CONCERNING A PROBLEM DUE TO SAM B. NADLER, JR.

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A function \( f: X \to X \) is called a contraction map if there is a positive number \( a < 1 \) such that

\[
q(f(x), f(y)) \leq aq(x, y) \quad \text{for all } x, y \in X.
\]

Nadler, Jr., asked (\(^1\)) the following question:

Is it true that, for every compactum \( X \) for which the identity map \( i_X: X \to X \) is a pointwise limit of contraction maps, all Čech cohomology groups of \( X \) (over integers) are trivial?

In order to give an affirmative answer to this question, let us recall first some notions belonging to the homology theory.

By a sequence of chains in a compactum \( X \) we understand a sequence \( \kappa = \{\kappa_i\} \) with

\[
\kappa_i = a_{i,1} \sigma_{i,1} + a_{i,2} \sigma_{i,2} + \ldots + a_{i,m_{i}} \sigma_{i,m_{i}},
\]

where \( a_{i,j} \) are elements of an abelian group \( \mathbb{A}_i \) (depending, in general, on \( i \)) and \( \sigma_{i,j} \) are oriented simplexes (i.e., finite systems of points (vertices) of \( X \)).

Let \( \text{mesh}(\kappa_i) \) denote the maximal diameter of the simplexes \( \sigma_{i,1}, \ldots, \sigma_{i,m_{i}} \). Let us prove the following

**Lemma 1.** If \( X \) is a compactum and \( f: X \to X \) is a map satisfying the condition \( q(f(x), f(y)) < q(x, y) \) for every \( x, y \in X \) with \( x \neq y \), and if \( \{\kappa_i\} \) is a sequence of chains in \( X \) with

\[
\lim_{i \to \infty} \text{mesh}(\kappa_i) \leq \epsilon, \quad \text{where } \epsilon > 0,
\]

then

\[
\lim_{i \to \infty} \text{mesh}(f(\kappa_i)) < \epsilon.
\]

**Proof.** Otherwise there would exist a sequence of indices \( i_1 < i_2 < \ldots \) such that in \( \kappa_{i_n} \) there is a simplex \( \sigma_{i_n,j_n} \) containing two vertices \( x_n, y_n \)

\(^1\) S. B. Nadler, Jr., *Some problems concerning stability of fixed points*, Colloquium Mathematicum 27 (1973), p. 263-268; see Problem 2.11 on p. 268.
with
\[ q(f(x_n), f(y_n)) \geq \varepsilon - \frac{1}{n}. \]

Since \( X \) is compact, we may assume \( x_n \to x \) and \( y_n \to y \), where \( x, y \in X \). Then \( q(f(x), f(y)) \geq \varepsilon \) and, consequently, \( q(x, y) > \varepsilon \). It follows that there exists a number \( \eta > \varepsilon \) such that the inequality \( q(x_n, y_n) \geq \eta \) is satisfied for almost all \( n \). Hence \( \limsup_{i \to \infty} \eta_i \geq \eta \geq \varepsilon \) for almost all \( n \), which contradicts our hypothesis that
\[ \lim_{i \to \infty} \text{mesh}(\kappa_i) \leq \varepsilon. \]

Thus the proof of Lemma 1 is complete.

If \( \kappa = \{\kappa_i\} \) is a sequence of chains in \( X \) satisfying the condition
\[ \lim_{i \to \infty} \text{mesh}(\kappa_i) = 0, \]
then we say that \( \kappa \) is an infinite chain in \( X \). An infinite chain \( \gamma = \{\gamma_i\} \) in \( X \) is said to be an infinite cycle in \( X \) if all chains \( \gamma_i \) are cycles, i.e. if their boundaries \( \partial \gamma_i \) vanish. If there is a sequence of chains \( \kappa = \{\kappa_i\} \) in \( X \) such that
\[ \lim_{i \to \infty} \text{mesh}(\kappa_i) \leq \varepsilon \quad \text{and} \quad \gamma_i = \partial \kappa_i \quad \text{for} \quad i = 1, 2, \ldots, \]
then the infinite cycle \( \gamma \) is said to be \( \varepsilon \)-homologous to zero in \( X \) and we write \( \gamma \sim 0 \) in \( X \).

If there is an infinite chain \( \kappa = \{\kappa_i\} \) in \( X \) such that \( \gamma_i = \partial \kappa_i \) for \( i = 1, 2, \ldots, \) then we write \( \gamma = \partial \kappa \) and we say that the infinite cycle \( \gamma \) is homologous to zero in \( X \) (notation: \( \gamma \sim 0 \) in \( X \)). A compactum \( X \) is said to be acyclic if every infinite cycle in \( X \) is homologous to zero in \( X \).

**Lemma 2.** An infinite cycle \( \gamma \) in \( X \) is homologous to zero in \( X \) if and only if \( \gamma \sim 0 \) in \( X \) for every \( \varepsilon > 0 \).

**Proof.** It is evident that the relation \( \gamma \sim 0 \) in \( X \) implies \( \gamma \sim 0 \) in \( X \) for every \( \varepsilon > 0 \). On the other hand, if \( \gamma \sim 0 \) in \( X \) for every \( \varepsilon > 0 \), then for every \( n = 1, 2, \ldots \) there is in \( X \) a sequence \( \{\kappa_i^{(n)}\} \) of chains such that
\[ \lim_{i \to \infty} \text{mesh}(\kappa_i^{(n)}) \leq \frac{1}{n} \quad \text{and} \quad \partial \kappa_i^{(n)} = \gamma_i \quad \text{for} \quad i = 1, 2, \ldots \]

Then for every \( n = 1, 2, \ldots \) there is an index \( i_n \) such that \( \text{mesh}(\kappa_i^{(n)}) \leq 2/n \) for \( i \geq i_n \). We may assume that \( i_{n+1} > i_n \) for \( n = 1, 2, \ldots \) Setting
\[ \kappa_i = \begin{cases} \kappa_i^{(1)} & \text{for} \ i = 1, 2, \ldots, i_1, \\ \kappa_i^{(n)} & \text{for} \ i_n \leq i < i_{n+1}, n = 1, 2, \ldots, \end{cases} \]
we get an infinite chain \( \kappa = \{ \kappa_i \} \) satisfying the condition \( \partial \kappa = \gamma \). Thus the proof of Lemma 2 is complete.

Now let us prove the following

**Theorem.** Let \( X \) be a compactum satisfying the following condition:

For every \( \varepsilon > 0 \) there exists a map \( f: X \to X \) such that \( \rho(f(x), x) < \varepsilon \)

for every \( x \in X \) and that

\[
\rho(f(x), f(y)) < \rho(x, y) \quad \text{if} \quad x, y \in X \quad \text{and} \quad x \neq y.
\]

Then \( X \) is acyclic.

**Proof.** If \( X \) is not acyclic, then there is an infinite cycle \( \gamma = \{ \gamma_i \} \)

in \( X \) such that \( \rho \sim 0 \) in \( X \). We infer, by Lemma 2, that there exist positive numbers \( \varepsilon \) such that

\[
(2) \quad \text{the relation} \quad \rho \sim 0 \quad \text{in} \quad X \quad \text{fails}.
\]

Let \( \varepsilon_0 \) denote the least upper bound of the set of all such numbers \( \varepsilon \).

We see easily that then there exists a sequence of chains \( \kappa = \{ \kappa_i \} \) in \( X \)

such that

\[
\partial \kappa_i = \gamma_i \quad \text{for} \quad i = 1, 2, \ldots \quad \text{and} \quad \lim_{i \to \infty} \text{mesh} \kappa_i \leq \varepsilon_0.
\]

By our hypothesis there is a map \( f: X \to X \) satisfying condition (1) and such that \( \rho(f(x), x) < \varepsilon_0 \) for every \( x \in X \). Then there is an infinite chain \( \lambda = \{ \lambda_i \} \) in \( X \) such that \( \partial \lambda_i = \gamma_i - f(\gamma_i) \) for \( i = 1, 2, \ldots \). Moreover, \( \{ f(\kappa_i) \} \) is a sequence of chains in \( X \) satisfying, by Lemma 1, the condition

\[
\lim_{i \to \infty} \text{mesh} f(\kappa_i) < \varepsilon_0.
\]

Setting \( \kappa'_i = \lambda_i + f(\kappa_i) \) for \( i = 1, 2, \ldots, \) we see that

\[
\lim_{i \to \infty} \text{mesh} \kappa'_i < \varepsilon_0.
\]

Since \( \partial(f(\kappa_i)) = f(\partial \kappa_i) = f(\gamma_i) \), we infer that

\[
\partial \kappa'_i = \gamma_i - f(\gamma_i) + \partial(f(\kappa_i)) = \gamma_i \quad \text{for} \quad i = 1, 2, \ldots
\]

But this contradicts (2). Thus the proof of the Theorem is complete.

Since every contractive map \( f: X \to X \) satisfies the condition \( \rho(f(x), f(y)) < \rho(x, y) \) for \( x, y \in X \) and \( x \neq y \), and since for an acyclic compactum \( X \) all Čech cohomology groups are trivial, the just proved theorem gives an affirmative answer to the problem of Sam B. Nadler, Jr.

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