ANNALES
POLONICI MATHEMATICI
XXVIII (1973)

Characteristic function of a meromorphic function and its derivative

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Abstract. In this paper relations between $T(r,f)$ and $T(r,f')$ have been obtained, where $T(r,f)$ and $T(r,f')$ are the Nevanlinna characteristic functions of the meromorphic functions $f(z)$ and $f'(z)$ respectively. Also results pertaining to Nevanlinna exceptional values have been established, and bounds for $k(f')$ in terms of Nevanlinna defects have been given, where

$$k(f') = \limsup_{r\to\infty} \frac{N(r,f')+N(r,1/f')}{T(r,f)}.$$

For instance it has been shown that if $\sum_i \sigma(ai) = a$ ($ai \neq \infty$ and $\sigma(\infty) = 2-a$), then

$$\frac{a-1}{a} \leqslant k(f') \leqslant \frac{2(a-1)}{a}.$$

1. Let $f(z)$ be a meromorphic function of order ϱ $(0 \leqslant \varrho < \infty)$. For a number a $(0 \leqslant |a| \leqslant \infty)$ let

$$\sigma(a,f) = \sigma(a) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)},$$

$$\theta(a,f) = \theta(a) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a)}{T(r,f)},$$

$$\Delta(a,f) = \Delta(a) = \limsup_{r\to\infty} \frac{m(r,a)}{T(r,f)} = 1 - \liminf_{r\to\infty} \frac{N(r,a)}{T(r,f)},$$

where $n(r,a)$, $N(r,a)$, $m(r,a)$, $\overline{n}(r,a)$, $\overline{N}(r,a)$ and $T(r,f)$ have the usual meaning, as in Nevanlinna Theory. See [9] and [10].

It is known that $0 \leqslant \sum_a \sigma(a) \leqslant 2$. If $\sum_a \sigma(a) = 2$, the function (f) is said to have the maximum defect. We also say that a is an e.v. N or e.v. V, according as $\sigma(a) > 0$ or $\Delta(a) > 0$. If $\sigma(a) = 1$, we say that a is an e.v. N with maximum defect. Similarly a is called an e.v. V with maximum defect if $\Delta(a) = 1$.

Let E denote a set of positive non-decreasing functions $\varphi(x)$ such that $\int_A^\infty \frac{dx}{x\varphi(x)} < \infty$. If for some $\varphi \in E$, $\liminf_{r\to\infty} \frac{T(r,f)}{n(r,a)\varphi(r)} > 0$, a is called an e.v. E for $f(z)$. See [11].

For meromorphic functions $f(z)$ of finite order we have

$$\liminf_{r\to\infty} \frac{T(r,f)}{T(r,f')} \geqslant \frac{1}{2}. \tag{1}$$

(1) follows from the fact that (i) $N(r,f') \leqslant 2N(r,f)$ since a pole of $f(z)$ of order m is a pole of $f'(z)$ of order $m+1 \leqslant 2m$ and (ii) $m(r,f') = m\left(r, \frac{f'}{f} f\right) \leqslant m\left(r, \frac{f'}{f}\right) + m(r,f)$ and $m\left(r, \frac{f'}{f}\right) = O(1)$ if $f(z)$ is a rational function and $m(r, f'/f) = O(\log r)$ if $f(z)$ is a transcendental meromorphic function (of finite order).

2. We prove:

THEOREM 1. *Let $f(z)$ be a meromorphic function of order ϱ $(0 < \varrho < \infty)$. Then*

$$\liminf_{r\to\infty} \frac{T(r,f)}{T(r,f')} < \infty, \tag{2}$$

$$\liminf_{r\to\infty} \frac{T(kr,f)}{T(r,f)} < \infty \tag{3}$$

for all positive constants k.

THEOREM 2. *Let $f(z)$ be a meromorphic function of finite order having $\{a_i\}$ as e.v. N such that*

$$\sum_i \sigma(a_i) = a \ (a_i \neq \infty) \quad \textit{and} \quad \sum_i \sigma(a_i) = 2,$$

where a_i's are distinct $(0 \leqslant |a_i| \leqslant \infty)$. Then

$$T(r,f') \sim aT(r,f).$$

COROLLARY 1. *If $\sum_i \sigma(a_i) = 1$, $a_i \neq \infty$ and $\sigma(\infty) = 1$, then*

$$T(r,f') \sim T(r,f).$$

COROLLARY 2. *If $\sum_i \sigma(a_i) = 2$, $a_i \neq \infty$, then*

$$T(r,f') \sim 2T(r,f).$$

See [11] and [14].

THEOREM 3. *Let $f(z)$ be a meromorphic function of finite order and let $T(r,f') \sim aT(r,f)$, where $a \geqslant 1$. Then*

$$\theta(\infty,f) \leqslant 2-a \quad \text{and} \quad \sum_i \sigma(a_i) \leqslant a, \quad a_i \neq \infty.$$

COROLLARY 3. *If $T(r,f') \sim 2T(r,f)$, then ∞ is neither e.v. N nor e.v. V for $f(z)$.*

COROLLARY 4. *If $\sum_i \{\sigma(a_i)+\mu(a_i)\} = a$ and $\sigma(\infty)+\mu(\infty) = 2-a$ $(1 \leqslant a \leqslant 2)$, then*

$$T(r,f') \sim aT(r,f),$$

where as usual $\mu(a)$ is defined by

$$\mu(a) = \liminf_{r\to\infty} \frac{N(r,a)-\bar{N}(r,a)}{T(r,f)}.$$

THEOREM 4. *If $f(z)$ is a meromorphic function of finite order with $\{a_i\}$ as e.v. N's $(a_i \neq \infty)$ such that $\sum_i \sigma(a_i) = a$, and $\mu(\infty) = 2-a$, then ∞ is not an e.v. N for $f'(z)$.*

COROLLARY 5. *If $f(z)$ has two finite e.v. E, then ∞ is not an e.v. E for $f'(z)$.* (See [14], Theorem 5.)

THEOREM 5. *If $f(z)$ is a meromorphic function of finite order with $\{a_i\}$ as e.v. N, such that $\sum_i \sigma(a_i) = a$, $a_i \neq \infty$ and $\sigma(\infty) = 2-a$, then*

$$\frac{a-1}{a} \leqslant k(f') \leqslant \frac{2(a-1)}{a},$$

where

$$k(f') = \limsup_{r\to\infty} \frac{N(r,f')+N(r,1/f')}{T(r,f')}.$$

See [2], [3] and [9], p. 51.

COROLLARY 6. *If $f(z)$ is a meromorphic function of finite order having* 0 *and ∞ as e.v. E, then $f'(z)$ has* 0 *and ∞ as e.v. N with maximum defect.*

See [14], Theorem 4.

Since $\sigma(0,f) = \sigma(\infty,f) = 1$ if 0 and ∞ are e.v. E so $a = 1$ and hence, by Theorem 5, $k(f') = 0$, which gives $\sigma(0,f') = \sigma(\infty,f') = 1$.

COROLLARY 7. *If $a = 1$, then ϱ must be a positive integer.*

3. Proof of Theorem 1. We know that

$$\text{(A)} \qquad T(r,f) < 40\frac{k}{k-1}\log\frac{ek}{k-1}T(kr,f')+\log^+(kr)+5+\log^+|f(0)| \qquad (k>1,\ r>0).$$

See [1], p. 171.

Combining (A) with (1) we immediately deduce, that the order of $f'(z)$ is equal to ϱ, the order of $f(z)$. Thus

$$\limsup_{r\to\infty} \frac{\log T(r,f')}{\log r} = 0.$$

Since $0 < \varrho < \infty$, it follows that there exists a proximate order $\varrho(r)$ relative to $T(r,f')$ such that $T(r,f') \leqslant r^{\varrho(r)}$ for $r \geqslant r_0$ and $T(r,f') = r^{\varrho(r)}$ for a sequence $r = r_n \to \infty$ as $n \to \infty$. Hence, by (A), we have

$$\begin{aligned} T(r,f) &< CT(kr,f') \quad (C \text{ constant, } k > 1) \\ &\leqslant C(kr)^{\varrho(kr)} \quad \text{for } r \geqslant r_0 \\ &\sim Ck^{\varrho} r^{\varrho(r)} \\ &= Ck^{\varrho} T(r,f') \quad \text{for a sequence } r = r_n. \end{aligned}$$

Hence

$$\liminf_{r\to\infty} \frac{T(r,f)}{T(r,f')} < \infty.$$

This proves (2). The proof of (3) is similar to that of (2). We consider a proximate order $\varrho(r)$ relative to $T(r,f)$ and use the fact that $T(kr,f) \leqslant (kr)^{\varrho(kr)}$ for $r \geqslant r_0$.

Remark 1. (1) is not true if $f(z)$ is of infinite order. Consider

$$f(z) = \sum_{1}^{\infty} \left(\frac{z}{3^n}\right)^{\lambda_n},$$

where λ_n is a rapidly increasing sequence of positive integers. If we define

$$r_N = \{3^{N\lambda_N} 2(N-1)\lambda_{N-1}\}^{\frac{1}{\lambda_N - \lambda_{N-1}}},$$

then $r_N \sim 3^N$ as $N \to \infty$, and

$$T(r_N, f) = \{N \log 3 + O(1)\}\lambda_{N-1} + \log(2N\lambda_{N-1}) + O(1), \tag{4}$$

$$T(r_N, f') > T(r_N, f) + \log \lambda_N + O(1) - \log r_N. \tag{5}$$

Hence, by (4), (5) and the choice of λ_N, it follows that

$$\lim_{N\to\infty} \frac{T(r_N, f)}{T(r_N, f')} = 0.$$

For details see [6], p. 17.

Proof of Theorem 2. In what follows we take $r \geqslant r_0$. From the inequality

$$pT(r,f) < T(r,f') + \sum_{1}^{p} N(r, a_i) - N\left(r, \frac{1}{f'}\right) + O(\log r)$$

of Milloux [8], p. 298, we have

$$p < \frac{T(r,f')}{T(r,f)} + \sum_{1}^{p} \frac{N(r,a_i)}{T(r,f)} - \frac{N(r,1/f')}{T(r,f)} + O(1)$$

$$= \frac{T(r,f')}{T(r,f)} \left\{ \frac{T(r,f') - N(r,1/f')}{T(r,f')} \right\} + \sum_{1}^{p} \frac{N(r,a_i)}{T(r,f)} + O(1)$$

$$= \frac{T(r,f')}{T(r,f)} \left\{ \frac{m(r,1/f') + O(1)}{T(r,f')} \right\} + \sum_{1}^{p} \frac{N(r,a_i)}{T(r,f)} + O(1).$$

Hence

$$p \leqslant \limsup_{r\to\infty} \left\{ \frac{m(r,1/f')}{T(r,f')} + O(1) \right\} \left\{ \liminf_{r\to\infty} \frac{T(r,f')}{T(r,f)} \right\} + \tag{6}$$

$$+ \sum_{1}^{p} \limsup_{r\to\infty} \frac{N(r,a_i)}{T(r,f)}.$$

Thus

$$\sum_{1}^{p} \sigma(a_i, f) \leqslant \Delta(0, f') \liminf_{r\to\infty} \frac{T(r,f')}{T(r,f)}. \tag{7}$$

Now, given an $\varepsilon > 0$, we can choose $a_1, a_2, \ldots, a_p$ $(p \geqslant 3)$ so that

$$\sum_{p+1}^{\infty} \sigma(a_i) < \varepsilon \qquad (a_i \neq \infty).$$

Hence

$$\sum_{1}^{p} \sigma(a_i) > a - \varepsilon \qquad (a_i \neq \infty).$$

Since $1 \leqslant a \leqslant 2$ and $\sigma(\infty) = 2 - a$, it follows that $\sum_{1}^{p} \sigma(a_i) > a - \varepsilon > 0$, so $\Delta(0,f') > 0$. Hence, by (7), we get

$$\frac{\sum_{1}^{p} \sigma(a_i)}{\Delta(0,f')} \leqslant \liminf_{r\to\infty} \frac{T(r,f')}{T(r,f)}. \tag{8}$$

Further we know ([9], p. 104) that

$$\limsup_{r\to\infty} \frac{T(r,f')}{T(r,f)} \leqslant 2 - \sigma(\infty) - \mu(\infty) \leqslant 2 - \sigma(\infty) = 2 - (2 - a) = a, \tag{9}$$

Hence, by (8) and (9),

$$\frac{a-\varepsilon}{\Delta(0,f')} \leqslant \frac{\sum_{1}^{p}\sigma(a_i)}{\Delta(0,f')} \leqslant \liminf_{r\to\infty}\frac{T(r,f')}{T(r,f)} \leqslant \limsup_{r\to\infty}\frac{T(r,f')}{T(r,f)} \leqslant a.$$

Thus $T(r,f') \sim aT(r,f)$ and $\Delta(0,f') = 1$, since ε is arbitrary.

Remark 2. The above theorem is not true for functions of infinite order. For instance, it is known that if $\psi(x)$ is any arbitrary positive increasing function of x for $x \geqslant 0$, then there exists an entire function $F(z)$ such that

$$\limsup_{r\to\infty}\frac{T(r,F')}{\psi(T(r,F))} = \infty.$$

See [5], Theorem 11.

Using the above theorem we can easily construct an entire function of infinite order for which Theorem 2 does not hold.

Remark 3. For any integer q $(1 \leqslant q \leqslant \infty)$ there exists a meromorphic function $f(z)$ for which $T(r,f') \sim (2-1/q)T(r,f)$, where for $q=\infty$ we interpret $2-1/q$ as 2.

If $q=1$, take $f(z) = e^z$.

If $q=\infty$, take $f(z) = e^z/(1+e^z)$ or $f(z) = \tan z$.

Then

$$T(r,f') \sim 2T(r,f).$$

So assume that $2 \leqslant q < \infty$.

Let $\varphi(z) = \int_0^z e^{-t^q}dt$,

$$a_k = \frac{2\pi ik}{e^q}\int_0^\infty e^{-t^q}dt, \qquad k = 1, 2, \ldots, q.$$

Let $f(z) = \dfrac{1}{\varphi(z)-a_1}$. Then

$$m(r,\varphi) \sim \frac{r^q}{\pi}, \qquad m\left(r, \frac{1}{\varphi-a_k}\right) \sim \frac{r^q}{q\pi}, \qquad k = 1, 2, \ldots, q,$$

$$N(r,f) = N(r,\infty,f) = N\left(r, \frac{1}{\varphi-a_1}\right) = \left(1-\frac{1}{q}\right)T(r,\varphi) \sim \left(1-\frac{1}{q}\right)T(r,f).$$

Hence

$$\sigma(\infty,f) = \frac{1}{q}.$$

Also, for $k \neq 1$,

$$N\left(r, \frac{1}{f - \dfrac{1}{a_k - a_1}}\right) = N\left(r, \frac{1}{\varphi - a_k}\right) = \left(1 - \frac{1}{q}\right)\left(T(r, f) + O(1)\right).$$

Hence

$$\sigma\left(\frac{1}{a_k - a_1}, f\right) = \frac{1}{q} \quad \text{for } k = 2, 3, \ldots, q, \quad \sigma(0, f) = 1.$$

Now if we put $a = (2 - 1/q)$, then the total defect is attained and so

$$T(r, f') \sim \left(2 - \frac{1}{q}\right) T(r, f).$$

Remark 4. In the above example the defect at infinity is positive and less than one and the total defect is attained and the function is of finite order. Let us note that there do exist meromorphic functions of infinite order for which the defect at infinity is positive and less than one and the total defect 2 is attained. For instance, consider the entire function $f(z)$ given in [4], Theorem 4.1. For that function $f(z)$, we have $\sigma(\infty) = 1$, $\sigma(a_\nu, f) = \sigma_\nu$ ($\nu = 1$ to ∞), where the a_ν's form an arbitrary sequence of distinct (finite) complex numbers and the σ_ν's form an arbitrary sequence of positive numbers such that $\sum_\nu \sigma_\nu = 1$.

Put $\sigma_1 = 2 - a$ (where $1 < a < 2$), $F(z) = \dfrac{1}{f(z) - a_1}$. Then

$$\sigma(\infty, f) = \sigma(a_1, f) = 2 - a$$

and

$$\sum_{a \neq \infty} \sigma(a, F) = \sum_{b \neq a_1} \sigma(b, f) = \sigma(\infty, f) + \sum_{2}^{\infty} \sigma_\nu = a.$$

Hence the total sum 2 is attained and $0 < \sigma(\infty, F) < 1$.

Remark 5. Theorem 2 is true even if

$$\sum_{a_i \neq \infty} \sigma(a_i) = a \quad \text{and} \quad \sum \sigma(a_i) + \mu(\infty) = 2.$$

This is an easy consequence of (9).

Remark 6. We have shown in Theorem 2 that 0 is an e.v. V for $f'(z)$ with maximum defect. Under the same hypothesis we can prove that 0 is an e.v. N for $f'(z)$ with maximum defect, since we could have written in (6)

$$1 \leqslant \liminf_{r \to \infty} \left\{\frac{m(r, 1/f')}{T(r, f')} + O(1)\right\} \left\{\limsup_{r \to \infty} \frac{T(r, f')}{T(r, f)}\right\} + \sum_{1}^{p} \limsup_{r \to \infty} \frac{N(r, a_i)}{T(r, f)},$$

which gives $\sigma(0, f') = 1$.

Proof of Theorem 3. Since $f(z)$ is of finite order,

$$m(r,f') = m\left(r,\frac{f'}{f}f\right) \leqslant m\left(r,\frac{f'}{f}\right)+m(r,f) \leqslant m(r,f)+O(\log r).$$

Also

$$N(r,f') = N(r,f)+\overline{N}(r,f).$$

Hence

$$\begin{aligned} T(r,f') &= N(r,f')+m(r,f') \leqslant N(r,f)+\overline{N}(r,f)+m(r,f)+O(\log r) \\ &= T(r,f)+\overline{N}(r,f)+O(\log r). \end{aligned}$$

Now, by assumption, $T(r,f') \sim aT(r,f)$. Hence

$$(a-1)T(r,f) \leqslant \overline{N}(r,f)+O(\log r).$$

Thus $\theta(\infty) \leqslant 2-a$.

Further we have from (7)

$$\sum_{a_i\neq\infty} \sigma(a_i) \leqslant \Delta(0,f') \liminf_{r\to\infty} \frac{T(r,f')}{T(r,f)}.$$

Hence, using the hypothesis again, we get

$$\sum_{a_i\neq\infty} \sigma(a_i) \leqslant \bar{a}.$$

Putting $a=2$ we get $\theta(\infty)=0$. Hence $\sigma(\infty,f)=0$. Thus ∞ is not an e.v. N for $f(z)$.

Also putting $a=2$ in $(a-1)T(r,f) \leqslant \overline{N}(r,f)+O(\log r)$ we get

$$\liminf_{r\to\infty} \frac{N(r,f)}{T(r,f)} \geqslant \liminf_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} = 1.$$

Hence $\Delta(\infty,f) = 0$. Thus ∞ is not an e.v. V for $f(z)$. This proves Corollary 3.

Also

(10) $$\limsup_{r\to\infty} \frac{T(r,f')}{T(r,f)} \leqslant 2-\sigma(\infty)-\mu(\infty) = 2-(2-a) = a$$

and [15], p. 21,

(11) $$\liminf_{r\to\infty} \frac{T(r,f')}{T(r,f)} \geqslant \sum_i \{\sigma(a_i)+\mu(a_i)\} = a.$$

Combining (10) and (11) we get $T(r,f') \sim aT(r,f)$. This proves Corollary 4.

Remark 7. From Corollary 4 we deduce that if $\sigma(\infty)=1$ and $\sum_{a_i\neq\infty} \{\sigma(a_i)+\mu(a_i)\}=1$, then $T(r,f') \sim T(r,f)$. Because then $a=1$, as $\sigma(\infty)+\mu(\infty)$ never exceeds 1.

This improves a result of S. M. Shah and S. K. Singh [14] which states that if 0 and ∞ are e.v. N with maximum defect, then $T(r,f') \sim T(r,f)$.

Proof of Theorem 4. From the second fundamental theorem of Nevanlinna [9] we have for $q \geqslant 3$ (q an integer),

$$(q-2)\,T(r,f)+2N(r,f)+N\left(r,\frac{1}{f'}\right)<\sum_{1}^{q}N(r,a_i)+N(r,f')+O(\log r).$$

Hence

$$(12)\qquad (q-2)+2\frac{N(r,f)}{T(r,f)}+\frac{N(r,1/f')}{T(r,f')}\cdot\frac{N(r,f')}{T(r,f)} < \sum_{1}^{q}\frac{N(r,a_i)}{T(r,f)}+\frac{N(r,f')}{T(r,f')}\cdot\frac{T(r,f')}{T(r,f)}+O(1).$$

By Remark 5 we have $T(r,f') \sim aT(r,f)$ and $\sigma(0,f')=1$ if $\sum\limits_{a_i\neq\infty}\sigma(a_i)=a$ and $\sum\limits_{i}\sigma(a_i)+\mu(\infty)=2$.

Hence, by (12), we get

$$(q-2)+2\frac{N(r,f)}{T(r,f)}+a\frac{N(r,1/f')}{T(r,f')}<\sum_{1}^{q}\frac{N(r,a_i)}{T(r,f)}+a\frac{N(r,f')}{T(r,f')}+O(1).$$

So

$$(13)\qquad (q-2)+2\limsup_{r\to\infty}\frac{N(r,f)}{T(r,f)}+a\liminf_{r\to\infty}\frac{N(r,1/f')}{T(r,f')} \leqslant \limsup_{r\to\infty}\sum_{1}^{q}\frac{N(r,a_i)}{T(r,f)}+a\limsup_{r\to\infty}\frac{N(r,f')}{T(r,f')}.$$

By assumption ∞ is not an e.v. N; hence

$$\limsup_{r\to\infty}\frac{N(r,f)}{T(r,f)}=1.$$

Further

$$\lim_{r\to\infty}\frac{N(r,1/f')}{T(r,f')}=0 \qquad \text{since} \qquad \sigma(0,f')=1.$$

Hence, by (13) and the hypothesis $\sum\limits_{1}^{q}\sigma(a_i)=a$, we deduce $\sigma(\infty,f')=0$. This proves the theorem.

Proof of Theorem 5. From $N(r,f') \leqslant 2N(r,f)$ we get

$$\frac{N(r,f')}{T(r,f')}\frac{T(r,f')}{T(r,f)} \leqslant \frac{2N(r,f)}{T(r,f)}. \tag{14}$$

The equalities $\sum\limits_{a_i \neq \infty} \sigma(a_i) = a$, $\sum \sigma(a_i) = 2$ imply that $T(r,f') \sim aT(r,f)$. Hence, from (14), we have

$$a\limsup_{r\to\infty}\frac{N(r,f')}{T(r,f')} \leqslant 2\limsup_{r\to\infty}\frac{N(r,f)}{T(r,f)} = 2\{1-\sigma(\infty,f)\} = 2(a-1). \tag{15}$$

In Remark 6 we have shown that

$$\lim_{r\to\infty}\frac{N(r,1/f')}{T(r,f')} = 0. \tag{16}$$

Hence

$$k(f') = \limsup\frac{N(r,f')+N(r,1/f')}{T(r,f')} \leqslant \limsup_{r\to\infty}\frac{N(r,f')}{T(r,f')} + \tag{17}$$

$$+\limsup_{r\to\infty}\frac{N(r,1/f')}{T(r,f')} \leqslant \frac{2(a-1)}{a}.$$

Further we have $N(r,f) \leqslant N(r,f')$ and

$$\frac{N(r,f)}{T(r,f)} \leqslant \frac{N(r,f')}{T(r,f')}\cdot\frac{T(r,f')}{T(r,f)} \sim a\frac{N(r,f')}{T(r,f')}.$$

Hence

$$\limsup_{r\to\infty}\frac{N(r,f')}{T(r,f')} \geqslant \frac{a-1}{a}. \tag{18}$$

(Since by hypothesis $\sigma(\infty) = 2-a$.)

Finally we have

$$k(f') \geqslant \limsup_{r\to\infty}\frac{N(r,f')}{T(r,f)} + \liminf_{r\to\infty}\frac{N(r,1/f')}{T(r,f')} \geqslant \frac{a-1}{a}$$

by (16) and (18).

Combining the above inequality with (17) we get the result. If $a = 1$, then $k(f') = 0$. Hence the order of $f'(z)$ must be an integer. See [9], p. 51, [4], p. 10, [12], [13]. This proves Corollary 7 since the order of $f(z)$ is the same as the order of $f'(z)$.

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Reçu par la Rédaction le 13. 6. 1970