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ON CLASSIFICATION, LOGICAL EDUCATIONAL MATERIALS, CATEGORIES, AND AUTOMATA

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The purpose of this paper is to propose an abstract mathematical concept of a kit motivated by some problems of classifying things according to their features and, in particular, by kits used in teaching logic and set theory at a kindergarten or primary school level.

The category of kits is discussed in Section 4 and an explicit form of a free kit is established.

In Section 5 there is a construction of a canonical functor from the category of Mealy automata to the category of kits, which is a left adjoint of the corresponding forgetful functor.

Section 6 is devoted to relations between kits and Brainerd's mathematical model of a phonetic system.

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1. First motivation: logical educational materials. One of the characteristic features of modern mathematics teaching in kindergartens and primary schools is an extensive use of various logical kits, i.e., factory-made or home-made kits designed for special activities to help the child to conceive some basic notions of set theory and logic. During a game or another activity the set of things of the given kit plays the role of a universe and the children are concerned with various subsets of this universe, their intersections, unions, complements, etc.

The most popular kit (the kit of Dienes logiblocks, cf. [4]) consists of 48 plastic blocks; each block has four easily distinguished features: colour (a block may be either red, or blue, or yellow), shape (circle, triangle, square, oblong), size (large, small), and thickness (thick, thin). Moreover, there is exactly one block having any given combination of these features. Thus, a mathematical model for this kit is the Cartesian product

$$X = Y_{\text{colour}} \times Y_{\text{shape}} \times Y_{\text{size}} \times Y_{\text{thickness}}, \tag{1.1}$$

where

$$Y_{\text{colour}} = \{\text{red, blue, yellow}\}, \quad Y_{\text{shape}} = \{\bigcirc, \triangle, \square, \square\},$$

$$Y_{\text{size}} = \{\text{large, small}\}, \quad Y_{\text{thickness}} = \{\text{thick, thin}\}.$$

Logical kits listed in catalogues of various firms may differ one from another in the choice of things and their features (they may be, e.g., kits of dolls classified according to sex, age, clothes, etc.), but, otherwise, they almost invariably follow the same pattern of a Cartesian product of two, three or four (rarely more) given sets. Also, in educational reports, too little attention seems to be paid to more general kits, though, from a didactical point of view, a systematic use of a Cartesian product as a universe is not desirable and the children should be given opportunities to work with other types of kits as well.

However, if one wants to start a systematic investigation of merits and demerits of various types of kits, a natural question arises: What is actually meant by a kit? Certainly, it is not just a set of things, since it is essential that these things could be classified according to some distinguishable features.

The following definition of a kit is motivated by, but not confined to, logical kits which have been used, or may be useful, in teaching mathematics.

Definition. A *kit* is a quadruple

$$\mathfrak{X} = \big(X, S, (Y_s)_{s\in S}, (\lambda_s)_{s\in S}\big), \tag{1.2}$$

where X is a set whose elements are called *things*, S is a set whose elements are called *features*, Y_s is the set of *admissible values* of the feature s, and

$$\lambda_s\colon X \to Y_s \tag{1.3}$$

is a function which assigns to each thing x the actual s-feature of this thing.

In the example described before, S is the set

$$S = \{\text{colour, shape, size, thickness}\} \tag{1.4}$$

and λ_s is just the s-th coordinate projection (e.g., if x is the blue-square-small-thin block, then $\lambda_{\text{colour}}(x) = \text{blue}$ and $\lambda_{\text{size}}(x) = \text{small}$).

It should be stressed that the set X itself does not determine the kit. We can, e.g., consider a kit in which the set of things is given by (1.1), and the set of features is given by (1.4), but one of the sets Y_s is different, namely

$$Y_{\text{colour}} = \{\text{warm, cold}\}$$

with $\lambda_{\text{colour}}(x) = \text{cold}$ if x is blue and $\lambda_{\text{colour}}(x) = \text{warm}$ if x is either red or yellow. Still another kit can be obtained if one considers the set (1.1) of things with another set of features, e.g.

$$S = \{\text{shape, size, thickness}\},$$

the colour being neglected as if the blocks were for colour-blind children. There are also two extreme cases (not interesting from an educational point of view): the set (1.1) of 48 elements regarded as a kit with 48 features (each feature having values 0 and 1) and the set (1.1) with one feature of 48 values.

The maps (1.3) need not be surjections; as an example, one can consider the set (1.1) of things with the set (1.4) of features and

$$Y_{\text{colour}} = \{\text{red, blue, yellow, green}\},$$

though none of the blocks is actually green.

2. Second motivation: classification according to certain data. The abstract notion of a kit can well turn out to be useful in some mathematical aspects of methodology of science and in theoretical problems concerning the use of computers for recognizing or classifying given things. In fact, related concepts (usually in non-formalized forms) have already been used in some papers.

There are numerous natural examples of kits in problems of applied mathematics. One of them is discussed in Section 6. Let us now consider three other examples.

Suppose, first, that X is a set of railway tickets,

$$S = \{\text{initial station, terminal station, distance, price, date}, \ldots\};$$

if s is, e.g., "distance", then Y_s is the set of possible distances between stations; if $x \in X$ and $s \in S$, then $\lambda(s, x)$ is the value of the feature s printed on the ticket x, e.g., if s is "terminal station", then $\lambda(s, x)$ is the name of the terminal station printed on the ticket x and Y_s is the set of admissible terminal stations (if there are alternative terminal stations printed on some tickets, then $\lambda(s, x)$ can be the set of terminal stations for which the ticket x is valid and Y_s can consist of sets of stations rather than of single stations). This yields a kit (1.2).

The responses to a questionnaire can also be regarded as a kit: X is the set of people who have filled up the inquiry sheets, S – the set of questions to be answered; if $s \in S$, then Y_s is the set of admissible answers (including, e.g., "not answered"); if $s \in S$ and $x \in X$, then $\lambda(s, x)$ is the answer to the question s written by the person x.

Another example of a kit is described as follows. S is a set consisting of 7 elements represented by bars in Fig. 1. For each s in S, the set Y_s

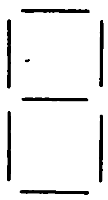


Fig. 1

consists of two numbers: 0 and 1. The set X consists of 10 elements which are subsets of S shown in Fig. 2 by heavy bars, and for each x in X the function $s \mapsto \lambda_s(x)$ is just the characteristic function of the set x.

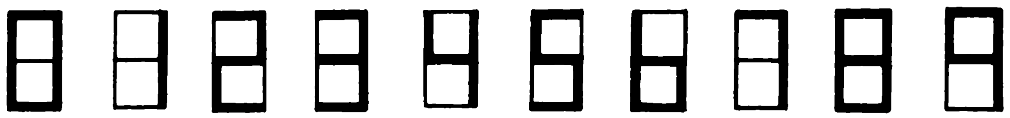


Fig. 2

The discussion of computer-science merits of kits was recently taken up in [5] and [9].

3. Some basic types of kits. Let $\mathfrak{X}$ be an arbitrary kit (1.2). The canonical map

$$(3.1) \qquad \lambda^{\nabla}: X \to \mathop{\mathsf{P}}_{s \in S} Y_s$$

is defined by $\lambda^{\nabla}(x) = \big(\lambda_s(x)\big)_{s\in S}$.

A kit $\mathfrak{X}$ will be called a *monokit* if the map (3.1) is an injection. This means that for any x, x' in X with $x \neq x'$ there is an s in S such that $\lambda_s(x) \neq \lambda_s(x')$, i.e., there are enough features to distinguish the things. Kits of railway tickets are not monokits (there are several tickets with the same text printed on them). The kit shown in Fig. 2 is, obviously, a monokit.

An *epikit* is a kit such that (3.1) is a surjection, i.e., for every family $(y_s)_{s\in S}$ with y_s in Y_s, there is an x in X such that $y_s = \lambda_s(x)$ for s in S. (F. W. Lawvere has pointed out that this notion is related to Platonism: every "idea of a thing" $(y_s)_{s\in S}$ is realized by a thing x.)

An epikit which is not a monokit can be obtained by taking two sets of plastic blocks described in Section 1 and putting them together. In such an epikit there can be two blocks not distinguishable by scrutinizing. One can also produce a kit similar to the above one, but having, e.g., several red-square-small-thin blocks such that a child can distinguish them since the colours of the red blocks are not identical (though all are classified as red).

A *product kit* is a kit which is both a monokit and an epikit, i.e., a kit such that (3.1) is a bijection. As we have already mentioned, most commercial logical kits for kindergartens are product kits.

4. The category of kits.

Definition. By a *morphism* from a kit (1.2) to a kit

$$\mathfrak{X}' = \big(X', S', (Y'_s)_{s\in S'}, (\lambda'_s)_{s\in S'}\big)$$

we mean a triple $\big(\xi, \sigma, (\eta_s)_{s\in S}\big)$, where

$$\xi\colon X\to X', \qquad \sigma\colon S\to S', \qquad \eta_s\colon Y_s\to Y'_{\sigma(s)} \tag{4.1}$$

are maps such that

$$\forall_{x\in X}\ \forall_{s\in S}\ \eta_s\lambda_s(x) = \lambda'_{\sigma(s)}\xi(x).$$

For instance, the map (3.1) yields a morphism from the kit $\mathfrak{X}$ to a product kit. The obvious function

$$\{\text{red, yellow, blue}\}\to\{\text{warm, cold}\}$$

also gives rise to a morphism of the corresponding kits.

The kits $\mathfrak{X}$ and $\mathfrak{X}'$ are *isomorphic* if the maps (4.1) are bijections; if this is the case, then the inverse maps

$$\big(\xi^{-1}, \sigma^{-1}, (\eta^{-1}_{\sigma^{-1}(s')})_{s'\in S'}\big)$$

also form a morphism of kits.

It is clear that the kits and their morphisms form a category. This category will be the subject of the rest of the paper. For unexplained categorical terms see, e.g., [6] and [8].

For categorical purposes, however, it will be convenient to make a formal modification of the definition of a kit. Let Y be the sum, i.e. the disjoint union, of the sets Y_s $(s\in S)$, let $\lambda(s, x)$ be the element $\lambda_s(x)$ regarded as an element of Y, and let p be the map which assings to each y in Y the corresponding index s. Thus

$$\forall_{x\in X}\ \forall_{s\in S}\ p\lambda(s, x) = s.$$

By a *kit* we now understand a quintuple

$$\mathfrak{X} = (X, S, Y, \lambda, p), \tag{4.2}$$

where X, S, Y are sets, and $\lambda\colon S\times X\to Y$ and $p\colon Y\to S$ are functions such that the diagram

$$\begin{tikzcd} S\times X \arrow[rr, "\lambda"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "p"] \\ & S & \end{tikzcd} \tag{4.3}$$

is commutative, i.e.,

$$p\lambda = \pi, \tag{4.4}$$

$\pi\colon S\times X\to S$ being the first-coordinate projection. If $X\neq 0$, then p is a surjection.

A morphism from a kit (4.2) to an analogous kit

$$\mathfrak{X}' = (X', S', Y', \lambda', p')$$

is a triple (ξ, σ, η) of maps $\xi\colon X\to X'$, $\sigma\colon S\to S'$ and $\eta\colon Y\to Y'$ such that the diagram

$$\begin{tikzcd} S\times X \arrow[rr, "\lambda"] \arrow[dr, "\pi"] \arrow[dd, "\sigma\times\xi"'] & & Y \arrow[dl, "p"'] \arrow[dd, "\eta"] \\ & S \arrow[dd, "\sigma"] & \\ S'\times X' \arrow[rr, "\lambda'"] \arrow[dr, "\pi'"'] & & Y' \arrow[dl, "p'"] \\ & S' & \end{tikzcd} \tag{4.5}$$

is commutative, where $(\sigma\times\xi)(s, x) = \big(\sigma(s), \xi(x)\big)$.

It is obvious that the new definition of a kit and that of a morphism of kits are essentially equivalent to the former ones. These new definitions have been suggested by F. W. Lawvere in order to make it clear that the category **Kt** of kits and their morphisms is an equationally definable category of algebras over three sets (X, S, Y) with the only equational axiom (4.4). From well-known theorems concerning such categories (see, e.g., [7] and [10]) it follows that **Kt** is complete (with respect to equalizers, coequalizers, products and coproducts of small families of objects) and the forgetful functor

$$U\colon \mathbf{Kt}\to \mathbf{Ens}^3, \tag{4.6}$$

defined by $U(X, S, Y, \lambda, p) = (X, S, Y)$ and $U(\xi, \sigma, \eta) = (\xi, \sigma, \eta)$, has a left adjoint

$$F\colon \mathbf{Ens}^3\to\mathbf{Kt}.$$

(**Ens** denotes the category of sets, and $\mathbf{Ens}^3$ is $\mathbf{Ens}\times\mathbf{Ens}\times\mathbf{Ens}$.) However, the general proof of existence gives a rather complicated description of F and it seems worth-while to establish directly an explicit form of a free kit

$$F(T_1, T_2, T_3) = (X^\odot, S^\odot, Y^\odot, \lambda^\odot, p^\odot). \tag{4.7}$$

By definition, a *free kit* is a kit (4.7) together with three maps

$$\tau_1\colon T_1\to X^\odot,\qquad \tau_2\colon T_2\to S^\odot,\qquad \tau_3\colon T_3\to Y^\odot$$

such that, for each kit (4.2) and each triple of maps

(4.8) $$\varphi_1\colon T_1 \to X, \qquad \varphi_2\colon T_2 \to S, \qquad \varphi_3\colon T_3 \to Y,$$

there is a unique morphism

(4.9) $$(\xi, \sigma, \eta)\colon (X^{\odot}, S^{\odot}, Y^{\odot}, \lambda^{\odot}, p^{\odot}) \to (X, S, Y, \lambda, p)$$

of kits such that (ξ, σ, η) composed with (τ_1, τ_2, τ_3) is $(\varphi_1, \varphi_2, \varphi_3)$, i.e.,

(4.10) $$\xi\tau_1 = \varphi_1, \qquad \sigma\tau_2 = \varphi_2, \qquad \eta\tau_3 = \varphi_3.$$

THEOREM 1. *The free kit $F(T_1, T_2, T_3)$ is (isomorphic to) the quintuple*

(4.11) $$\big(T_1,\ T_2+T_3,\ [(T_2+T_3)\times T_1]+T_3,\ \lambda^{\odot},\ p^{\odot}\big)$$

with the obvious functions

(4.12) $$\lambda^{\odot}\colon (T_2+T_3)\times T_1 \to [(T_2+T_3)\times T_1]+T_3,$$

(4.13) $$p^{\odot}\colon [(T_2+T_3)\times T_1]+T_3 \to T_2+T_3,$$

(4.14) $$\tau_1\colon T_1 \to T_1,$$

(4.15) $$\tau_2\colon T_2 \to T_2+T_3,$$

(4.16) $$\tau_3\colon T_3 \to [(T_2+T_3)\times T_1]+T_3.$$

Here $+$ denotes the sum of sets (coproduct in **Ens**). The "obvious" map (4.13) is the unique map which yields the first-coordinate projection on $(T_2+T_3)\times T_1$ and the identity on T_3; the maps (4.12), (4.15) and (4.16) are the canonical embeddings into the respective sums while (4.14) is the identity on T_1.

Proof. Let $\mathfrak{X}$ be any kit (4.2) and let (4.8) be any triple of maps. We are looking for a unique morphism (4.9) satisfying (4.10). The condition that (4.9) be a morphism in **Kt** is that the diagram

(4.17)
$$\begin{tikzcd}
(T_2+T_3)\times T_1 \arrow[rr, "\lambda^{\odot}"] \arrow[dr, "\pi^{\odot}"] \arrow[dd, "\sigma\times\xi"'] & & {[(T_2+T_3)\times T_1]+T_3} \arrow[dl, "p^{\odot}"'] \arrow[dd, "\eta"] \\
 & T_2+T_3 \arrow[dd, "\sigma"] & \\
S\times X \arrow[rr, "\lambda"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "p"] \\
 & S &
\end{tikzcd}$$

be commutative. The commutativity of the bottom triangle follows from the assumption that $\mathfrak{X}$ is a kit; the commutativity of the upper triangle, i.e., the condition that (4.11) is a kit, is obvious. We are to find maps

$$\xi\colon T_1 \to X, \qquad \sigma\colon T_2+T_3 \to S, \qquad \eta\colon [(T_2+T_3)\times T_1]+T_3 \to Y$$

such that $\eta\lambda^{\odot} = \lambda(\sigma\times\xi)$ and $p\eta = \sigma p^{\odot}$ (the condition $\sigma\pi^{\odot} = \pi(\sigma\times\xi)$ is satisfied for any choice of ξ and σ). We can assume, for simplicity, that the sets T_1, T_2, T_3 are pairwise disjoint and the sums in (4.11)-(4.16) are just unions. Write

(4.18) $$\xi(t_1) = \varphi_1(t_1) \quad \text{for } t_1 \text{ in } T_1,$$

(4.19) $$\sigma(t_2) = \varphi_2(t_2) \quad \text{for } t_2 \text{ in } T_2,$$

(4.20) $$\sigma(t_3) = p\varphi_3(t_3) \quad \text{for } t_3 \text{ in } T_3,$$

(4.21) $$\eta(t_2, t_1) = \lambda\big(\varphi_2(t_2), \varphi_1(t_1)\big) \quad \text{for } (t_2, t_1) \text{ in } T_2\times T_1,$$

(4.22) $$\eta(t_3, t_1) = \lambda\big(p\varphi_3(t_3), \varphi_1(t_1)\big) \quad \text{for } (t_3, t_1) \text{ in } T_3\times T_1,$$

(4.23) $$\eta(t_3) = \varphi_3(t_3) \quad \text{for } t_3 \text{ in } T_3.$$

A straightforward verification shows that the maps ξ, σ and η are well defined and yield the desired morphism (4.9), which is unique. Specifically, equations (4.18), (4.19) and (4.23) imply, and are implied by, equations (4.10) while (4.20), (4.21) and (4.22) imply, and are implied by, the condition that the diagram (4.17) be commutative.

COROLLARY 1. *The free kits generated by the objects* $(1, 0, 0)$, $(0, 1, 0)$ *and* $(0, 0, 1)$ *of* $\mathbf{Ens}^3$ *are*

$$F(1,0,0) = (1,0,0,\ldots), \qquad F(0,1,0) = (0,1,0,\ldots),$$
$$F(0,0,1) = (0,\ 1,1,\ldots).$$

Here 0 denotes the empty set, $1 = \{0\}$, and the dots stand for the corresponding maps $\lambda^{\odot}$ and $p^{\odot}$.

Let us now consider the three forgetful functors

(4.24) $$U_1\colon \mathbf{Kt}\to\mathbf{Ens}, \qquad U_2\colon \mathbf{Kt}\to\mathbf{Ens}, \qquad U_3\colon \mathbf{Kt}\to\mathbf{Ens}$$

obtained by composing the functor (4.6) with three projection functors from $\mathbf{Ens}^3$ to $\mathbf{Ens}$. Thus

$$U_1(X, S, Y, \lambda, p) = X, \qquad U_2(X, S, Y, \lambda, p) = S,$$
$$U_3(X, S, Y, \lambda, p) = Y.$$

Since the projection functors have left adjoints determined by

$$T_1 \mapsto (T_1, 0, 0), \qquad T_2 \mapsto (0, T_2, 0), \qquad T_3 \mapsto (0, 0, T_3),$$

respectively, we get the following corollary:

COROLLARY 2. *The left adjoints of the functors* (4.24) *are determined by the correspondences*:

$$T_1 \mapsto F(T_1, 0, 0) = (T_1, 0, 0, \ldots), \quad T_2 \mapsto F(0, T_2, 0) = (0, T_2, 0, \ldots),$$
$$T_3 \mapsto F(0, 0, T_3) = (0, T_3, T_3, \ldots).$$

In a similar way, we infer that a left adjoint of the forgetful functor from **Kt** to **Ens**^{2} determined by $(X, S, Y, \lambda, p) \mapsto (X, S)$ is the functor from **Ens**^{2} to **Kt** determined by

$$(T_1, T_2) \mapsto F(T_1, T_2, 0) = (T_1, T_2, T_2 \times T_1, \ldots).$$

Recently, Wiweger [11] found explicit forms of coproducts and coequalizers in **Kt** (products and equalizers are obvious); in [12] he applied a similar technique to find coproducts of automata.

5. Relations between kits and automata. By an *automaton* we mean a deterministic Mealy automaton, i.e., a quintuple

$$(X, S, Y, \lambda, \delta), \tag{5.1}$$

where X, S, Y are sets, and $\lambda: S \times X \to Y$ and $\delta: S \times X \to S$ are any functions (see [1], p. 57, [6] and [3]). X is called the *input alphabet*, S the *set of states*, Y the *output alphabet*, λ the *next-output function*, and δ the *next-state function*.

A morphism from (5.1) to an automaton $(X', S', Y', \lambda', \delta')$ is a triple (ξ, σ, η) such that the diagram

$$\begin{tikzcd} S \arrow[d, "\sigma"'] & S \times X \arrow[l, "\delta"'] \arrow[r, "\lambda"] \arrow[d, "\sigma \times \xi"] & Y \arrow[d, "\eta"] \\ S' & S' \times X' \arrow[l, "\delta'"] \arrow[r, "\lambda'"'] & Y' \end{tikzcd} \tag{5.2}$$

is commutative. In this way we get the category **Au** of automata and their morphisms.

Any kit (4.2) determines an automaton

$$\square (X, S, Y, \lambda, p) = (X, S, Y, \lambda, \pi) \tag{5.3}$$

in which the role of the function δ is now played by the first-coordinate projection $\pi: S \times X \to S$. Therefore, the automaton (5.3) has a very special property: no input signal can change the state. If the automaton is in a state s (i.e., a feature s has been fixed), then, for any given thing x in the input alphabet, the automaton gives the signal $\lambda(s, x)$ at the output.

Thus, from a formal point of view, a kit can be interpreted as an automaton which can be set in any given state s and then, if a thing x appears as an input signal, the automaton "reads" the s-feature of this thing and the result appears at the output. In other words, a kit is an automaton to recognize certain features of given things.

We can say that a kit is a commutative diagram (4.3). If we erase the map p in that diagram, we get an automaton. Similarly, a morphism in **Kt** is a commutative diagram (4.5); if we erase p and p' in (4.5), we get a diagram of the form (5.2). Consequently, there is a forgetful functor

$$\square\colon \mathbf{Kt} \to \mathbf{Au}. \tag{5.4}$$

The rest of this section will be devoted to an explicit construction of a left adjoint of the functor (5.4).

Suppose that (5.1) is an automaton. We define a kit

$$(X, S^*, Y^*, \lambda^*, p^*) \tag{5.5}$$

as follows. The set S^* is the sum

$$S^* = (Y \setminus Y_1) + S/\sim,$$

where $Y_1 = \{\lambda(s, x)\colon (s, x) \in S \times X\}$, and $\sim$ is the smallest equivalence relation on S satisfying the conditions

$$\Big[\underset{x', x'' \in X}{\exists} \lambda(s, x') = \lambda(s', x'')\Big] \Rightarrow (s \sim s'), \tag{5.6}$$

$$\underset{x \in X}{\forall}\ \underset{s \in S}{\forall}\ \delta(s, x) \sim s. \tag{5.7}$$

The set Y^* is the quotient $Y/\approx$, where $\approx$ is the smallest equivalence relation on Y such that

$$\underset{s, s' \in S}{\forall} (s \sim s') \Rightarrow \Big[\underset{x \in X}{\forall} \lambda(s, x) \approx \lambda(s', x)\Big]. \tag{5.8}$$

Let us note that the relation $\approx$ does not identify any points of $Y \setminus Y_1$; more precisely,

$$\underset{y \in Y \setminus Y_1}{\forall}\ \underset{y' \in Y}{\forall} (y \approx y') \Rightarrow (y = y').$$

The function $\lambda^*\colon S^* \times X \to Y^*$, i.e.,

$$\lambda^*\colon [(Y \setminus Y_1) + S/\sim] \times X \to Y/\approx,$$

is defined by

$$\begin{aligned} \lambda^*(y, x) &= \{y\} && \text{for } y \text{ in } Y \setminus Y_1,\ x \text{ in } X, \\ \lambda^*(s/\sim, x) &= \lambda(s, x)/\approx && \text{for } s \text{ in } S,\ x \text{ in } X. \end{aligned} \tag{5.9}$$

By (5.8), this definition does not depend on the choice of s in the equivalence class $s/\sim$.

The function $p^*\colon Y^* \to S^*$ is defined by

$$(5.10) \qquad p^*(y/\approx) = \begin{cases} y & \text{if } y \in Y \setminus Y_1, \\ s/\sim & \text{if } y = \lambda(s, x) \in Y_1. \end{cases}$$

This definition is independent of the choice of y in $y/\approx$ and of the choice of (s, x) such that $\lambda(s, x) = y$. Indeed, suppose that $(s', x') \in S \times X$ and $\lambda(s, x) \approx \lambda(s', x')$. Since $\approx$ is the smallest equivalence relation satisfying (5.8), there exist finite sequences

$$(s_0, s_1, \ldots, s_n), \quad (s_0', s_1', \ldots, s_n') \quad \text{and} \quad (x_0, x_1, \ldots, x_n)$$

of elements of S and X, respectively, such that

$$(s_0, x_0) = (s, x), \quad (s_n', x_n) = (s', x'),$$

$$s_0 \sim s_0', \quad s_1 \sim s_1', \quad \ldots, \quad s_n \sim s_n',$$

$$\lambda(s_0', x_0) = \lambda(s_1, x_1), \quad \lambda(s_1', x_1) = \lambda(s_2, x_2),$$

$$\ldots, \quad \lambda(s_{n-1}', x_{n-1}) = \lambda(s_n, x_n).$$

Consequently, by (5.6),

$$s_0' \sim s_1, \quad s_1' \sim s_2, \quad \ldots, \quad s_{n-1}' \sim s_n,$$

and hence $s_0 \sim s_n'$, i.e., $s \sim s'$.

From (5.9) and (5.10) it follows that $p^*\lambda^*(s, x) = s$ for s in S^*. This means that (5.5) is a kit.

The construction of the kit (5.5) can be interpreted as follows. (5.6) means that two states s and s' of the given automaton (5.1) are identified in each case where they have at least one output signal in common (even for different input signals). According to (5.7), if the automaton can change from a state s to a state s' (after having received some input signal x), then s and s' are also to be identified. As a consequence, if states s and s' have been identified in the way described above and the output signals y and y' are obtained from the same input signal in states s and s', respectively, then y is to be identified with y'.

(5.5) can be regarded as an automaton classifying the elements of the input alphabet depending on the way the automaton (5.1) acts upon them.

The above-given construction of the automaton (5.5) from the automaton (5.1) can be simulated as follows: keep the input alphabet without any change; switch off the device δ which changes the states of the automaton (5.1); leave precisely one state s of each equivalence class $s/\sim$ (and abandon the other states); regard those elements of the original output alphabet which can never be output signals (for any x and s whatsoever) as additional states of the new automaton; and identify (with a coding device) those pairs of elements of the output alphabet which satisfy $y \approx y'$.

THEOREM 2. *The correspondence*

$$(X, S, Y, \lambda, \delta) \mapsto (X, S^*, Y^*, \lambda^*, p^*)$$

is the object transformation of a covariant functor from **Au** *to* **Kt** *which is a left adjoint of the functor* (5.4).

Proof. The canonical morphism from $(X, S, Y, \lambda, \delta)$ to the automaton $\square(X, S^*, Y^*, \lambda^*, p^*)$ is the triple (τ_1, τ_2, τ_3), where $\tau_1\colon X \to X$ is the identity while

$$\tau_2\colon S \to (Y \setminus Y_1) + S/\sim \quad \text{and} \quad \tau_3\colon Y \to Y/\approx$$

are quotient maps. This triple is a morphism in **Au** since the diagram

$$\begin{tikzcd} S \arrow[d, "\tau_2"'] & S \times X \arrow[l, "\delta"'] \arrow[r, "\lambda"] \arrow[d, "\tau_2 \times \tau_1"] & Y \arrow[d, "\tau_3"] \\ S^* & S^* \times X \arrow[l, "\pi^*"] \arrow[r, "\lambda^*"'] & Y^* \end{tikzcd}$$

is commutative (the commutativity of the left-hand square follows from (5.7)).

Let $(X', S', Y', \lambda', p')$ be any kit and let

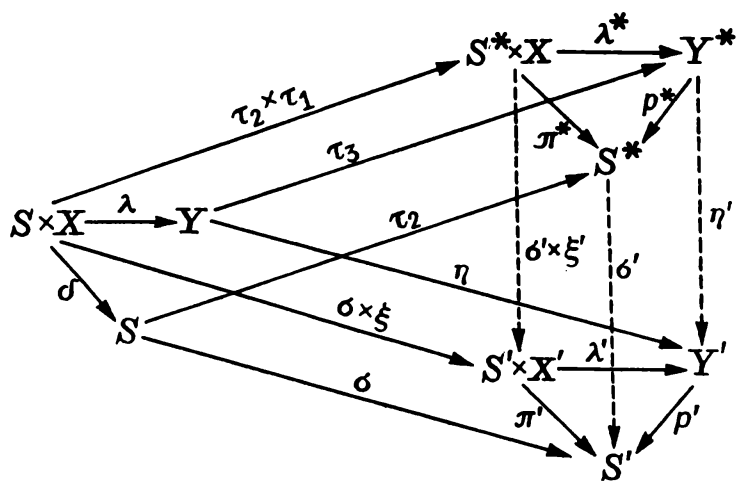
(5.11) $$(\xi, \sigma, \eta)\colon (X, S, Y, \lambda, \delta) \to \square(X', S', Y', \lambda', p')$$

be an **Au**-morphism. We claim that there is a unique **Kt**-morphism

$$(\xi', \sigma', \eta')\colon (X, S^*, Y^*, \lambda^*, p^*) \to (X', S', Y', \lambda', p')$$

such that the diagram

(5.12)



is commutative, i.e., $\xi'\tau_1 = \xi$, $\sigma'\tau_2 = \sigma$, $\eta'\tau_3 = \eta$. The maps ξ', σ', η' are defined as follows:

$$\xi'(x) = \xi(x) \quad \text{for } x \text{ in } X,$$

$$\sigma'(y) = \eta(y) \quad \text{for } y \text{ in } Y \setminus Y_1,$$

$$\sigma'(s/\sim) = \sigma(s) \quad \text{for } s \text{ in } S,$$

$$\eta'(y/\approx) = \eta(y) \quad \text{for } y \text{ in } Y.$$

We have to show that the function σ' is well defined, i.e.,

$$s \sim s' \text{ implies } \sigma(s) = \sigma(s'). \tag{5.13}$$

We recall that $s \sim s'$ is determined by (5.6) and (5.7). Suppose, first, that $\lambda(s, x') = \lambda(s', x'')$ for some x', x'' in X. Since (5.11) is a morphism of automata, from the commutativity of the right-hand square of the diagram (5.2) it follows that

$$\lambda'(\sigma(s),\ \xi(x')) = \eta\lambda(s, x') = \eta\lambda(s', x'') = \lambda'(\sigma(s'),\ \xi(x'')),$$

and hence

$$\sigma(s) = p'\lambda'(\sigma(s),\ \xi(x')) = p'\lambda'(\sigma(s'),\ \xi(x'')) = \sigma(s').$$

Suppose now that $\delta(s, x) = s'$. We again make use of the assumption that (5.11) is a morphism. In view of $\sigma\delta = \pi'(\sigma \times \xi)$, we infer that

$$\sigma\delta(s, x) = \pi'(\sigma(s),\ \xi(x)) = \sigma(s), \tag{5.14}$$

i.e., $\sigma(s) = \sigma(s')$. Since $\sim$ is the smallest equivalence relation satisfying (5.6) and (5.7), we get (5.13).

The function η' is also well defined. Indeed, by (5.8), the relation $\approx$ is the smallest equivalence relation satisfying the following conditions:

$$\Big[\underset{x', x'' \in X}{\exists} \lambda(s, x') = \lambda(s', x'')\Big] \Rightarrow \Big[\underset{x \in X}{\forall} \lambda(s, x) \approx \lambda(s', x)\Big], \tag{5.15}$$

$$\underset{x' \in X}{\forall}\ \underset{x \in X}{\forall} \lambda(s, x) \approx \lambda(\delta(s, x'), x). \tag{5.16}$$

Suppose that $y \approx y'$. If $y = \lambda(s, x)$, $y' = \lambda(s', x)$ and $\lambda(s, x') = \lambda(s', x'')$, then, in virtue of (5.13),

$$\eta(y) = \eta\lambda(s, x) = \lambda'(\sigma(s),\ \xi(x)) = \lambda'(\sigma(s'),\ \xi(x)) = \eta\lambda(s', x) = \eta(y').$$

On the other hand, if $y = \lambda(s, x)$ and $y' = \lambda(\delta(s, x'), x)$, then, by equation (5.14),

$$\begin{aligned}\eta(y') &= \eta\lambda(\delta(s, x'), x) = \lambda'(\sigma\delta(s, x'),\ \xi(x)) \\ &= \lambda'(\sigma(s),\ \xi(x)) = \eta\lambda(s, x) = \eta(y).\end{aligned}$$

Thus, if y is identified with y', in view of (5.15) or (5.16), we have $\eta(y) = \eta(y')$. Therefore, η is constant on each equivalence class $y/\approx$ and the function η' is well defined.

It is now trivial to verify that ξ', σ' and η' defined above are unique functions for which (5.12) is commutative.

6. Relations between kits and some mathematical models of classical linguistic structures. Brainerd ([2], p. 72) defines an *etic system* ([1]) as a quadruple

(6.1) $$(X, Y, \boldsymbol{P}, \varphi),$$

where X is a non-empty finite set of *signs*, Y is a non-empty finite set of *features*, $\boldsymbol{P}$ is a partition of Y, and $\varphi: X \to 2^Y$ is a function which assigns to each x in X a subset $\varphi(x)$ of Y such that, for each P in $\boldsymbol{P}$, the intersection $\varphi(x) \cap P$ is either empty or contains exactly one element.

A *weak homomorphism* from an etic system (6.1) to an etic system

(6.2) $$(X', Y', \boldsymbol{P}', \varphi')$$

is a pair of maps $\xi\colon X \to X'$ and $\eta\colon Y \to Y'$ such that

(6.3) $$\underset{x \in X}{\forall} \varphi'\big(\xi(x)\big) \cap \eta(Y) = \eta\big(\varphi(x)\big)$$

and

(6.4) $$\underset{y_1, y_2 \in Y}{\forall} [p(y_1) = p(y_2)] \Rightarrow [p'\eta(y_1) = p'\eta(y_2)],$$

where $p\colon Y \to \boldsymbol{P}$ and $p'\colon Y' \to \boldsymbol{P}'$ are the respective quotient maps.

We shall show that finite kits can be interpreted as etic systems of a special kind. Technically, let $\mathbf{Kt}_\mathrm{f}$ denote the full subcategory of $\mathbf{Kt}$ consisting of those kits (4.2) for which X, S, Y are finite and non-empty and let $\mathbf{Et}$ denote the category of etic systems and weak homomorphisms. We define a functor

(6.5) $$\mathscr{K}\colon \mathbf{Kt}_\mathrm{f} \to \mathbf{Et}$$

as follows: Let $\mathscr{K}(X, S, Y, \lambda, p)$ denote the etic system $(X, Y, \boldsymbol{P}, \varphi)$, where $\boldsymbol{P}$ is the partition $\{p^{\leftarrow}(s)\}_{s \in S}$ of Y and

$$\varphi(x) = \{\lambda(s, x)\}_{s \in S} \quad \text{for } x \text{ in } X.$$

Let (ξ, σ, η) be a morphism from (X, S, Y, λ, p) to $(X', S', Y', \lambda', p')$ in $\mathbf{Kt}_\mathrm{f}$. Then (ξ, η) is a weak homomorphism of the corresponding etic systems. Indeed, suppose that y' belongs to the left-hand side of (6.3).

([1]) An *etic system* is meant to be an abstract model for phonetic systems while an *emic system* is meant in [2] to be a model for phonemic systems.

Then there is a y in Y such that $y' = \eta(y)$ and there is an s' in S' such that $y' = \lambda'(s', \xi(x))$. Hence

$$s' = p'\lambda'(s', \xi(x)) = p'\eta(y) = \sigma p(y)$$

and

$$y' = \lambda'(\sigma p(y), \xi(x)) = \eta\lambda(p(y), x) \in \eta(\varphi(x)).$$

Consequently, (6.3) must hold. Condition (6.4) is an immediate consequence of the condition $p'\eta = \sigma p$. This yields the desired functor (6.5), which is obviously faithful.

Conversely, let $\mathbf{Et}_1$ denote the full subcategory of $\mathbf{Et}$ consisting of those etic systems (6.1) which satisfy the condition $\varphi(x) \cap P \neq 0$ for all x in X and P in $\boldsymbol{P}$, i.e., such that each set $\varphi(x) \cap P$ is a singleton $\{\lambda(P, x)\}$. In this case, let $\mathscr{U}_1(X, Y, \boldsymbol{P}, \varphi)$ denote the kit $(X, \boldsymbol{P}, Y, \lambda, p)$; if (ξ, η) is a weak homomorphism from (6.1) to (6.2), then it follows from (6.4) that there is a unique map $\sigma\colon \boldsymbol{P} \to \boldsymbol{P}'$ such that $\sigma p = p'\eta$ (specifically, $\sigma(P) = p'\eta(y)$ for any y in P). We claim that (ξ, σ, η) is a $\mathbf{Kt}$-morphism from $\mathscr{U}_1(X, Y, \boldsymbol{P}, \varphi)$ to $\mathscr{U}_1(X', Y', \boldsymbol{P}', \varphi')$. Indeed, if $x \in X$ and $P \in \boldsymbol{P}$, then, by (6.3), $\eta\lambda(P, x)$ belongs to $\varphi'(\xi(x))$, i.e., $\eta\lambda(P, x) = \lambda'(P', \xi(x))$ for some P' in $\boldsymbol{P}'$; moreover,

$$P' = p'\lambda'(P', \xi(x)) = p'\eta\lambda(P, x) = \sigma p\lambda(P, x) = \sigma(P).$$

Hence $\eta\lambda(P, x) = \lambda'(\sigma(P), \xi(x))$. This construction yields a functor

$$\mathscr{U}_1\colon \mathbf{Et}_1 \to \mathbf{Kt}_{\mathbf{f}}.$$

Clearly, $\mathscr{K}\mathscr{U}_1$ is the identity functor on $\mathbf{Et}_1$, and $\mathscr{U}_1\mathscr{K}$ is naturally equivalent to the identity on $\mathbf{Kt}_{\mathbf{f}}$. Thus, the categories $\mathbf{Et}_1$ and $\mathbf{Kt}_{\mathbf{f}}$ are equivalent.

We are now going to show that $\mathbf{Et}_1$ is a reflective subcategory of $\mathbf{Et}$. Let (6.1) be any etic system. Write

$$M = \{(P, x)\colon P \in \boldsymbol{P} \;\&\; x \in X \;\&\; \varphi(x) \cap P = 0\}$$

and

$$\mathscr{U}(X, Y, \boldsymbol{P}, \varphi) = (X, \boldsymbol{P}, Y + M, \lambda, p), \tag{6.6}$$

where the function $\lambda\colon \boldsymbol{P} \times X \to Y + M$ is defined by

$$\lambda(P, x) = \begin{cases} \text{the unique element of } \varphi(x) \cap P & \text{if } \varphi(x) \cap P \neq 0, \\ (P, x) & \text{otherwise}, \end{cases}$$

and the function $p\colon Y + M \to \boldsymbol{P}$ is defined by $p(P, x) = P$ for (P, x) in M and by $p(y) = P$ for y in P.

THEOREM 3. *The correspondence* (6.6) *yields a left adjoint of the functor* (6.5).

Specifically, for any kit $\mathfrak{X}' = (X', S', Y', \lambda', p')$ and any Et*-morphism (ξ, η) from $(X, Y, \boldsymbol{P}, \varphi)$ to $\mathscr{K}(\mathfrak{X}')$, there is a unique* Kt*-morphism $(\xi^*, \sigma^*, \eta^*)$ from $\mathscr{U}(X, Y, \boldsymbol{P}, \varphi)$ to $\mathfrak{X}'$ such that the diagram*

$$\begin{tikzcd}
(X, Y, \boldsymbol{P}, \varphi) \arrow[r, "(\iota_X, \varepsilon)"] \arrow[dr, "(\xi,\eta)"'] & \mathscr{K}\mathscr{U}(X, Y, \boldsymbol{P}, \varphi) \arrow[d, "\mathscr{K}(\xi^*,\sigma^*,\eta^*)"] \\
& \mathscr{K}(\mathfrak{X}')
\end{tikzcd}$$

is commutative, where $\varepsilon\colon Y \to Y + M$ is the canonical injection.

Proof. A straightforward verification shows that (ι_X, ε) is a weak homomorphism. Suppose that $(\xi^*, \sigma^*, \eta^*)$ is a **Kt**-morphism such that the above-given diagram is commutative. The condition $\xi^*\iota_X = \xi$ implies that $\xi^* = \xi$ while $\eta^*\varepsilon = \eta$ implies that

(6.7) $$\eta^*(y) = \eta(y) \quad \text{for } y \text{ in } Y.$$

If $P \in \boldsymbol{P}$, then from the condition $p'\eta^* = \sigma^* p$ it follows that

(6.8) $$\sigma^*(P) = \sigma^*\big(p(y)\big) = p'\eta(y) \quad \text{for } y \text{ in } P.$$

If $(P, x) \in M$, then

(6.9) $$\eta^*(P, x) = \eta^*\lambda(P, x) = \lambda'\big(p'\eta(y), \xi(x)\big).$$

We have thus shown the uniqueness of $(\xi^*, \sigma^*, \eta^*)$. On the other hand, it follows from (6.4) that $\sigma^*(P)$ is well defined by formula (6.8). It is now clear that formulas (6.7)-(6.9) define the desired **Kt**-morphism; in particular, the condition $\eta^*\lambda = \lambda'(\sigma^* \times \xi^*)$ follows from (6.3) and (6.9).

Remark. The assumption that the involved sets are finite plays no role in the above-given argument. The assumption that $X \neq 0$, however, is essential. For example, the kit $F(0, 1, 0)$ described in Section 4 cannot be interpreted as an etic system, since in this case the function $p\colon 0 \to 1$ is not a surjection.

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