an impressive career as it happens frequently with simple but useful results. It stimulated quite a few authors and is now known widely as the Opial inequality.

Opial devoted much attention and research effort to ordinary differential equations with the right-hand side almost periodic, see [24], [40], [42], [3]. In particular, he was interested in the question whether the existence of a bounded solution implies the existence of almost periodic solutions of such equations. For a long time he believed that the answer is positive but finally he constructed a difficult example of a differential equation $y' = f(x, y)$, where f is almost periodic in x, all solutions of which are bounded in both directions but none of them is almost periodic (see [45]).

Perhaps one of the most important of Opial's results from that period is contained in [35]. He proved there the continuous dependence of solutions on the right-hand side of the equation with respect to L_1 topology of the right-hand sides. This is rather a fundamental result for the theory of ODE and it is not surprising that this paper of Opial is more frequently referred to in the literature than others. He himself applied it to almost autonomous systems, that is, the systems of the form $x' = f(x) + g(t, x)$, where the perturbation $g(t, x)$, for x from any fixed bounded set, is bounded by an integrable function of t. He proved a beautiful result that any solution x, bounded for $t > 0$, has the limiting set $x(\infty)$ composed of full trajectories of the corresponding autonomous system $x' = f(x)$.

The period of Opial's mathematical life we were discussing up to now was not chosen accidentally. It forms rather a closed time interval since his one year study stay in Paris influenced him a lot and changed his evaluation of mathematics in general and of his own activity in particular. In one of his letters from Paris he wrote to the second of the authors "... mathematics in Kraków differs so much from mathematics in Paris or in New York ..." and later "... a stay abroad opens one's eyes and it is fruitful if one is able to assimilate this previously unknown mathematics. This requires some effort, arouses a lot of psychological resistance and does not bring results quickly."

He was absorbing this new mathematics as nobody else among us could. His stay in Paris explains the fact that after his return to Kraków he started first of all change the substance and the ways of teaching mathematics on the university level with great enthusiasm and energy so typical for him. It was then that he wrote an excellent text-book on algebra, at the time quite "revolutionary" in Poland, and which quickly became very popular throughout the country. It had nine editions in eleven years. In other words, research in the strict sense of the word became one of a few equally important activities of Opial. Another one was teaching in the best sense of the word, teaching understood as the highest possible responsibility for the future generations of mathematicians, for the future of mathematics in the country.

But, of course, he continued to enrich mathematics itself and some more examples of that are presented below.

In the sixties, Felix Browder discovered an interesting fact that a non-expansive mapping on a closed, convex and bounded subset of a Hilbert space has a fixed point. For some time there was a question if or when the successive approximations converge, and hence to a fixed point. A simple example, rotation of a circle on a plane, shows that the non-expansiveness alone,

$$\|Tx-Ty\| \leqslant \|x-y\|,$$

is not sufficient. The problem was: what else is needed? There were several attempts made but it was Opial who gave an answer which was both simple and elegant: it suffices to assume that T is asymptotically regular, that is,

$$\lim_{n\to\infty} (T^{n+1}x - T^n x) = 0.$$

Thus he proved ([62]) that if $T\colon\ C \to C$ is an asymptotically regular non-expansive map of a closed and convex subset C of a Hilbert space, then the set of fixed points of T is not empty and for each x in C the sequence $T^n x$ converges to a fixed point. Notice that he need not assume that C is bounded on one hand and on the other the condition above is necessary for such x that $T^n x$ converges. In the same paper he extended his result to a class of Banach spaces. This class of Banach spaces is apparently of interest and they are sometimes given the name "Banach spaces satisfying Opial's condition." It is worth mentioning at this point that he became acquainted with the subject during his one year stay at Brown University and that he gave a course there on it. The lecture notes from this course *Non-expansive and monotone mappings in Banach spaces* published as a preprint by the Lefschetz Center for Dynamical Systems is still a very valuable source of information.

More or less at the same time, Opial returned to the problem of the continuous dependence of solutions on the right-hand side of the equation. This time he was working on linear systems. Even in this question, seemingly fully answered, he was able to discover new phenomena. We would like to present here one of his theorems in this direction, a simple one.

Consider a sequence of linear differential equations

$$x' = A_k(t)x, \qquad x \in \boldsymbol{R}^n, \tag{6}$$

with initial conditions

$$x(0) = r_k. \tag{7}$$

The matrices $A_k(t)$ and initial values r_k converge as k tends to infinity to $A_0(t)$ and r_0, respectively. We have then the limiting equation

$$x' = A_0(t)x, \quad x(0) = r_0. \tag{8}$$

Opial proved ([62]) that solutions of (6)–(7) converge uniformly on [0, 1] to the solution of (8) provided $r_k \to r_0$, A_k, A_0 are integrable and

$$\Big(1+\int_0^1 |A_k(t)|dt\Big)\Big(\max\Big\{\Big|\int_0^t (A_k(s)-A_0(s))ds\Big|:\ 0 \leqslant t \leqslant 1\Big\}\Big) \to 0. \tag{9}$$

In particular, the above holds true if A_k converges to A_0 in L_1 norm, or if the first term of (9) is bounded (A_k is bounded in L_1 norm) while the second term converges to zero. He proved in the same paper that the latter condition alone is not sufficient to claim convergence of solutions.

Opial was a very skillful analyst. If he studied a paper, one always could expect him to come out with a simplification, a new, more elegant proof of the result if not with a new stronger theorem. A typical example of such a situation is paper [60] concerned with the multipoint boundary value problem for the equation

$$x^{(n)} = a_1(t)x^{(n-1)} + \ldots + a_n(t)x. \tag{10}$$

We say that equation (10) has property $I(a, b)$ if for each sequence of real numbers

$$a < t_1 < \ldots < t_n < b \tag{11}$$

the function $x(t) \equiv 0$ is the only solution of (10) satisfying the boundary conditions

$$x(t_i) = 0, \quad i = 1, \ldots, n. \tag{12}$$

Equation (10) has property $I^*(a, b)$ if for each sequence of points

$$a < t_1 < \ldots < t_m < b \quad (m \leqslant n) \tag{13}$$

and for each sequence of positive integers

$$p_1, \ldots, p_m, \quad p_1 + \ldots + p_m = n \tag{14}$$

the function $x(t) \equiv 0$ is the only solution of (10) satisfying the conditions

$$x^{(j)}(t_i) = 0 \quad \text{for } i = 1, \ldots, m,\ j = 0, \ldots, p_i - 1. \tag{15}$$

It is clear that property $I^*(a, b)$ implies property $I(a, b)$. A Romanian mathematician O. Arama proved that if the coefficients in (10) are locally integrable on (a, b), then both those properties are equivalent; in other words also $I(a, b)$ implies $I^*(a, b)$. The original proof was long and complicated. Opial not only found a short and elegant proof ([60]) but also noticed several interesting implications of it. Here is one of them. Consider a non-linear n-th order differential equation

$$x^{(n)} = f(t, x^{(n-1)}, \ldots, x), \tag{16}$$

where f is continuous and satisfies the inequality

$$|f(t, x_1, \ldots, x_n)| \leqslant \sum a_i(t)|x_i| + b(t). \tag{17}$$

Here $a_i(t)$ and $b(t)$ are non-negative locally integrable functions on the interval (a, b). Together with equation (16) consider the differential inequality

$$|x^{(n)}| \leqslant \sum a_i(t)|x^{(n-i)}|, \tag{18}$$

where the coefficients $a_i(t)$ are the same as in (17).

Opial proved that if $x(t) \equiv 0$ is the only solution of inequality (18) which satisfies (12) for each choice of points t_i satisfying (11), then for each sequence of points in (13) and each sequence of positive integers satisfying (14) there is at least one solution of (16) satisfying the boundary conditions

$$x^{(j)}(t_i) = r_{ij}, \quad i = 1, \ldots, m;\ j = 0, \ldots, p_i - 1,$$

where r_{ij} are arbitrary but fixed real numbers.

What is particularly worth noticing in this result is the implication: the uniqueness of the solution of the homogeneous problem (18)–(12) implies the existence for a rather large class of non-linear boundary value problems.

Opial's results and ideas have stimulated many mathematicians in the world and his influence on the Kraków mathematical center cannot be overestimated.

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