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REMARKS ON JOINT DISTRIBUTIONS OF OBSERVABLES

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In the quantum probability theory the σ-field of random events is replaced by the lattice of orthogonal projectors in a separable infinite-dimensional Hilbert space H. A countably additive function from this lattice to the unit interval constitutes a state, the non-commutative analogue of a probability measure. The Theorem of Gleason [3] asserts that every state is of the form

$$\Pi \to \operatorname{tr} \Pi T, \tag{1}$$

where Π runs over all projectors and T is a probability operator on H, i.e., a positive operator of unit trace. Conversely, every probability operator determines a state by (1). From now onwards let $\mathfrak{P}$ stand for the set of all probability operators on H. We shall denote by $\mathfrak{T}_1$ the space of all nuclear operators acting in H with the norm $\|T\|_1 = \operatorname{tr}(TT^*)^{1/2}$. Of course, $\mathfrak{P}$ is a convex and closed subset of $\mathfrak{T}_1$. Further, by $\mathfrak{T}_2$ we shall denote the space of all Hilbert–Schmidt operators on H with the norm $\|T\|_2 = (\operatorname{tr} TT^*)^{1/2}$. Obviously, $\mathfrak{T}_1 \subset \mathfrak{T}_2$, $\|T\|_2 \leqslant \|T\|_1$ for $T \in \mathfrak{T}_1$, and

$$\|TU\|_1 \leqslant \|T\|_2 \|U\|_2 \tag{2}$$

for $T, U \in \mathfrak{T}_2$.

In quantum theory, to every physical quantity or observable there corresponds a self-adjoint not necessarily bounded linear operator on H. Let A be such an operator. The probability distribution of A at the state T is defined for all Borel subsets E of the real line R by the formula

$$p_T^A(E) = \operatorname{tr} \Pi_A(E)\, T,$$

where Π_A is the projector-valued spectral measure associated with A, i.e.,

$$A = \int_R \lambda \Pi_A(d\lambda).$$

The characteristic function of p_T^A, i.e., its Fourier transform $\hat{p}_T^A$, is then given by the formula

$$\hat{p}_T^A(t) = \operatorname{tr} e^{itA}\, T.$$

A system $A_1, A_2, \ldots, A_k$ $(k \geqslant 2)$ of observables is said to be *regular* if there exists a dense linear manifold D in H such that for arbitrary real numbers $a_1, a_2, \ldots, a_k$ the operator $\sum_{j=1}^{k} a_j A_j$ is well defined on D and is essentially self-adjoint, so that the probability distribution $p_T^{a_1 A_1 + \ldots + a_k A_k}$ is well defined at every state T. Of course, all systems of bounded observables are regular.

In [9] I introduced the concept of the joint probability distribution for regular systems of observables. Namely, a Borel probability measure p on the k-dimensional Euclidean space R^k is said to be the *joint probability distribution* at the state T of a regular system $A_1, A_2, \ldots, A_k$ of observables if for every k-tuple $a_1, a_2, \ldots, a_k$ of real numbers the projection of p onto the real line defined by

$$(x_1, x_2, \ldots, x_k) \to \sum_{j=1}^{k} a_j x_j$$

coincides with $p_T^{a_1 A_1 + \ldots + a_k A_k}$. It is clear that the joint probability distribution is uniquely determined provided it exists. In the sequel it will be denoted by $p_T^{A_1, \ldots, A_k}$. Then we have the equation

$$\hat{p}_T^{A_1, \ldots, A_k}(t_1, t_2, \ldots, t_k) = \operatorname{tr} \exp\Big(i \sum_{j=1}^{k} t_j A_j\Big)_T. \tag{3}$$

Further, by $\mathfrak{P}(A_1, A_2, \ldots, A_k)$ we shall denote the set of all states T for which $p_T^{A_1, \ldots, A_k}$ exists. It is evident that $T \in \mathfrak{P}(A_1, A_2, \ldots, A_k)$ if and only if the function

$$(t_1, t_2, \ldots, t_k) \to \operatorname{tr} \exp\Big(i \sum_{j=1}^{k} t_j A_j\Big)_T$$

is continuous and positive definite on R^k. Consequently, $\mathfrak{P}(A_1, A_2, \ldots, A_k)$ is a convex and closed in the topology of $\mathfrak{T}_1$ subset of $\mathfrak{P}$.

A relation between the existence of joint probability distribution at every state and the commutability of observables is given by the following statement:

Let $A_1, A_2, \ldots, A_k$ be a regular system of observables. Then

$$\mathfrak{P}(A_1, A_2, \ldots, A_k) = \mathfrak{P}$$

if and only if $A_1, A_2, \ldots, A_k$ commute with one another.

For observables with purely point spectrum this statement was proved in [9]. Recently, an elementary proof was given by Ruymagaart [8]. Without any restriction on the spectrum a proof can be found in [4] and [6]. In the more general framework of quantum logics the theorem was proved by Varadarajan [11].

We say that a regular system $A_1, A_2, \ldots, A_k$ of observables fulfils the *probabilistic commutation condition* if there exists a regular system $B_1, B_2, \ldots, B_k$ of commuting observables such that

$$p_T^{A_1,\ldots,A_k} = p_T^{B_1,\ldots,B_k} \tag{4}$$

for every $T \in \mathfrak{P}(A_1, A_2, \ldots, A_k)$. Recently, I proved in [10] that every system of bounded observables with purely point spectrum fulfils the probabilistic commutation condition. The aim of the present note is to show that this result cannot be extended to an arbitrary regular system of observables. Namely, we shall prove that any pair of canonical observables does not fulfil the condition in question.

Given a subset $\mathfrak{X}$ of $\mathfrak{T}_1$, we shall denote by $[\mathfrak{X}]$ the linear subspace of $\mathfrak{T}_1$ spanned by $\mathfrak{X}$.

First we shall prove the following simple statement:

PROPOSITION 1. *If a regular system $A_1, A_2, \ldots, A_k$ of observables fulfils the probabilistic commutation condition, then*

$$\mathfrak{P}(A_1, A_2, \ldots, A_k) = \mathfrak{P} \cap [\mathfrak{P}(A_1, A_2, \ldots, A_k)].$$

Proof. Let $B_1, B_2, \ldots, B_k$ be a regular system of commuting observables satisfying equation (4). Let $\mathfrak{S}$ be the set of all operators T from $\mathfrak{T}_1$ for which the equation

$$\operatorname{tr} \exp\Big(i \sum_{j=1}^{k} t_j A_j\Big) T = \operatorname{tr} \exp\Big(i \sum_{j=1}^{k} t_j B_j\Big) T \tag{5}$$

holds for all $(t_1, t_2, \ldots, t_k) \in R^k$. It is clear that $\mathfrak{S}$ is a linear subspace of $\mathfrak{T}_1$ and, by (3) and (4),

$$[\mathfrak{P}(A_1, A_2, \ldots, A_k)] \subset \mathfrak{S}.$$

Since for every $T \in \mathfrak{P}$ the right-hand side of (5) is continuous and positive definite on R^k, we infer that for every $T \in \mathfrak{P} \cap [\mathfrak{P}(A_1, A_2, \ldots, A_k)]$ the left-hand side of (5) is also continuous and positive definite on R^k. In other words, we have the inclusion

$$\mathfrak{P}(A_1, A_2, \ldots, A_k) \supset \mathfrak{P} \cap [\mathfrak{P}(A_1, A_2, \ldots, A_k)].$$

The converse inclusion is obvious, which completes the proof.

By a *pair of canonical observables* we mean a pair P, Q for which there exists a dense linear manifold D in H contained in the domains of definition of P, Q and invariant under P, Q. When restricted to D, the observables P, Q satisfy the Heisenberg commutation relation

$$PQ-QP = -iI,$$

where I is the identity operator. Moreover, the operator P^2+Q^2 on D is essentially self-adjoint. From von Neumann [7] and Dixmier [2] results it follows that the system P, Q is regular and the function $(x, y) \to \operatorname{tr} e^{i(xP+yQ)} T$ is continuous on R^2 for every $T \in \mathfrak{T}_1$ ([1], Proposition 3). For each point $z = (x, y)$ of R^2 and $T \in \mathfrak{T}_1$ we put

$$\tilde{T}(z) = \operatorname{tr} e^{i(xP+yQ)} T.$$

Then, by (3), $\hat{p}_T^{P,Q} = \tilde{T}$ for all $T \in \mathfrak{P}(P, Q)$. Consequently, $T \in \mathfrak{P}(P, Q)$ if and only if $\tilde{T}$ is positive definite on R^2. Further, it is well known ([5], Chapter 5) that the map $T \to \tilde{T}$ ($T \in \mathfrak{T}_1$) extends uniquely to a linear isometric transformation from $\mathfrak{T}_2$ onto the space $L^2(R^2)$ of all complex-valued functions on R^2 which are square integrable with respect to the Lebesgue measure. Moreover,

$$\tilde{T}^*(z) = \bar{\tilde{T}}(-z) \tag{6}$$

and

$$\widetilde{TU}(z) = \frac{1}{2\pi} \int_{R^2} \tilde{T}(w)\, \tilde{U}(z-w) \exp\left(\frac{i}{2}\Delta(w, z)\right) dw, \tag{7}$$

where $\Delta(w, z) = w_1 z_2 - w_2 z_1$ for $w = (w_1, w_2)$ and $z = (z_1, z_2)$.

Let $\mathfrak{B}$ be the subspace of $\mathfrak{T}_2$ consisting of all operators T for which $\tilde{T}(z) = f(|z|^2)$, where $|z|^2 = x^2+y^2$ if $z = (x, y)$. The map $T \to f$ is an isomorphism between Hilbert spaces $\mathfrak{B}$ and $L^2(R_+)$, where R_+ denotes the positive half-line.

LEMMA. *For every* $T \in \mathfrak{B}$ *with* $\|T\|_2 = 1$ *we have* $TT^* \in \mathfrak{P} \cap [\mathfrak{P}(P, Q)]$.

Proof. We define the operators G_a ($a \geqslant 1$) by the formula

$$\tilde{G}_a(z) = \exp\left(-\frac{a}{4}|z|^2\right).$$

It is known that G_a are Gaussian states ([5], Chapter 5), which yields $G_a \in \mathfrak{P}$ ($a \geqslant 1$). Since $\tilde{G}_a$ are positive definite on R^2, we have also

$$G_a \in \mathfrak{P}(P, Q) \quad (a \geqslant 1). \tag{8}$$

Moreover, taking into account (7), we get by a simple calculation the formula

$$G_a G_b = \frac{2}{a+b} G_{c(a,b)} \quad (a, b \geqslant 1),$$

where $c(a, b) = (1+ab)/(a+b) \geqslant 1$. Consequently, by (8),

$$G_a G_b \in [\mathfrak{P}(P, Q)]. \tag{9}$$

We recall that the linear span of the sequence $\{e^{-nt/4}\}$ $(n \geqslant 1)$ is dense in $L^2(R_+)$. Consequently, by the isomorphism between $\mathfrak{B}$ and $L^2(R_+)$, the linear span of the sequence $\{G_n\}$ $(n \geqslant 1)$ is dense in $\mathfrak{B}$. Hence it follows that for every operator $T \in \mathfrak{B}$ there exists a sequence $\{T_n\}$ of linear combinations of $G_1, G_2, \ldots$ such that $\|T_n - T\|_2 \to 0$. By (9),

$$T_n T_n^* \in [\mathfrak{P}(P, Q)] \quad (n = 1, 2, \ldots), \tag{10}$$

and, by (2),

$$\|T_n T_n^* - TT^*\|_1 \leqslant \|T_n\|_2 \|T_n - T\|_2 + \|T\|_2 \|T_n - T\|_2,$$

which yields $\|T_n T_n^* - TT^*\|_1 \to 0$ and, by (10), $TT^* \in [\mathfrak{P}(P, Q)]$. If in addition $\|T\|_2 = 1$, then $TT^* \in \mathfrak{P}$, which completes the proof.

Proposition 2. *We have*

$$\mathfrak{P}(P, Q) \neq \mathfrak{P} \cap [\mathfrak{P}(P, Q)].$$

Proof. Let U be the operator from $\mathfrak{T}_2$ defined by the equation

$$\tilde{U}(z) = (1 - \tfrac{1}{2}|z|^2)\exp(-\tfrac{1}{4}|z|^2).$$

By (6) the operator U is self-adjoint. Moreover, using formula (7), by a simple calculation we get $\tilde{U}^2 = \tilde{U}$, which yields $U^2 = U$. Thus U is a projector and, consequently, $\|U\|_2 = \|U\|_1 = \tilde{U}(0) = 1$. Applying the Lemma we have the relation

$$U \in \mathfrak{P} \cap [\mathfrak{P}(P, Q)].$$

Suppose that $U \in \mathfrak{P}(P, Q)$. Then $\tilde{U}$ is positive definite on R^2 and, consequently, its Fourier transform h is continuous and non-negative. But, by a simple calculation, we have

$$h(0) = \int_{R^2} (1 - \tfrac{1}{2}|z|^2)\exp(-\tfrac{1}{4}|z|^2)\,dz = -4\pi,$$

which shows that $U \notin \mathfrak{P}(P, Q)$. The proposition is thus proved.

From Propositions 1 and 2 we get the following

Corollary. *The pair P, Q of canonical observables does not fulfil the probabilistic commutation condition.*

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