ANNALES
POLONICI MATHEMATICI
XXX (1974)

On the regular solutions of a linear functional equation

by Marek Cezary Zdun (Katowice)

Abstract. In this paper we investigate the solutions of equation (1) wich are differentiable at a fixed point of the function f, and the relation between these solutions and C^1-solutions of equation (2).

In the present paper we shall deal with the regular solutions of the functional equation

$$\varphi(f(x)) = f'(x)\varphi(x), \tag{1}$$

where φ is the unknown function, and with the relation between the regular solutions of equation (1) and C^1-solutions of the Schröder equation.

$$\gamma(f(x)) = s\gamma(x). \tag{2}$$

By $f_n(x)$ we denote here the n-th iterate of the function $f(x)$, i.e.

$$f_0(x) = x, \quad f_{n+1}(x) = f(f_n(x)), \quad n = 0, 1, 2, \ldots$$

We assumme the following hypotheses:

(i) f belongs to the class C^1 in $[0, a)$ and $0 < f(x) < x$, for $x \in (0, a)$.

The above conditions imply that for every $x \in (0, a)$ there exists $\lim_{n\to\infty} f_n(x) = 0$ and $f(0) = 0$.

(ii-a) f is convex.

(ii-b) f is concave and $f'(x) \neq 0$ for $x \in (0, a)$.

A solution φ of equation (1) which is continuous in $[0, a)$ and differentiable at 0 shall be called a *regular solution* (cf. [1]).

Let us write $f'(0) = s$.

1. Theorem 1. *If hypotheses* (i) *and* (ii-a) *or* (ii-b) *are fulfilled and the condition* $0 < s < 1$ *holds, then equation* (1) *has exactly one-parameter family of regular solutions in* $[0, a)$ *given by the formula*

$$\varphi(x) = \eta \lim_{n\to\infty} \frac{f_n(x)}{f_n'(x)}. \tag{3}$$

Proof. Let φ be a regular solution of equation (1). Then we have $\varphi(0) = 0$. Let us write $\frac{\varphi(x)}{x} = \psi(x)$ for $x \neq 0$ and $\varphi'(0) = \psi(0)$. Equation (1) may be written in the form

$$\psi(f(x)) = \frac{xf'(x)}{f(x)}\psi(x) \quad \text{for } x \in (0, a), \tag{4}$$

where ψ is continuous in $[0, a)$.

The problem of regular solutions of equation (1) is equivalent to the problem of continuous solutions of equation (4).

From equation (4) we get by induction

$$\psi(f_n(x)) = \frac{xf_n'(x)}{f_n(x)}\psi(x). \tag{5}$$

We shall consider the sequence of functions

$$G_n(x) = \frac{xf_n'(x)}{f_n(x)} \quad \text{for } x \neq 0 \tag{6}$$

$$\text{and} \quad G_n(0) = 1 \quad \text{for } n = 0, 1, 2, \ldots$$

The functions $G_n(x)$ are continuous in $[0, a)$. We shall show that for $x \in [0, a)$ there exists $\lim_{n\to\infty} G_n(x) = G(x)$, $G(x) \neq 0$ for $x \in [0, a)$ and G is continuous.

The function f belongs to the class C^1, therefore $f(x) = \int_0^x f'(t)\,dt$. If the function f is convex, then f' is increasing, whence $f(x) \leqslant xf'(x)$ and thus we have

$$\frac{xf'(x)}{f(x)} \geqslant 1 \quad \text{for } x \in (0, a). \tag{7}$$

If the function f is concave, then

$$\frac{xf'(x)}{f(x)} \leqslant 1 \quad \text{for } x \in (0, a). \tag{8}$$

By inequality (7) or (8) we get

$$\frac{G_{n+1}(x)}{G_n(x)} = \frac{xf_{n+1}'(x)}{f_{n+1}(x)}\,\frac{f_n(x)}{xf_n'(x)} = \frac{f'(f_n(x))f_n'(x)f_n(x)}{f_n'(x)f_{n+1}(x)} \tag{9}$$

$$= \frac{f'(f_n(x))f_n(x)}{f(f_n(x))} \underset{(\leqslant)}{\geqslant} 1.$$

If f is convex (concave), then by (9) the sequence $G_n(x)$ is increasing (decreasing).

Let f be convex. We have the equality $\int_{f(x)}^{x} f'(t)\,dt = f(x)-f_2(x)$. Hence

$$(x-f(x))f'(f(x)) \leqslant f(x)-f_2(x) \leqslant (x-f(x))f'(x).$$

Putting $f_{i-1}(x)$ instead of x we have

$$(10)\qquad (f_{i-1}(x)-f_i(x))f'(f_i(x)) \leqslant f_i(x)-f_{i+1}(x) \leqslant (f_{i-1}(x)-f_i(x))f'(f_{i-1}(x))$$

for integers $i \geqslant 1$.

It is easily seen from (10) by induction that

$$(x-f(x))f'(f(x)) \dots f'(f_n(x)) \leqslant f_n(x)-f_{n+1}(x) \leqslant (x-f(x))f'(x) \dots f'(f_{n-1}(x)),$$

whence

$$(x-f(x))\frac{f'_{n+1}(x)}{f'(x)} \leqslant f_n(x)-f_{n+1}(x) \leqslant (x-f(x))f'_n(x),$$

and therefore

$$\frac{f'_{n+1}(x)f_{n+1}(x)}{f_{n+1}(x)f_n(x)f(x)} \leqslant \frac{1-\dfrac{f_{n+1}(x)}{f_n(x)}}{x-f(x)} \leqslant \frac{f'_n(x)}{f_n(x)}.$$

Thus

$$(11)\qquad G_{n+1}(x)\frac{f_{n+1}(x)}{f_n(x)}\frac{1}{f'(x)} \leqslant \frac{x\left(1-\dfrac{f_{n+1}(x)}{f_n(x)}\right)}{x-f(x)} \leqslant G_n(x).$$

If f is concave, then we get the opposite inequalities

$$(12)\qquad G_n(x) \leqslant \frac{x\left(1-\dfrac{f_{n+1}(x)}{f_n(x)}\right)}{x-f(x)} \leqslant G_{n+1}(x)\frac{f_{n+1}(x)}{f_n(x)}\frac{1}{f'(x)}.$$

By the condition $f'(0) = s$ it follows that

$$\lim_{n\to\infty}\frac{f_{n+1}(x)}{f_n(x)} = \lim_{n\to\infty}\frac{f(f_n(x))}{f_n(x)} = s.$$

If f is convex, then the function $x/f(x)$ is decreasing. Then inequality (11) implies

$$(13)\qquad G_{n+1} \leqslant \frac{f'(x)x}{x-f(x)}\left(\frac{f_n(x)}{f_{n+1}(x)}-1\right) \leqslant \frac{\left(\dfrac{1}{s}-1\right)f'(x)x}{x-f(x)}.$$

Likewise, if f is concave, then the sequence $G_n(x)$ is bounded from below by the function $\dfrac{(1/s-1)f(x)x}{x-f(x)}$ for $x>0$.

In both cases there exists the limit $\lim_{n\to\infty} G_n(x) = G(x)$ and $G(x)>0$.

Fix an arbitrary $x_0 \in (0, a)$. We shall show that the sequence G_n converges uniformly in $[0, x_0]$. Then

$$0 \leqslant G_{m+k}(x)-G_m(x) = \frac{xf'_{m+k}(x)}{f_{m+k}(x)} - \frac{xf'_m(x)}{f_m(x)} = \frac{xf'_k\big(f_m(x)\big)f'_m(x)f_m(x)}{f_k\big(f_m(x)\big)f_m(x)} - \frac{xf'_m(x)}{f_m(x)}$$
$$= \frac{xf'_m(x)}{f_m(x)}\left[\frac{f'_k\big(f_m(x)\big)f_m(x)}{f_k\big(f_m(x)\big)} - 1\right] = G_m(x)\big(G_k[f_m(x)]-1\big).$$

From (13) and the above inequality we have

$$(14)\quad |G_{m+k}(x)-G_m(x)| \leqslant \frac{x\left(\frac{1}{s}-1\right)f'(x)}{x-f(x)}\left[\frac{\left(\frac{1}{s}-1\right)f'\big(f_m(x)\big)f_m(x)}{f_m(x)-f_{m+1}(x)} - 1\right]$$
$$= \frac{\left(\frac{1}{s}-1\right)f'(x)}{1-\dfrac{f(x)}{x}}\left[\frac{\left(\frac{1}{s}-1\right)f'\big(f_m(x)\big)}{1-\dfrac{f_{m+1}(x)}{f_m(x)}} - 1\right].$$

The function f is convex and $f'(0)>0$, so f is increasing and for every positive integer m the function $f_{m+1}(x)/f_m(x)$ is also increasing. We have for $x\in(0, x_0]$

$$(15)\qquad \frac{\left(\frac{1}{s}-1\right)f'\big(f_m(x)\big)}{1-\dfrac{f_{m+1}(x)}{f_m(x)}} \leqslant \frac{\left(\frac{1}{s}-1\right)f'\big(f_m(x_0)\big)}{1-\dfrac{f_{m+1}(x_0)}{f_m(x_0)}}.$$

Since

$$\lim_{n\to\infty}\frac{f_{n+1}(x_0)}{f_n(x_0)} = s \quad\text{and}\quad \lim_{n\to\infty} f'\big(f_n(x_0)\big) = s,$$

it results from inequalities (11) and (12) that for every $\varepsilon>0$ there exists an integer N such that for $n \geqslant M$ and for every integer $k>0$ we have the inequality

$$|G_{m+k}(x)-G_m(x)| < \varepsilon \quad \text{for } x\in[0, x_0].$$

By the uniform Cauchy condition the sequence $G_n(x)$ tends uniformly to $G(x)$ in $[0, x_0]$, so G is continuous in $[0, \rho^{\backprime}$

It follows from Theorem 2.2 in [3], p. 48, and from the remark that the problem of regular solutions of equation (1) is equivalent to the problem of continuous solutions of equation (4), that there exists exactly one-parameter family of regular solutions of (1) and it is given by formula (3). Moreover, the solution given by formula (3) fulfils the condition $\varphi'(0) = \eta$.

COROLLARY 1. *Let conditions* (i), (ii-a) [*or* (ii-b)] *be fulfilled*, $0 < s < 1$ *and let* φ *be the regular solution of equation* (1) *such that* $\varphi'(0) = 1$. *Then, for* $n = 0, 1, 2, \ldots$, φ *fulfils the inequalities*

$$(16)\qquad \frac{s}{f'(f_n(x))}\frac{g(f_n(x))}{f_n'(x)} \underset{(\geqslant)}{\leqslant} \varphi(x) \underset{(\geqslant)}{\leqslant} \frac{g(f_n(x))}{f_n'(x)} \quad \textit{for } x \in [0, a),$$

where

$$g(x) = \frac{x-f(x)}{1-s}.$$

Proof. Let the function f be convex. Thus inequality (11) holds and we get, letting $n \to \infty$,

$$(17)\qquad G(x)\frac{s}{f'(x)} \leqslant \frac{x(1-s)}{x-f(x)} \leqslant G(x).$$

In view of Theoreme 1 we have

$$\varphi(x) = \frac{x}{G(x)},$$

thus inequality (17) implies

$$(18)\qquad \frac{s}{f'(x)}\frac{x-f(x)}{1-s} \leqslant \varphi(x) \leqslant \frac{x-f(x)}{1-s}.$$

Putting $f_n(x)$ instead of x in (18) we obtain

$$\frac{s}{f'(f_n(x))}g(f_n(x)) \leqslant \varphi(f_n(x)) \leqslant g(f_n(x)).$$

By equation (1) we get

$$\varphi(f_n(x)) = f_n'(x)\varphi(x) \quad \text{for } n = 0, 1, 2, \ldots,$$

whence inequality (16) results.

Remark 1. If hypothesis (i) is fulfilled, f is concave, $0 < s < 1$, and there exists an $x_0 \in (0, a)$ such that $f'(x_0) = 0$, then the only regular solution of equation (1) is the function $\varphi(x) \equiv 0$, Proof. Let ψ be a continuous solution of equation (4) and $\psi(x) \not\equiv 0$. Let $y = \inf\{x\colon f'(x) = 0\}$. Evidently $f'(y) = 0$ and $y > 0$. If $0 < x < y$, then $G_n(x) \geqslant \frac{(1/s-1)f'(x)}{1-f(x)/x} > 0$. There exists an $\bar{x}$ such that $0 < \bar{x} < y$ and $\psi(\bar{x}) \neq 0$. From relation (4) we have

$$|\psi(f_n(\bar{x}))| \geqslant \frac{(1/s-1)f'(\bar{x})}{1-f(\bar{x})/\bar{x}} |\psi(\bar{x})| > 0,$$

hence $\lim_{n\to\infty} \psi(f_n(\bar{x})) \neq 0$.

On the other hand, $\lim_{n\to\infty} \psi(f_n(x_0)) = 0$, since $f_n'(x_0) = 0$ for $n = 1, 2, \ldots$, but this is a contradiction to the continuity of the function ψ at 0, whence $\psi(x) \equiv 0$.

THEOREM 2. *Let hypotheses* (i), (ii-b) *be filfilled and* $s = 1$. *Then equation* (1) *has in* $[0, a)$ *a regular solution* φ *depending on an arbitrary function, and* φ *fulfils the conditions* $\varphi(0) = 0$ *and* $\varphi'(0) = 0$.

Proof. From the concavity of f and the left-hand side of inequality (12) it follows that the sequence $G_n(x)$ converges almost uniformly in $(0, a)$ to 0. It is easily seen that the sequence $f_n'(x)$ tends almost uniformly in $(0, a)$ to 0. Hence and from Theorem 2.2 in [3] results that for every continuous solution φ of equation (1) we have $\varphi(0) = 0$. Similarly, any continuous solution ψ of equation (4) depends on an arbitrary function and $\psi(0) = 0$. This implies the assertion of the theorem.

THEOREM 3. *If conditions* (ii), (ii-a) *are fulfilled and* $s = 0$, *then the only regular solution of equation* (1) *is* $\varphi(x) \equiv 0$.

Proof. The funbtion f is convex, thus the sequence $f_n'(x)/f_n(x)$ is increasing and $f_n'(x)/f_n(x) > 0$. Let ψ be a continuous solution of equation (4). If there exists an x_0 such that $\psi(x_0) \neq 0$, then from relation (5) it follows that the sequence $\psi(f_n(x))$ is monotonic and goes away from 0, whence $\psi(0) \neq 0$.

If φ is a regular solution of equation (1) and $\varphi \not\equiv 0$, then $\varphi'(0) = \eta \neq 0$ and $\varphi(x) \neq 0$ for $x \in (0, a)$.

We can write formula (3) in the form

$$\frac{\eta}{\varphi(x)} = \lim_{n\to\infty} \frac{f_n'(x)}{f_n(x)}.$$

From the monotonicity of the sequence $f_n'(x)/f_n(x)$ we have

$$0 < \frac{f_n'(x)}{f_n(x)} < \frac{\eta}{\varphi(x)}.$$

Let $x_0 \in (0, a)$. We have the following inequality

$$\int_{f(x_0)}^{x_0} \frac{\eta}{\varphi(t)}\, dt \geqslant \int_{f(x_0)}^{x_0} \frac{f_n'(t)}{f_n(t)}\, dt = \log f_n(x_0) - \log f_{n+1}(x_0) = \log \frac{f_n(x_0)}{f_{n+1}(x_0)} .$$

Since $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = 0$, we have $\lim\limits_{n\to\infty} \log \dfrac{f_n(x_0)}{f(f_n(x_0))} = \infty$, so we get $\int_{f(x_0)}^{x_0} \frac{\eta}{\varphi(t)}\, dt = \infty$, but this is a contradiction to the continuity of φ in $[f(x_0), x_0]$ and the condition that $\varphi(x) \neq 0$ for $x \in (0, a)$.

2. Let conditions (i) and $0 < s < 1$ be fulfilled and suppose that equation (2) has a C^1-solution γ in $[0, a)$ such that $\gamma'(0) \neq 0$.

It is easy to verify that the function $\varphi(x) = \eta \dfrac{\gamma(x)}{\gamma'(x)}$ is a regular solution of equation (1) (see also Dubuc [2]).

The converse theorem is not true. From the existence of a regular solution of equation (1) such that $\varphi'(0) = 0$ one cannot conclude anything about the C^1-solutions of the Schröder equation. This may be seen from the following examples.

Let $f(x) = s\int_0^x \left(1 - \dfrac{1}{\log t}\right) dt$. Then it is known (see Kuczma [3] or [4]) that equation (2) has a C^1-solution in $[0, a)$ depending on an arbitrary function. On the other hand, this function f is convex and hence it results from Theorem 1 that there exists the unique one-parameter family of regular solutions of equation (2) such that $\varphi'(0) \neq 0$. Let

$$f(x) = s\int_0^x \left(1 + \frac{1}{\log t}\right) dt.$$

Then the unique C^1-solution in $[0, a)$ of equation (2) is $\gamma(x) \equiv 0$. The function f is concave, so, by Theorem 1, that there exist a regular solutions of equation (1) such that $\varphi'(0) \neq 0$.

However, under the assumption of the convexity of f there exists a close connection between regular solutions of equation (1) and convex solutions of equation (2).

THEOREM 4. *Let conditions* (i), (ii-a) [(ii-b)] *be fulfilled and* $0 < s < 1$. *Then there exists a convex* [*concave*] *solution* γ *of equation* (2). *This solution belongs to the class* C^1 *in the interval* $[0, a)$ [*in the interval* $[0, a)$, *or in* $(0, a)$ *with* $\lim\limits_{x\to 0} \gamma'(x) = \infty$]. *Moreover, the following relation*

$$\varphi(x) = \varphi'(0)\frac{\gamma(x)}{\gamma'(x)} \quad \textit{for } x \in (0, a) \tag{19}$$

holds, where φ *is the regular solution of equation* (2).

Proof. Let φ be a regular solution of equation (1) and $\varphi'(0) = \eta \neq 0$. From Theorem 1 we know that $\varphi(x) \neq 0$ and

$$\frac{\varphi'(0)}{\varphi(x)} = \lim_{n\to\infty} \frac{f_n'(x)}{f_n(x)},$$

the convergence being almost uniform in $(0, a)$.

Let $x_0 \in (0, a)$; then for $0 < x < x_0$ we have

$$\varphi'(0) \int_x^{x_0} \frac{dt}{\varphi(t)} = \lim_{n\to\infty} \int_x^{x_0} \frac{f_n'(t)}{f_n(t)}\, dt = \lim_{n\to\infty} \left(\log f^n(x_0) - \log f^n(x)\right) = -\log \frac{f_n(x)}{f_n(x_0)}.$$

Hence we get

$$\lim_{n\to\infty} \frac{f_n(x)}{f_n(x_0)} = \exp\left[-\varphi'(0) \int_x^{x_0} \frac{dt}{\varphi(t)}\right]. \tag{20}$$

It is easily seen that relation (20) is true also for $x \geqslant x_0$.

Write $\lim\limits_{n\to\infty} \dfrac{f_n(x)}{f_n(x_0)} = \gamma(x)$. Condition (ii-a) implies the convexity of γ.

Relation (20) may be written as

$$\gamma(x) = \exp \varphi'(0) \int_{x_0}^{x} \frac{dt}{\varphi(t)}. \tag{21}$$

The function γ fulfils equation (2). By formula (21) it follows that γ belongs to the class C^1 in $(0, a)$ and relation (19) holds. The function γ is convex and $\gamma(0) = 0$, so $\gamma(x)/x$ is increasing. Thus there exists $\lim\limits_{x\to 0+} \gamma(x)/x = c$ and $0 \leqslant c < \infty$. Moreover, from (19) we get

$$\lim_{x\to 0+} \gamma'(x) = \varphi'(0) \lim_{x\to 0+} \frac{\gamma(x)}{\varphi(x)} = \varphi'(0) \lim_{x\to 0+} \frac{\gamma(x)x}{\varphi(x)x} = c.$$

This shows that γ belongs to the class C^1 in $[0, a)$.

If f is concave, then γ is concave, but then c can be finite or infinite.

References

[1] B. Choczewski, *Regular solutions of a linear functional equation of the first order*, Nehézipari Müszaki Egetem Kozleményei Miskolc 30 (1970), p. 255–262.

[2] S. Dubuc, *Problèmes relatifs a l'itération de fonctions suggérés par les processus en cascade*, Annales de L'Institut Fourier de L'Université de Grenoble 31 (1971), p. 172–251.

[3] M. Kuczma, *Functional equation in a single variable*, Warszawa 1968 (Monografie Mat. 46).

[4] — *On the Schröder equation*, Rozprawy Matematyczne 34 (1963).

Reçu par la Rédaction le 27. 9. 1972