Some extremal problems for certain families
of analytic functions I

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Abstract. Let $\Omega$ be the class of functions $\theta(z)$, $\theta(0) = 0$, $|\theta(z)| < 1$ regular in the disc $K = \{z: |z| < 1\}$, $A$ and $B$ — arbitrary fixed numbers, $A \in (-1, 1)$, $B \in [-1, A)$, $\varphi(A, B)$ — the class of functions $P(z)$, $P(0) = 1$, regular in $K$ such that $P(z) \in \varphi(A, B)$ if and only if $P(z) = (1 + A\theta(z))(1 + B\theta(z))^{-1}$ for some function $\theta(z) \in \Omega$ and every $z \in K$, and $S^*(A, B)$ — the class of functions $f(z)$, $f(0) = 0$, $f'(0) = 1$, regular in $K$ satisfying the condition: $f(z) \in S^*(A, B)$ if and only if $zf'(z)f(z)^{-1} = P(z)$ for some $P(z) \in \varphi(A, B)$ and all $z$ in $K$.

In the present paper the author determines the bounds for $\text{re}(P(z) + zP''(z)) (P(z))^{-1}$ and $\text{re}(zP'(z)(P(z))^{-1})$ on $|z| = r < 1$ within $\varphi(A, B)$, the bounds of $|f(z)|$ and $|f'(z)|$ in $S^*(A, B)$ and the exact value of the radius of convexity for $S^*(A, B)$.

1. Introduction. Let $\Omega$ be the family of functions $\theta(z)$ regular in the disc $K = \{z: |z| < 1\}$ and satisfying the conditions $\theta(0) = 0$, $|\theta(z)| < 1$ for $z \in K$.

Next, for arbitrary fixed numbers $A$, $B$, $-1 < A \leq 1$, $-1 \leq B < A$, denote by $\varphi(A, B)$ the family of functions

\begin{equation}
P(z) = 1 + b_1 z + \ldots
\end{equation}

regular in $K$ and such that $P(z)$ is in $\varphi(A, B)$ if and only if

\begin{equation}
P(z) = \frac{1 + A\theta(z)}{1 + B\theta(z)}
\end{equation}

for some function $\theta(z) \in \Omega$ and every $z \in K$.

Moreover, let $S^*(A, B)$ denote the family of functions

\begin{equation}
f(z) = z + a_2 z^2 + \ldots
\end{equation}

regular in $K$ and such that $f(z)$ is in $S^*(A, B)$ if and only if

\begin{equation}
\frac{zf'(z)}{f(z)} = P(z)
\end{equation}

for some $P(z)$ in $\varphi(A, B)$ and all $z$ in $K$. 
Finally, we consider the following classes of functions defined in \( K \) (the first five of them consisting of functions of form (1.1) the remaining ones — of form (1.2)): \( \varphi \) — the class of Carathéodory functions, i.e. of functions \( P(z) \) for which \( \text{re} P(z) > 0 \) in \( K \); \( \varphi_a \) — the class of Carathéodory functions of order \( a \), \( 0 < a < 1 \), i.e. such that \( \text{re} P(z) > a \) for \( z \in K \); \( \varphi(M) \), \( M > \frac{1}{2} \), \( \varphi^{(b)} \) and \( \varphi^{(e)} \), \( 0 < \beta \leq 1 \) — the classes of functions satisfying the conditions

\[
|P(z) - M| < M \quad [1], \quad \left| \frac{P(z) - 1}{P(z) + 1} \right| < \beta, \quad |P(z) - 1| < \beta
\]

for \( z \in K \), respectively and \( S^* \) — the class of functions starlike w.r.t. the origin; \( S_a^* \) — the class of functions starlike of order \( a \) [7]; \( S^*(M) \), \( S^{(b)} \) and \( S^{(e)} \) — the classes of functions satisfy (1.3), where \( P(z) \) belong to \( \varphi(M) \), \( \varphi^{(b)} \) and \( \varphi^{(e)} \) for \( z \in K \), respectively.

The classes \( S^*(M) \), \( S^{(b)} \) and \( S^{(e)} \) have been introduced in [1], [6] and [4].

It is easy to prove that

\[
\varphi(A, B) \leq \varphi_{1 - A, 1 - B}, \quad \varphi(A, B) \leq \varphi \left( \frac{1}{1 + B} \right),
\]

\[
\varphi(A, -1) \equiv \varphi_{-A, -1}, \quad \varphi(1, B) \equiv \varphi \left( \frac{1}{1 + B} \right), \quad \varphi(1, 1) \equiv \varphi,
\]

and

\[
\varphi(A, -A) \equiv \varphi^{(-A)}, \quad \varphi(A, 0) \equiv \varphi^{(A)}.
\]

Analogous relations hold for the corresponding classes of starlike functions.

In this paper we give the greatest lower bound and the smallest upper bound for \( \text{re} \left[ P(z) + \frac{zP'(z)}{P(z)} \right] \) and \( \text{re} \left[ \frac{zP'(z)}{P(z)} \right] \) on \( |z| = r < 1 \) within \( \varphi(A, B) \), the bounds of \( |f(z)| \) and \( |f'(z)| \) in \( S^*(A, B) \) and the exact value of the radius of convexity for \( S^*(A, B) \) for every admissible \( A \) and \( B \). As corollaries we obtain certain results given by the present author [1], Libera [2], Mac Gregor [3], [4], Nevanlinna [5], Padmanabhah [6], Robertson [7], [8] and Zmorović [9].

2. Auxiliary lemmas. From the definitions of the classes \( \varphi \) and \( \varphi(A, B) \) we easily obtain the following

**Lemma 1.** If \( P(z) \in \varphi(A, B) \), then

\[
(2.1) \quad P(z) = \frac{(1+A)p(z) + 1-A}{(1+B)p(z) + 1-B}
\]

for some \( p(z) \in \varphi \) and conversely.
Let $\zeta$ be an arbitrary fixed point of $K$. We consider the functional

\[(2.2) \quad F(P) = P(\zeta), \quad P(z) \in \wp(A, B).\]

**Lemma 2.** The set of values of the functional (2.2) is the closed disc with centre $c$ and radius $\varrho$, where

\[(2.3) \quad c = c(r) = \frac{1 - ABr^2}{1 - B^2r^2}, \quad \varrho = \varrho(r) = \frac{(A - B)r}{1 - B^2r^2}, \quad r = |\zeta|.

**Proof.** Every boundary function $P_0(z)$ of $\wp(A, B)$ w.r.t. the functional (2.2) is of form (2.1), where

\[p(\zeta) = \frac{1 + \varepsilon z}{1 - \varepsilon z}, \quad |\varepsilon| = 1\]

[8]. Hence

\[(2.4) \quad P_0(z) = \frac{1 + A\varepsilon z}{1 + B\varepsilon z}.

Since for $z = r e^{i\varphi}, 0 \leq \varphi \leq 2\pi$,

\[(2.5) \quad P_0(z) = c + \varrho \eta_0,

where

\[(2.6) \quad \eta_0 = \varepsilon r e^{i\varphi} \frac{1 + B r e^{-i\varphi}}{1 + B r e^{i\varphi}},

the lemma has been proved.

Denote by $\wp_2(A, B)$ the subclass of $\wp(A, B)$ containing all functions of form (2.1), where

\[(2.7) \quad p(z) = \frac{1 + \lambda}{2} p_1(z) + \frac{1 - \lambda}{2} p_2(z),

\[(2.8) \quad p_k(z) = \frac{1 + \varepsilon_k z}{1 - \varepsilon_k z} \quad \text{for } k = 1, 2\]

and

\[(2.9) \quad |\varepsilon_k| = 1, \quad -1 \leq \lambda \leq 1.

Next let $F(u, v)$ be an analytical function in the $v$-plane and in the half-plane $\text{re } u > 0$, such that

\[|F_u|^2 + |F_v|^2 > 0\]

at every point $(u, v)$.

Since every boundary function of $\wp$ w.r.t. the functional $F[p(z), zp'(z)], |z| = R$, is of form (2.7) [8], every boundary function of $\wp(A, B)$ w.r.t. the functional $F[P(z), zP'(z)], |z| = R$, belongs to $\wp_2(A, B)$. Thus, the extremal problem for $\text{re } F[P(z), zP'(z)], |z| = R$, in $\wp(A, B)$ can be replaced by an analogous problem for this functional in the class $\wp_2(A, B)$. 

LEMMA 3. If $P(z) \in \wp_{\alpha}(A, B)$, then for $z = re^{i\varphi}$, $0 \leq r < 1$, $0 \leq \varphi \leq 2\pi$, we have

$$P(z) = c + x\varphi,$$

where

$$x = \frac{A(1 + \lambda)\eta_1 + (1 - \lambda)\eta_2}{(1 + \lambda)\eta_1 + (1 - \lambda)\eta_2}, \quad \varphi = \frac{(1 + \lambda)h_1 \eta_1 + (1 - \lambda)h_2 \eta_2}{|(1 + \lambda)h_1 \eta_1 + (1 - \lambda)h_2 \eta_2|},$$

$$h_k = \frac{g + (1 + B)\varphi \eta_{k-1}}{v}, \quad v = 2g + (1 + B)[(1 + \lambda)\eta_2 + (1 - \lambda)\eta_1]\varphi,$$

$$g = (1 + B)c - 1 - A, \quad \eta_k = e^{x \varphi} \frac{1 + B \varepsilon_k re^{-i\varphi}}{1 + B \varepsilon_k re^{i\varphi}} \quad \text{for} \quad k = 1, 2,$$

c and $\varphi$ are given by (2.3) and $0 \leq x \leq \varphi$.

Proof. Assume

$$x(p(z); \mu) = (1 + \mu)p(z) + 1 - \mu$$

for every function $p(z)$ of $\wp$ and every number $\mu$. Next let $P(z) \in \wp_{\alpha}(A, B)$. Then

$$P(z) = \frac{(1 + \lambda)x(p_1(z); A) + (1 - \lambda)x(p_2(z); A)}{(1 + \lambda)x(p_1(z); B) + (1 - \lambda)x(p_2(z); B)}$$

for some functions $p_k(z)$ of form (2.8).

Let

$$P_k(z) = \frac{x(p_k(z); A)}{x(p_k(z); B)}, \quad Q_k(z) = (1 + B)P_k(z) - 1 - A,$$

$$V(z) = (1 + \lambda)Q_2(z) + (1 - \lambda)Q_1(z), \quad H_k(z) = \frac{Q_{k-1}(z)}{V(z)}, \quad k = 1, 2.$$

Since

$$x(p_k(z); B) = 2(B - A)Q_k^{-1}(z) \quad \text{and} \quad (1 + \lambda)H_1(z) + (1 - \lambda)H_2(z) = 1,$$

we find after some calculation that $P(z)$ can be represented in the form

$$P(z) = (1 + \lambda)P_1(z)H_1(z) + (1 - \lambda)P_2(z)H_2(z).$$

Since

$$P_k(re^{i\varphi}) = c + \varphi \eta_k$$

(comp. (2.4)-(2.6)), we have

$$Q_k(re^{i\varphi}) = g + (1 + B)\varphi \eta_k, \quad V(re^{i\varphi}) = v$$

and

$$H_k(re^{i\varphi}) = h_k.$$
Therefore
\[ P(re^{i\theta}) = c \left[ (1 + \lambda) \bar{h}_1 + (1 - \lambda) h_1 \right] + \sigma \left[ (1 + \lambda) h_1 \eta_1 + (1 - \lambda) h_2 \eta_2 \right]. \]

The first term of the last sum is equal to \( c \), and thus \( P(re^{i\theta}) \) is of form (2.10), where
\[ (2.17) \quad \kappa^2 = \sigma \left[ (1 + \lambda) h_1 \eta_1 + (1 - \lambda) h_2 \eta_2 \right]. \]

Equality (2.17) implies
\[ \kappa^2 = \sigma^2 \left| (1 + \lambda) h_1 \eta_1 + (1 - \lambda) h_2 \eta_2 \right|^2. \]

Assuming \( \eta_k = e^{i\phi_k}, k = 1, 2 \), we find hence the relationship
\[ \kappa^2 = \sigma^2 \left[ (1 + \lambda) h_1 \bar{\eta}_1 + (1 - \lambda) h_2 \bar{\eta}_2 + (1 - \lambda^2)(h_1 \bar{\eta}_2 \eta_1 + h_2 \bar{\eta}_1 \eta_2) \right] \]
and because of
\[ (1 + \lambda) \bar{h}_1 = 1 - (1 - \lambda) \bar{h}_2, \quad (1 - \lambda) \bar{h}_2 = 1 - (1 + \lambda) \bar{h}_1 \]
we get
\[ \kappa^2 = \sigma^2 \left( 1 - (1 - \lambda^2) \left| (1 - \eta_1 \bar{\eta}_2) h_1 \bar{\eta}_2 + (1 - \bar{\eta}_1 \eta_2) \bar{h}_1 h_2 \right| \right). \]

Finally we obtain
\[ (2.19) \quad \kappa^2 = \sigma^2 \left[ 1 - 4(1 - \lambda^2) \frac{g^2 -(1 + B)^2 \sigma^2}{|\eta|^2} \sin^2 \frac{\gamma_1 - \gamma_2}{2} \right]. \]

Since
\[ (2.20) \quad g^2 -(1 + B)^2 \cdot \sigma^2 = \frac{(A-B)^2(1 - \sigma^2)}{1 - B^2 \sigma^2} > 0, \]
we have \( \kappa \leq \sigma \), which ends the proof.

**Lemma 4.** If \( P(z) \in \mathcal{O}_2(A, B) \), then on \( |z| = r < 1 \)
\[ (2.21) \quad zP'(z) = \frac{-BP^2(z) + (A + B)P(z) - A}{A - B} - \frac{1}{2} \frac{\sigma^*}{\sigma} \left[ g^2 - |P(z) - c|^2 \right] \eta^*, \]
where \( c, \sigma \) are given by (2.3),
\[ (2.21') \quad \sigma^* = \frac{2r}{1 - r^2} \quad \text{and} \quad |\eta^*| = 1. \]

**Proof.** The differentiation in (2.14) yields
\[ (2.22) \quad zP'(z) = U(z) + W(z), \]
where
\[ U(z) = z[(1 + \lambda) P_1(z) H'_1(z) + (1 - \lambda) P_2(z) H'_2(z)], \]
and
\[ W(z) = z[(1 + \lambda) H_1(z) P'_1(z) + (1 - \lambda) H_2(z) P'_2(z)]. \]
Using (2.14), we obtain after simplification

\[
U(z) = \frac{(1 - \lambda^2)(1 + B)}{(A - B) \xi^2(z)} \left[ P_1(z) - P_2(z) \right]^2 N(z),
\]

where

\[
N(z) = A^2 + B + B(1 + B)P_1(z)P_2(z) - B(1 + A)[P_1(z) + P_2(z)],
\]

and

\[
W(z) = \frac{-A + (A + B)P_1(z) - BT(z)}{A - B},
\]

where

\[
T(z) = (1 + \lambda)P_1^2(z) \cdot H_1(z) + (1 - \lambda)P_2^2(z) \cdot H_2(z).
\]

Because of (2.18) we have

\[
[(1 + \lambda)P_1(z)H_1(z) + (1 - \lambda)P_2(z)H_2(z)]^2
\]

\[= (1 + \lambda)P_1^2(z)H_1(z) + (1 - \lambda)P_2^2(z)H_2(z) - (1 - \lambda^2)H_1(z)H_2(z)[P_1(z) - P_2(z)]^2;
\]

thus

\[
T(z) = P_1^2(z) + (1 - \lambda^2)[P_1(z) - P_2(z)]^2 H_1(z)H_2(z).
\]

From (2.22) - (2.26) we conclude that (2.21) may be represented in the form

\[
\pi P'(x) = \frac{-BP_2(x) + (A + B)P_1(x) - A}{A - B} + \frac{(1 - \lambda^2)(A - B)}{\xi^2(z)} \left[ P_1(x) - P_2(x) \right]^2
\]

for every \( P(x) \in \phi_\lambda(A, B) \) and \( x \in K \).

Let \( x = re^{i\phi}, 0 \leq r < 1, 0 \leq \phi \leq 2\pi \). Then from (2.15) it follows that

\[
[P_1(re^{i\phi}) - P_2(re^{i\phi})]^2 = -4g^2\eta\sin^2 \frac{\gamma_1 - \gamma_2}{2},
\]

where

\[
\eta = \eta_1 \cdot \eta_2.
\]

Hence in view of (2.19) and because of

\[
\frac{A - B}{g^2 - (1 + B)^2 g^2} = \frac{1}{2} \frac{g^*}{\ell},
\]

we obtain

\[
\frac{(1 - \lambda^2)(A - B)}{\xi^2(re^{i\phi})} \left[ P_1(re^{i\phi}) - P_2(re^{i\phi}) \right]^2 = -\frac{1}{2} \frac{g^*}{\ell} (g^2 - \kappa^2) \eta^*,
\]
where
\[ \eta^* = \frac{V(re^{i\varphi})}{V(re^{i\varphi})} \eta, \]

which ends the proof.

**COROLLARY.** If \( P(z) \in \mathcal{G}_2(1, -1) \), then on \( |z| = r < 1 \)

\[ zP'(z) = \frac{1}{2} [P^2(z) - 1] - \frac{1}{2} [q^2 - |P(z) - c|^2] \eta, \]

where

\[ c^* = \frac{1 + r^2}{1 - r^2}, \]

cf. [9].

3. **An extremum problem over \( \mathcal{G}(A, B) \).**

I. Let \( P(z) \in \mathcal{G}_2(A, B) \). Thus, because of (2.1), (2.7)-(2.9), where
\[ \varepsilon_k = e^{i\theta_k} \ (k = 1, 2), \] and in view of Lemma 4, the expression

\[ \omega(r) = \min \left\{ \text{re} \left[ P(z) + \frac{zP'(z)}{P(z)} \right] : |z| = r < 1, P \in \mathcal{G}_2(A, B) \right\} \]

may be represented for \( z = re^{i\varphi}, 0 \leq r < 1, 0 \leq \varphi \leq 2\pi \), as follows:

\[ \omega(r) = \min_{\lambda, \delta, \eta_1, \eta_2} L(P(re^{i\varphi})), \]

where

\[ L(w) = \frac{(A - 2B)w^2 + (A + B)w - A}{(A - B)w} - \frac{c^*}{2\varepsilon} \left[ q^2 - |w - c|^2 \right] w^{-1} \eta^*, \]

\[ -1 \leq \lambda \leq 1, \ 0 \leq \delta_k \leq 2\pi \text{ for } k = 1, 2 \text{ and } c, q, q^*, \eta^* \text{ are given by } (2.3), (2.21'), \text{ and } (2.28), \text{ respectively.} \]

Let

\[ (3.2) \quad P(re^{i\varphi}) = se^{it}, \quad s > 0, \ \text{im} \ t = 0. \]

Since

\[ (3.3) \quad r \text{e}^{i\eta^*} e^{-it} \leq 1, \]

we obtain because of (3.1)

\[ (3.4) \quad \omega(r) \geq \tau(r), \]

where

\[ (3.5) \quad \tau(r) = \min_{s, t} \Phi(s, t) \]

and

\[ (3.6) \quad \Phi(s, t) = \Phi(s, t; r) = (E_1s + E_2 + E_3s^{-1}) \cos t + E_4s + E_5 + E_6s^{-1} \]
with
\[
E_1 = \frac{A - 2B}{A - B}, \quad E_2 = -c \frac{q^*}{q} = -2 \frac{1 - ABr^2}{(A - B)(1 - r^2)}, \\
E_3 = \frac{A}{A - B}, \quad E_4 = \frac{q^*}{2q} = \frac{1 - B^2r^2}{(A - B)(1 - r^2)}, \\
E_5 = \frac{A + B}{A - B}, \quad E_6 = \frac{q^*(c^2 - q^2)}{2q} = \frac{1 - A^2r^2}{(A - B)(1 - r^2)}.
\]

(3.7)

In view of Lemmas 2 and 3 the function \( \Phi(s, t) \) is defined in the region
\[
D = \{(s, t): c - q < s < c + q, -\psi(s) < t < \psi(s)\}
\]
and on its boundary \( \partial D \), where
\[
(3.9) \quad \psi(s) = \arccos \frac{s^2 + c^2 - q^2}{2cs}, \quad 0 \leq \psi(s) \leq \psi(s_0)
\]
with \( s_0 = \sqrt{c^2 - q^2} \).

If, at some point \( (s_1, t_1) \) of the region \( D \), \( \Phi(s, t) \) attains its minimum, then \( s_1 \) and \( t_1 \) are the solutions of the system of equations

\[
\frac{\partial \Phi(s, t)}{\partial s} = 0, \quad \frac{\partial \Phi(s, t)}{\partial t} = 0
\]

with the unknowns \( s \) and \( t \), i.e. of the system
\[
(3.10) \quad (E_1 - E_3 s^{-2}) \cos t + E_4 - E_6 s^{-2} = 0, \quad \sin t = 0
\]
or
\[
(3.11) \quad (E_1 - E_3 s^{-2}) \cos t + E_4 - E_6 s^{-2} = 0, \quad E_1 s + E_2 + E_6 s^{-1} = 0,
\]
where \( |\cos t| \neq 1 \).

The numbers \( s_1 \) and \( t_1 \) do not satisfy (3.11) (2); thus, in view of (3.8)-(3.10), the minimum problem for \( \Phi(s, t) \) if \( (s, t) \in D \) is equivalent to an analogous problem for
\[
(3.12) \quad \tilde{\Phi}(s) = \tilde{\Phi}(s; r) = \Phi(s, 0, r),
\]
where \( \tilde{\Phi}(s) = \tilde{\Phi}(s; r) \) is defined in the interval \( I = \{s: c - q < s < c + q\} \).

(1) In the sequel \( |\overline{a}| \), for \( a > 0 \) will be denoted by \( \overline{a} \).

(2) In fact, putting
\[
\delta(s, t) = \frac{\partial^2 \Phi(s, t)}{\partial s^2} \frac{\partial^2 \Phi(s, t)}{\partial t^2} - \left( \frac{\partial^2 \Phi(s, t)}{\partial s \partial t} \right)^2,
\]
we obtain \( \delta(s_1, t_1) = -(E_2 - E_3 s_1^{-2}) \sin^2 t_1 \). Supposing \( \delta(s_1, t_1) = 0 \) for \( \sin t_1 \neq 0 \), we would have, because of (3.11), \( E_1 E_4 = E_2 E_6 \), whence in view of (3.7) and (3.3) we would obtain \( 2 - A(A - B)r^2 = 0 \), which is impossible (cf. (1.2)).
Lemma 5. The function $\tilde{\Phi}(s) = \tilde{\Phi}(s; r)$ attains its minimum at the point

$$s_1 = s_1(r) = \sqrt{\frac{(1 - A)(1 + Ar^2)}{A - 2B + 1 - (A - 2B + B^2)r^2}}$$  \hspace{1cm} (3.13)$$

of $I$ only for $r^* < r < 1$, where $r^* = r^*(A, B)$ is the unique root of the polynomial

$$g(r; A, B) = A(A - B)r^4 - 2A(1 - B)r^3 - (A^2 - AB + 2A + 2B - 2)r^2 + 2(1 + A)r - 2$$  \hspace{1cm} (3.14)$$
in the interval $(0, 1]$.

Proof. Differentiating (3.12) w.r.t. $s$ we obtain

$$\tilde{\Phi}'(s) = E_1 + E_4 - (E_3 + E_5)s^{-2},$$

where

$$E_1 + E_4 = \frac{A - 2B + 1 - (A - 2B + B^2)r^2}{(A - B)(1 - r^2)},$$  \hspace{1cm} (3.15)$$

$$E_3 + E_5 = \frac{(1 - A)(1 + Ar^2)}{(A - B)(1 - r^2)}.$$

Since $E_1 + E_4 > 0$ and $E_3 + E_5 > 0$ for every admissible $A, B, r$, the function $\tilde{\Phi}(s)$ attains its minimum in $s_1$ if $s_1 \in I$.

For $A \neq 1$ put

$$k(r) = [c(r) - g(r)]^2, \hspace{0.5cm} l(r) = s_1^2(r), \hspace{0.5cm} n(r) = [c(r) + g(r)]^2.$$  \hspace{1cm} (3.16)$$

It is easy to verify that the function $k(r)$ decreases and $l(r)$ increases for $0 < r < 1$. Since $k(0) > l(0)$ and $k(1) < l(1)$, we have $s_1 > c - g$ for $r^* < r < 1$, where $r^*, 0 < r^* \leq 1$, is the root of the equation $k(r) - l(r) = 0$, i.e. the root of the polynomial $g(r; A, B)$.

At the same time $n(r) - l(r) > 0$ for $0 < r < 1$. In fact, if $A + B > 0$, then $l(1) < n(0)$; hence $l(r) < n(r)$ for $0 < r < 1$. If $A + B < 0$ and $A > 0$, then $B < 0$. Thus

$$l(r) - n(r) < \frac{(1 + Ar)^2}{1 - B^2r^2} \left[ (1 - A) \frac{1 + Ar^2}{(1 + Ar)^2} - \frac{1 - Br}{1 + Br} \right]$$

$$< \frac{(1 + Ar)^2}{1 - B^2r^2} \left( 1 - \frac{1 - Br}{1 + Br} \right) < 0.$$ 

Finally, for $A + B < 0$ and $A < 0$, because of $n(r) > c(r) + g(r)$, we have

$$l(r) - n(r) < \frac{(B - A)\chi(r)}{(1 + Br)[A - 2B + 1 - (A - 2B + B^2)r^2]},$$
where
\[ \chi(r) = -A(1-B)r^3 - (2-A-B)r^2 + (1+A)r + 2 > 0. \]

Hence we always have \( s_1 < c + \varrho \) for \( r^* < r < 1 \), which ends the proof of the lemma.

**Corollary.** If \( 0 < r \leq r^* \), then \( \Phi(s, t) \) attains its minimum at a point of \( \partial D \).

**Remark.** If \( A = 1 \) and only in this case we have \( r^* = 1 \).

Therefore, if \( A = 1 \), \( \hat{\Phi}(s) \) does not attain its minimum in \( I \).

Assuming
\[ s_2 = s_2(r) = c(r) - \varrho(r) \]
and \( \hat{\Phi}(s_2) = \hat{\Phi}(s_2, 0) \), we have
\[ \hat{\Phi}(s_1) < \hat{\Phi}(s_2) \quad \text{for} \quad r^* < r < 1. \]

Let \( (s, t) \in \partial D \). Then, because of \( \Phi(s, t) = \Phi(s, -t) \), we have
\[ \Phi(s, \varphi(s)) = \Phi(s, -\varphi(s)) = \hat{\Phi}(s), \]
where \( \varphi(s) \) is given by (3.9) and \( s \in J \), where \( J = \{ s : c - \varrho \leq s \leq c + \varrho \} \).

Thus
\[ \hat{\Phi}(s) = (E_1 s + E_3 s^{-1}) \cos \varphi(s) + E_6 \]
with
\[ \cos \varphi(s) = -\frac{E_4 s + E_8 s^{-1}}{E_2}. \]

**Lemma 6.** Let
\[ Z_1 = \{(A, B) : -1 < A < 0, \quad -1 \leq B < A \}, \]
\[ Z_2 = \{(A, B) : 0 \leq A \leq 1, \quad -1 \leq B < \frac{A}{2} \}, \]
\[ Z_3 = \{(A, B) : 0 < A \leq 1, \quad \frac{A}{2} \leq B < A \}, \]
\[ s' = s'(r) = \sqrt{\frac{E_2 - E_6}{E_1 - E_4}} \quad \text{for} \quad (A, B) \in Z_1 \]
and \( I = \{ s : c - \varrho < s < c + \varrho \} \).

Then
\[
\min_{(s, t) \in \partial D} \Phi(s, t) = \min_{s \in I} \hat{\Phi}(s) = \begin{cases} 
\hat{\Phi}(c - \varrho), & \text{if} \ (A, B) \in Z_2 \cup Z_3 \ \text{or} \ (A, B) \in Z_1, \ s' \notin I, \\
\hat{\Phi}(s'), & \text{if} \ (A, B) \in Z_1, \ s' \in I.
\end{cases}
\]
Proof. Differentiating (3.19) w.r.t. $s$, we find by (3.20) that
\[
\hat{\phi}'(s) = -\frac{1}{E_2} \left[ (E_1 - E_2 s^{-2})(E_4 s + E_6 s^{-1}) + (E_4 s + E_6 s^{-1})(E_4 - E_6 s^{-2}) \right],
\]
i.e.
\[
\hat{\phi}'(s) = -\frac{2(E_1 E_4 s^4 - E_2 E_6)}{E_2 s^3}.
\]

For any admissible $A$, $B$ and $r$ we find from (3.7) that $E_2 < 0$, $E_4 > 0$ and $E_6 > 0$. If $(A, B) \in Z_1$, then $E_1 > 0$, $E_3 > 0$; for $(A, B) \in Z_2$ we have $E_1 > 0$, $E_3 < 0$ and the condition $(A, B) \in Z_3$ implies $E_1 < 0$ and $E_3 < 0$. Thus, in view of $\hat{\phi}(c - \varphi) < \hat{\phi}(c + \varphi)$, the lemma has been proved.

Lemma 7. If $s' \in I$, where $s'$ is given by (3.21), then
\[
\hat{\phi}(s') < \hat{\phi}(s').
\]

Proof. Since $\hat{\phi}(s) = \Phi(s, 0)$, then, in view of (3.6) and because of (3.19), we obtain
\[
\hat{\phi}(s) - \hat{\phi}(s) = U(s)[1 - \cos \psi(s)] + V(s),
\]
where
\[
U(s) = E_1 s + E_2 s^{-1}, \quad V(s) = E_4 s + E_6 s^{-1}.
\]
Since
\[
1 - \cos \psi(s) = \frac{V(s)}{E_2},
\]
equality (3.24) now becomes
\[
\hat{\phi}(s) - \hat{\phi}(s) = \frac{V(s)}{E_2} [U(s) + E_2].
\]

Since $E_2 < 0$, we find from (3.36) that $V(s) < 0$. Because of (3.22) and (3.25) we obtain for $s = s'$
\[
U(s') + E_2 = E_2 \left[ \frac{E_1 s'^2 - E_2}{E_4 s'^2 - E_6} \cos \psi(s') + 1 \right].
\]
From (3.7) we get $E_4 < E_1$, and thus
\[
E_1 s'^2 - E_2 > E_1(s'^2 - 1).
\]
On the other hand, basing ourselves on Lemma 6, we conclude that $E_2 > 0$ in (3.28); thus we obtain $A^2 < B^2$. Hence $E_4 > E_6$. Therefore
\[
E_4 s'^2 - E_6 < E_4(s'^2 - 1).
\]
From (3.29) and (3.30) because of $E_2 < 0$ we have $U(s') + E_2 < 0$. Thus, in view of (3.27), inequality (3.23) is true.
Let \( \Phi \) be the minimal value of \( \Phi(s, t) \) in \( D \cup \partial D \). In view of Lemmas 5–7 and inequality (3.18) \( \Phi = \Phi(s_2, 0) \) for \( 0 < r \leq r^* \) and \( \Phi = \Phi(s_1, 0) \) for \( r^* < r < 1 \).

We shall now prove the following

**Theorem 1.** For all \( P(z) \) in \( \varrho(A, B) \) and \( |z| = r, 0 < r < 1 \)

\[
(3.31) \quad \text{re} \left[ P(z) + \frac{zP'(z)}{P(z)} \right] \geq \begin{cases} 
X_1(r; A, B) & \text{for } 0 < r \leq r^*, \\
X_2(r; A, B) & \text{for } r^* < r < 1,
\end{cases}
\]

where

\[
(3.32) \quad X_1(r; A, B) = \frac{A^2 r^2 - (3A - B)r + 1}{(1 - Ar)(1 - Br)},
\]

\[
(3.33) \quad X_2(r; A, B) = 2 \frac{\sqrt{\mathfrak{B}} - (1 - ABr^2)}{(A - B)(1 - r^2)} + \frac{A + B}{A - B},
\]

\[
(3.34) \quad \mathfrak{H} = \mathfrak{H}(r; A, B) = A - 2B + 1 - (A - 2B + B^2)r^2,
\]

\[
(3.35) \quad \mathfrak{B} = \mathfrak{B}(r; A, B) = (1 - A)(1 + Ar^2)
\]

and \( r^* = r^*(A, B) \) is the unique root the polynomial

\[
(3.36) \quad g(r; A, B) = A(A - B)r^4 - 2A(1 - B)r^3 - \quad
- (A^2 - AB + 2A + 2B - 2)r^2 + 2(1 + A)r - 2
\]

in the interval \((0, 1]\).

These bounds are sharp, being attained at the point \( z = re^{\theta} \), \( 0 \leq \varphi \leq 2\pi \), by

\[
(3.37) \quad P^*(z; A, B) = \frac{1 - Ae^{-i\varphi} z}{1 - Be^{-i\varphi} z} \quad \text{for } 0 < r \leq r^*
\]

and by

\[
(3.38) \quad P^{**}(z; A, B) = \frac{1 - (1 - A)de^{-i\varphi} z - Ae^{-2i\varphi} z^2}{1 - (1 - B)de^{-i\varphi} z - Be^{-2i\varphi} z^2} \quad \text{for } r^* < r < 1
\]

respectively, where

\[
(3.39) \quad \hat{d} = \hat{d}(r; A, B) = \frac{1}{r} \frac{(1 - Br^2)s_1 - (1 - Ar^2)}{(1 - B)s_1 - (1 - A)}, \quad s_1 = \sqrt{\mathfrak{B}^{-1}}.
\]

**Proof.** For \( s_2 \) and \( s_1 \) given by (3.17) and (3.13), respectively, we obtain \( \Phi(s_2, 0) = X_1(r) \) and \( \Phi(s_1, 0) = X_2(r) \); thus, in view of (2.9), (3.4), (3.5) and Lemmas 5–7, inequality (3.31) is true. We shall prove that this estimation is sharp.

To this end we observe first that if a function \( P^*(z) \) of the family \( G_2(A, B) \) satisfies condition (3.2) at some point \( re^{\theta} \), \( 0 < r \leq r^* \), \( 0 \leq \varphi \leq 2\pi \), with \( s = s_2 \) and \( t = 0 \), then

\[
(3.40) \quad P^*(re^{\theta}) = s_2.
\]
To make notation simpler, we denote the values of the parameters appearing in Lemma 3 by the same letters as the parameters themselves.

Since \( s_2 = s - q \), from (2.10) we obtain \( z = q \) and \( \psi = -1 \). Therefore from (2.19) it follows that \( \lambda^2 = 1 \) or \( \gamma_1 = \gamma_2 \). If \( \lambda^2 = 1 \), then because of (2.7), (2.8) and (2.1) we get

\[
P^*(z) = \frac{1 + Az}{1 + Bz}
\]

for some \( |e| = 1 \).

If \( \gamma_1 = \gamma_2 \), then in view of \( \eta_k = e^{iy_k} \) \( (k = 1, 2) \) and because of (2.11) we obtain \( e_1 = e_2 \). Thus, from (2.7) and (2.8) and because of (2.1) we infer that \( P^*(z) \) is also of form (3.41).

We find \( s \). For \( z = re^{i\psi} \) we have

\[
P^*(re^{i\psi}) = \frac{1 + Arre^{i\psi}}{1 + Bre^{i\psi}}.
\]

Equating (3.40) and (3.42) we obtain

\[
\varepsilon = \frac{1}{r} \frac{s_2 - 1}{A - B} e^{-i\psi}
\]

and because of (3.17) we obtain \( \varepsilon = -e^{-i\psi} \). Thus \( P^*(z) \) is of form (3.37). Evidently \( P^*(z) \in \varphi(A, B) \). It is easy to verify that for \( z = re^{i\psi} \)

\[
P^*(z) + \frac{zP''(z)}{P'(z)} = X_1(r; A, B).
\]

Next, if a function \( P^{**}(z) \) of \( \varphi(z, A, B) \) satisfies condition (3.2) at some point \( r = r^* \), \( r^* < r < 1 \), \( 0 \leq \varphi \leq 2\pi \), with \( s = s_1 \) and \( t = 0 \), then

\[
P^{**}(re^{i\psi}) = s_1.
\]

We accept the foregoing agreement concerning the notation of values of the parameters corresponding to the function \( P^{**}(z) \).

Since \( t = 0 \) (comp. (3.2) and (3.12)), by (3.3)-(3.6) we have \( \eta^* = 1 \). Therefore, in view of (2.27)

\[
\frac{\Omega(re^{i\psi})}{\Omega(re^{i\psi})} \eta_1 \eta_2 = 1
\]

i.e.

\[
\frac{\Omega}{\varphi} \eta_1 \eta_2 = 1
\]
(cf. (2.16)). Hence

\begin{equation}
\frac{2g + (1 + B)[(1 + \lambda) \bar{\eta}_2 + (1 - \lambda) \bar{\eta}_1]}{2g + (1 + B)[(1 + \lambda) \eta_2 + (1 - \lambda) \eta_1]} \eta_1 \eta_2 = 1
\end{equation}

(cf. (2.11)).

We conclude from (3.44) that

\begin{equation}
g(\eta_1 \eta_2 - 1) + \lambda (1 + B) \varphi(\eta_1 - \eta_2) = 0.
\end{equation}

Moreover, since \( P^{**}(re^{i\varphi}) \) is real, by (2.10) and (2.11) we have \( \psi = \bar{\psi} \), i.e.

\begin{equation}
(1 + \lambda) h_1 \eta_1 + (1 - \lambda) h_2 \eta_2 = (1 + \lambda) \bar{h}_1 \bar{\eta}_1 + (1 - \lambda) \bar{h}_2 \bar{\eta}_2.
\end{equation}

By (3.46), in view of (2.11), an easy calculation yields

\begin{equation}
\lambda g(\eta_1 - \eta_2) + (1 + B) \varphi(\eta_1 \eta_2 - 1) = 0.
\end{equation}

We shall solve the system of equations (3.45) and (3.47) with the unknowns \( \lambda, \eta_1 \) and \( \eta_2 \).

Supposing that for \( \eta_1 = \eta_2 \) we would have \( s_1 = s_2 \); then because of (2.13), (2.7), (2.8) and (2.16) we would obtain \( h_1 = h_2 = \frac{1}{2} \); hence, in view of (2.17), we would get \( \kappa \psi = \varphi \cdot \eta_1 \) where \( \eta = \eta_1 = \eta_2 \). Therefore, because of (2.12) we would obtain \( P^{**}(re^{i\varphi}) = c + \varphi \eta \) and because of the equalities \( P^{**}(re^{i\varphi}) = \bar{P}^{**}(re^{i\varphi}) \) and (3.43) we would find that \( s_1 = c - \varphi \) or \( s_1 = c + \varphi \), which is impossible. Thus, \( \eta_1 \neq \eta_2 \). From (3.47) we find

\begin{equation}
\lambda = -\frac{(1 + B) \varphi(\eta_1 \eta_2 - 1)}{g(\eta_1 - \eta_2)}.
\end{equation}

Substituting \( \lambda \) from (3.48) into (3.45), we obtain

\begin{equation}
(\eta_1 \eta_2 - 1)[g^2 - (1 + B)^2 \cdot \varphi^2] = 0.
\end{equation}

Since \( g^2 - (1 + B)^2 \cdot \varphi^2 \neq 0 \) (cf. (2.20)), we have

\begin{equation}
\eta_1 \eta_2 = 1.
\end{equation}

It follows from (3.49) and (3.47) that

\begin{equation}
\lambda = 0.
\end{equation}

Because of (3.49) we find from the equality

\begin{equation}
\eta_k = \varepsilon_k e^{i\varphi} \frac{1 + B \tilde{e}_k re^{-i\varphi}}{1 + B \tilde{e}_k r e^{i\varphi}} \quad (k = 1, 2)
\end{equation}

(cf. (2.11)) that

\begin{equation}
\varepsilon_1 \varepsilon_2 = e^{-2i\varphi}.
\end{equation}
Thus, because of (3.50), (2.12) and (2.8),

\[ P^{**}(z) = \frac{1 - \frac{1}{2}(1-A)(\varepsilon_1 + \varepsilon_2)z - A\varepsilon_1 \varepsilon_2 z^2}{1 - \frac{1}{2}(1-B)(\varepsilon_1 + \varepsilon_2)z - B\varepsilon_1 \varepsilon_2 z^2}. \]

Let

\[ 2\hat{d} = \varepsilon_1 e^{i\varphi} + \varepsilon_1 e^{-i\varphi}. \]

From (3.51) and (3.52) we obtain \( \varepsilon_1 + \varepsilon_2 = 2e^{-i\varphi} \cdot \hat{d} \); thus

\[ P^{**}(z) = \frac{1 - (1-A)\hat{d} e^{-i\varphi} z - A\varepsilon_2 z^2}{1 - (1-B)\hat{d} e^{-i\varphi} z - B\varepsilon_2 z^2}. \]

It follows from (3.43) and (3.53) that

\[ s_1 = \frac{1 - (1-A)\hat{d} r - Ar^2}{1 - (1-B)\hat{d} r - Br^2}. \]

Therefore \( P^{**}(z) \) is of form (3.38) with \( \hat{d} \) given by (3.39). Evidently \( P^{**}(z) \in \mathcal{G}(A,B) \).

Finally we prove that, for \( z = re^{i\varphi} \),

\[ P^{**}(z) + \frac{zP^{**'}(z)}{P^{**}(z)} = X_4(r; A, B). \]

Differentiating the function \( P^{**}(z) \), we obtain

\[ P^{**'}(z) = (A-B)e^{-i\varphi} - \frac{\hat{d} - 2e^{-i\varphi} z + e^{-2i\varphi} \hat{d} z^2}{[1 - (1-B)\hat{d} e^{-i\varphi} z - B\varepsilon_2 z^2]^2}. \]

Therefore, for \( z = re^{i\varphi} \),

\[ P^{**}(z) + \frac{zP^{**'}(z)}{P^{**}(z)} = s_1 + \frac{A - B}{s_1} \frac{dr(1 + r^2) - 2r^3}{[1 - (1-B)\hat{d} r - Br^2]^2}, \]

and by (3.39) we get

\[ P^{**}(z) + \frac{zP^{**'}(z)}{P^{**}(z)} = \frac{\mathcal{U}(r; A, B)s_1^2 - [2 - A - B + (B - 2AB + A)r^2]s_1 + \mathcal{B}(r; A, B)}{(A-B)(1-r^2)s_1}, \quad z = re^{i\varphi}. \]

Since \( s_1 = \sqrt{\mathcal{B}\mathcal{B}^{-1}} \), we have, for \( z = re^{i\varphi} \),

\[ P^{**}(z) + \frac{zP^{**'}(z)}{P^{**}(z)} = 2 \frac{\mathcal{B}(r) - (1-ABr^2)s_1}{(A-B)(1-r^2)s_1} + \frac{A + B}{A - B} = X_4(r; A, B), \]

which ends the proof of Theorem 1.
II. Let

$$\omega_1(r) = \max_{P(z) \neq 0, |z| = r < 1} \text{re} \left[ \frac{P(z) + zP'(z)}{P(z)} \right].$$

Proceeding as in part I of this section and preserving the same notation, we obtain first $\omega_1(r) \leq \tau_1(r)$, where

$$\tau_1(r) = \max_{(s, t) \in D \cap \partial D} \phi_1(s, t)$$

and

$$\phi_1(s, t) = \phi_1(s, t; r) = (E_1s - E_2 + E_3s^{-1}) \cos t - E_4s + E_5 - E_6s^{-1}.$$  

Next we prove that if $\phi_1(s, t)$ attains its maximum at a point $(\hat{s}, \hat{t}) \in D$, then $\hat{t} = 0$. Let

$$\hat{\phi}_1(s) = \bar{\phi}_1(s; r) = \phi_1(s, 0, r).$$

Since

$$\hat{\phi}_1'(s) = E_1 - E_4 + (E_6 - E_5)s^{-2}$$

and $E_6 - E_5 > 0$, the point $\hat{s}$ exists only if $E_1 < E_4$. Hence

$$\hat{s} = \hat{s}_1(r) = \sqrt{\frac{E_6 - E_5}{E_4 - E_1}},$$

if $1^0 E_1 < E_4$ and $2^0 [(E_6 - E_5)(E_4 - E_1)^{-1}]^{1/2} \in I$.

$E_1 < E_4$ only if $1^0 A - 2B - 1 \leq 0$, $0 < r < 1$ or $2^0 A - 2B - 1 > 0$, $r_0 < r < 1$, where $r_0 = \sqrt{(A - 2B - 1)/(A - 2B - B^2)}$.

Putting $\hat{l}(r) = \hat{s}_1^2(r)$ in these cases, we find that $\hat{l}(r)$ decreases for $0 < r < 1$ and for $r_0 < r < 1$, respectively. The function $k(r)$ defined by (3.16) decreases for $0 < r < 1$ and $k(0) = 1$. Next, we obtain $\hat{l}(r) > k(r)$ for $0 < r < 1$ and for $r_0 < r < 1$, respectively (*) and $\hat{l}(r) < n(r)$ (cf. (3.16)) for $r^** < r < 1$, where $r^**$ is the unique root of the polynomial $g(r; -A, -B)$, in the intervals $(0, 1)$ and $(r_0, 1)$, respectively.

Summing, we obtain

**Lemma 5'.** The function $\hat{\phi}_1(s) = \bar{\phi}_1(s; r)$ attains its maximum at the point

$$\hat{s}_1 = \hat{s}_1(r) = \sqrt{\frac{(1 + A)(1 - Ar^2)}{(A - 2B - B^2)r^2 - (A - 2B - 1)}}$$

(*) In fact, if $A + B < 0$, then $\hat{l}(1) > k(0)$; hence $\hat{l}(r) > k(r)$ in this case. If $A + B > 0$, then $A > 0$ and $\hat{l}(r) - k(r) > (A - B)\hat{\chi}(r)/(1 - Br)[(A - 2B - B^2)r^2 - (A - 2B - 1)]$, where $\hat{\chi}(r) = A(1 + B)r^2 - (A + B + 2)r + (1 - A)r + 2 > 0$ for $0 < r < 1$. 

of $I$ only for $r^{**} < r < 1$, where $r^{**} = r^{**}(A, B)$ is the unique root of the polynomial
\[ A(A - B)r^4 + 2A(1 + B)r^3 - (A^2 - AB - 2A - 2B - 2)r^2 + 2(1 - A)r - 2 \]
in the interval $(0, 1)$.

**Corollary.** If $0 < r < r^{**}$, then $\Phi_1(s, t)$ attains its maximum at a point of $\partial D$.

**Remark.** If $B = -1$ and only in this case we have $r^{**} = 1$. Therefore, if $B = -1$, then $\Phi_1(s)$ does not attain its maximum in $I$.

We see that for $r^{**} < r < 1$
\[ \Phi_1(\delta_1) > \Phi_1(c + q). \]

As in part I we obtain

**Lemma 5'.** Let
\[ \Phi_1(s, \psi)(s) = \Phi_1(s), \]
where $\psi(s)$ is given by (3.7), $s \in J, J = \{s: c - q \leq s \leq c + q\}$ and let
\[ s' = s'(r) = \sqrt{\frac{E_3 E_6}{E_1 E_4}} \quad \text{for} \quad (A, B) \in \mathbb{Z}_3 \]
(cf. Lemma 6). Then
\[ \max_{(s, t) \in \partial D} \Phi_1(s, t) = \max_{s \in J} \Phi_1(s) = \begin{cases} \Phi_1(c + q), & \text{if} \quad (A, B) \in \mathbb{Z}_1 \cup \mathbb{Z}_2 \text{ or} \\ \Phi_1(s'), & (A, B) \in \mathbb{Z}_3, s' \notin I; \\ \Phi_1(s'), & \text{if} \quad (A, B) \in \mathbb{Z}_3, s' \in I. \end{cases} \]

**Remark.** If $(A, B) \in \mathbb{Z}_3$, then $A - 2B - 1 < 0$.

We prove the following

**Lemma 7'.** If $s' \in I$, then
\[ \Phi_1(s') > \Phi_1(s'). \]

**Proof.** Preserving the notation adopted in Lemma 7, we easily obtain the equality
\[ \Phi_1(s') - \Phi_1(s') = V(s') [T(s') \cos \psi(s') - 1], \]
where
\[ T(s') = \frac{E_1 s'^2 - E_3}{E_4 s'^2 - E_6}. \]

Since $(A, B) \in \mathbb{Z}_3$, we have $E_3 < E_1 < 0$ and $E_4 > E_6 > 0$.

Putting $h(r) = E_6 E_4 - E_1 E_3$, we get
\[ h(r) = \frac{h_1(r)}{(A - B)^2 (1 - r^2)^2}, \]

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where
\[ h_1(r) = A(AB^2 - 2B + A)r^4 + (A - B)(B - 3A)r^2 + A^2 - 2AB + 1. \]

If \( h_1(r) > 0 \) for \( 0 < r < 1 \), then \( h(r) > 0 \) and because of \( E_1E_6 - E_3E_4 > 0 \) we obtain after some calculation the inequality
\[ s'^2 > \frac{E_6 - E_3}{E_4 - E_1}. \]

Hence \( T(s') < 1 \); thus, because of \( \cos \psi(s') > 0 \) and \( V(s') < 0 \), in case of \( h_1(r) > 0 \), the lemma is proved.

The relation \( (A, B) \in Z_1 \) implies \( 0 < A \leq 2B \). Thus \( h_2(0) > 0 \); moreover, \( h_1(1) \geq 0 \). Therefore \( h_1(r) > 0 \) for \( 0 < r < 1 \) if \( AB^2 - 2B + A < 0 \). Similarly for \( AB^2 - 2B + A > 0 \) this inequality is true, which ends the proof.

Basing ourselves on Lemmas 5'-7', we obtain

**Theorem 2.** For all \( P(z) \) in \( g(A, B) \) and \( |z| = r \), \( 0 < r < 1 \)

\[(3.54) \quad \text{re} \left[ \frac{zP'(z)}{P(z)} \right] \leq \begin{cases} X_1(r; -A, -B) & \text{for } 0 < r \leq r^*; \\ X_2(r; -A, -B) & \text{for } r^* < r < 1, \end{cases} \]
equality holding in \( z = re^{i\phi} \) for \( P^*(r; -A, -B) \) if \( 0 < r \leq r^* \) and for \( P^*(z; -A, -B) \) if \( r^* < r < 1 \), respectively, where \( r^* = r^*(A, B) \) is the unique root of the polynomial \( g(r; -A, -B) \) in the interval \( (0, 1] \) (cf. Theorem 1).

**III.** Similarly we prove the following

**Theorem 3.** For all \( P(z) \) in \( g(A, B) \) and \( |z| = r \), \( 0 < r < 1 \)

\[ 1^o \quad \text{re} \frac{zP'(z)}{P(z)} \geq \begin{cases} Y_1(r; A, B) & \text{for } 0 < r \leq \hat{r}^*, \\ Y_2(r; A, B) & \text{for } \hat{r}^* < r < 1, \end{cases} \]

where
\[ Y_1(r; A, B) = -\frac{(A - B)r}{(1 - Ar)(1 - Br)}, \]
\[ Y_2(r; A, B) = 2\frac{\hat{\Phi}B^2 - (1 - ABr^2)}{(A - B)(1 - r^2)} + \frac{A + B}{A - B}, \]
\[ \hat{\Phi} = \hat{\Phi}(r; A, B) = (1 - B)(1 + Br^2), \quad \hat{B} = B \]
(cf. Theorem 1) and \( \hat{r}^* = \hat{r}^*(A, B) \) is the unique root of the polynomial \( g(r; A, B) = AB^2 - 2AB + 2A + 2B - A^2 + 1 \) in the interval \( (0, 1] \). Functions \( (3.37) \) and \( (3.38) \), where \( d \) is given by \( (3.39) \) and \( s_1 = V.B^{\hat{\Phi}^{-1}} \) shows this result to be sharp.

\[ 2^o \quad \text{re} \frac{zP'(z)}{P(z)} \leq \begin{cases} Y_1(r; -A, -B) & \text{for } 0 < r \leq \hat{r}^{**}, \\ Y_2(r; -A, -B) & \text{for } r^{**} < r < 1, \end{cases} \]
equality holding for functions $P^*(z; -A, -B)$ and $P^{**}(z; -A, -B)$, respectively; $\hat{\gamma}^{**}$ is the unique root of the polynomial $\hat{g}(r; -A, -B)$ in the interval $[0, 1]$.

Remark. If $A = 1$ and only in this case we have $\hat{\gamma}^* = 1$, $\hat{\gamma}^{**} = 1$ only for $B = -1$.

Applying Theorem 3 to the special cases where $A = 1 - 2\alpha$ and $B = -1$, $A = 1$ and $B = \frac{1}{M} - 1$, $A = \beta$ and $B = -\beta$, $A = \beta$ and $B = 0$, we obtain the corresponding theorems on $\text{re} \frac{zP'(z)}{P(z)}$ in the families $\varphi_\alpha$, $\varphi(M)$, $\varphi^{(\beta)}$ and $\varphi_{[\beta]}$, respectively. If $A = 1$ and $B = -1$, then we obtain a result of Libera [2].

4. The estimations of $|f(z)|$ and $|f'(z)|$ in $S^*(A, B)$.

**Theorem 4.** If $f(z) \in S^*(A, B)$, then for $|z| = r, 0 \leq r < 1$

\[
C(r; -A, -B) \leq |f(z)| \leq C(r; A, B),
\]

where

\[
C(r; A, B) = \begin{cases} 
(1 + Br)^{(A-B)/B} & \text{for } B \neq 0, \\
re^{4r} & \text{for } B = 0.
\end{cases}
\]

These bounds are sharp, being attained at the point $z = re^{i\varphi}, 0 \leq \varphi \leq 2\pi$, by

\[
f_*(z) = z \cdot f_0(z; -A, -B)
\]

and

\[
f^*(z) = z \cdot f_0(z; A, B),
\]

respectively, where

\[
f_0(z; A, B) = \begin{cases} 
(1 + Be^{-i\varphi}z)^{(A-B)/B} & \text{for } B \neq 0, \\
e^{4e^{-i\varphi}z} & \text{for } B = 0.
\end{cases}
\]

**Proof.** Since $f(z) \in S^*(A, B)$, we have

\[
f(z) = z \cdot \exp \left( \int_0^z \frac{P(\zeta) - 1}{\zeta} d\zeta \right), \quad P(z) \in \varphi(A, B).
\]

Therefore

\[
|f(z)| = |z| \exp \left( \text{re} \int_0^z \frac{P(\zeta) - 1}{\zeta} d\zeta \right).
\]

Substituting $\zeta = zt$, we obtain

\[
|f(z)| = |z| \exp \left( \text{re} \int_0^1 \frac{P(zt) - 1}{t} dt \right).
\]
Hence

\[ |f(z)| \leq |z| \exp \left( \int_0^1 \max_{|z|=rt} \left( \operatorname{re} \frac{P(zt)}{t} - 1 \right) \, dt \right). \]

From Lemma 2 it follows that

\[ \max_{|z|=rt} \frac{P(zt) - 1}{t} = \frac{(A - B)r}{1 + Br}; \]

then, after integration, we obtain the upper bounds in (4.1). Similarly, we obtain the bounds on the left-hand side of (4.1), which ends the proof.


**Theorem 5.** If $f(z) \in S^*(A, B)$, then for $|z| = r$, $0 \leq r < 1$,

\[ \tilde{L}(r) \leq |f'(z)| \leq L(r), \]

where

\[ L(r) = \begin{cases} D(r), & \text{if } 0 < r \leq r^*, \\ D(r^*) \frac{\exp H(r)}{\exp H(r^*)}, & \text{if } r^* < r < 1, \end{cases} \]

(4.4)

\[ \tilde{L}(r) = \begin{cases} \tilde{D}(r), & \text{if } 0 < r \leq r^*, \\ \tilde{D}(r^*) \frac{\exp \tilde{H}(r)}{\exp \tilde{H}(r^*)}, & \text{if } r^* < r < 1, \end{cases} \]

$r^*$ and $r^{**}$ are the roots of the polynomial $g(r; A, B)$ and $g(r; -A, -B)$, respectively (cf. Theorems 2 and 1),

\[ D(r) = D(r; A, B) = \begin{cases} (1 + Ar)(1 + Br)^{(d - 2m)n}, & \text{if } B \neq 0, \\ (1 + Ar)^{e^{dr}}, & \text{if } B = 0, \end{cases} \]

\[ H(r) = H(r; A, B) = \frac{2}{A - B} \int \frac{1 + B - B(1 + A)r - \sqrt{(1 + A)(1 - Ar^2)(a_1 - a_2 r^2)}}{r(1 - r^2)} \, dr \]

(4) After integration we obtain $H(r) = 2 \frac{1 + B}{A - B} \log r - \frac{1 - AB}{A - B} \log (1 - r^2) + \sum_{k=1}^3 J_k \log \text{const}$, where

\[ J_1 = \begin{cases} 0, & \text{for } A = 0 \text{ or } a_2 = 0, \\ -2b_1 \arctan t_1^{-1}, & \text{for } a_2 < 0 \text{ and } A > 0, \\ b_1 \log \frac{1 - t_1}{1 + t_1}, & \text{for } a_2 > 0 \text{ or } a_2 < 0 \text{ and } A < 0, \end{cases} \]
\[ a_1 = a_1(A, B) = -A + 2B + 1, \quad a_2 = a_2(A, B) = -A + 2B + B^2 \]

and
\[
\tilde{D}(r) = D(r, -A, -B), \quad \tilde{H}(r) = H(r, -A, -B).
\]

The upper bound \( L(r) \) for \( 0 < r \leq r^{**} \) and the lower bound \( \tilde{L}(r) \) for \( 0 < r \leq r^* \) are sharp, being attained by functions (4.3) and (4.2), respectively.

Proof. If \( f(z) \in S^*(A, B) \), then because of (1.3) an easy calculation yields
\[
1 + \frac{zf''(z)}{f'(z)} = P(z) + \frac{zP'(z)}{P(z)}
\]
for some \( P(z) \) in \( \varphi(A, B) \). On the other hand, we have
\[
\operatorname{re} \frac{zf''(z)}{f'(z)} = r \frac{\partial}{\partial r} \log |f'(z)|, \quad |z| = r
\]
then, using (3.54), (3.32) and (3.33), we obtain
\[
\frac{\partial}{\partial r} \log |f'(z)| \leq \frac{(A-B)(Ar+2)}{(1+Ar)(1+Br)}
\]
for \( 0 < r \leq r^{**} \)
\[
\frac{\partial}{\partial r} \log |f'(z)| \leq 2 \frac{1+B-B(1+A)r^2 - \sqrt{(1+A)(1-Ar^2)(a_1-a_2r^2)}}{(A-B)r(1-r^2)}
\]
for \( r^{**} < r < 1 \).

Integrating both sides of inequality (4.6) from 0 to \( r \), we obtain
\[
|f'(z)| \leq D(r),
\]
where \( D(r) \) is given by (4.4).

\[
J_2 = \begin{cases} 0 & \text{for } a_1 = 0, \\ 2b_2 \arctan t_2^{-1} & \text{for } a_1 < 0, \\ -b_2 \log \left(\frac{1-t_2}{1+t_2}\right) & \text{for } a_1 > 0, \end{cases}
\]
\[
J_3 = \begin{cases} 0 & \text{for } A = 1, \\ b_4 \log \left(\frac{1-t_3}{1+t_3}\right) & \text{for } A \neq 1, \end{cases}
\]

\[
b_1 = \frac{\sqrt{1-A} |A|^{-1/2} |a_2|}{A-D}, \quad b_2 = \frac{\sqrt{1+A} |a_2|}{A-D}, \quad b_3 = \frac{\sqrt{(1-A^2)(1-B^2)}}{A-B},
\]

\[
t_1 = \sqrt{\frac{a_2}{A}} \cdot t \quad \text{for } A \neq 0, \quad t_2 = \sqrt{|a_1|} \cdot t, \quad t_3 = \sqrt{\frac{1-B^2}{1-A^2}} \cdot t \quad \text{for } A \neq 1
\]
and
\[
t = \sqrt{\frac{1-A^2}{a_1-a_2r^2}}.
\]
Let \( r^{**} < r < 1 \). Denoting by \( I_1(r) \) and \( I_2(r) \) the right-hand sides of inequalities (4.6) and (4.7), respectively, we get the inequalities

\[
\log |f'(z)| \leq \int_0^{r^{**}} I_1(r) \, dr + \int_{r^{**}}^r I_2(r) \, dr.
\]

We easily obtain

\[
\int_0^{r^{**}} I_1(r) \, dr = \log D(r^{**})
\]

and

\[
\int_{r^{**}}^r I_2(r) \, dr = H(r) - H(r^{**}),
\]

where \( H(r) \) is given by (4.4).

By (4.8)-(4.11) the first part of the theorem on the upper estimation of \( |f'(z)| \) has been proved. Similarly, the second part of the theorem on the lower estimation of \( |f'(z)| \) can be proved.

The lower bound of \( |f'(z)| \) in the classes \( S^*_a, S^*(M), S^*(\beta) \) and \( S(\beta) \) is sharp in the following intervals of \( r: (0, r_a); (0, 1) \) [1]; \( (0, r_1(\beta)]; (0, r_2(\beta)); (0, r_\beta) \], respectively, where \( r_a = r^*(1 - 2\alpha, -1), \, r_1(\beta) = r^*(\beta, -\beta), \, r_2(\beta) = r^*(\beta, 0) \) and the upper bound — in the intervals: \( (0, 1) \) [7]; \( (0, 1) \) for \( M \geq 1 \) [1] and in \( (0, R(M)) \) for \( M < 1 \); \( (0, 1) \) for \( \beta \geq \frac{1}{2} \) and in \( (0, r_2(\beta)) \) for \( \beta < \frac{1}{2} \); \( (0, r_\beta) \), respectively, where \( R(M) = r^{**} \left(1, \frac{1}{M} - 1\right), \, r_2(\beta) = r^{**}(\beta, -\beta), \, r_\beta = r^{**}(\beta, 0) \).

5. The radius of convexity for the family \( S^*(A, B) \). Let \( S \) be the family of functions (1.2) regular and univalent in \( K \) and \( T \) an arbitrary subclass of \( S \). If \( f \) is in \( T \), then r.c. \( \{f\} \), the radius of convexity of \( f \), is

\[
\text{r.c. } \{f\} = \sup \left\{ r: \text{re} \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, |z| < r \right\}
\]

and r.c. \( T \), the radius of convexity of \( T \), is

\[
\text{r.c. } T = \inf_{f \in T} \text{r.c. } \{f\}.
\]

If \( T \) is compact, then the problem of finding r.c. \( T \) is reduced to finding the greatest value of \( r, 0 < r \leq 1 \), for which

\[
\text{re} \left(1 + \frac{zf''(z)}{f'(z)}\right) \geq 0
\]

for every \( |z| \leq r \) and every function \( f(z) \in T \).
Some extremal problems

Since $S^*(A, B)$ is compact, it follows immediately that r. c. $S^*(A, B)$
equals the smallest root $r_0$, $0 < r_0 \leq 1$, of the equation $\omega(r) = 0$, where

$$\omega(r) = \min \left\{ \mathfrak{R} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] : |z| = r < 1, f \in S^*(A, B) \right\}.$$

Let $f(z)$ be an arbitrary function of $S^*(A, B)$. Then, in view of (1.3), (4.5) and because of Theorem 1,

(5.1) \[ \omega(r) = \begin{cases} u(r) / u_1(r) & \text{for } 0 < r \leq r^*, \\ v(r) / v_1(r) & \text{for } r^* < r < 1, \end{cases} \]

where

(5.2) \[ u(r) = A^2 r^2 - (3A - B)r + 1, \quad u_1(r) = (1 - Ar)(1 - Br) > 0, \]

(5.3) \[ v(r) = c_1 r^4 - 2c_2 r^3 + c_3, \]

\[ v_1(r) = (A - B)(1 - r^2) \left[ 2\sqrt{3A} + 2(1 - ABr^2) - (A + B)(1 - r^2) \right] > 0, \]

\[ c_1 = 4A^2 - 5A + B, \quad c_2 = 2A^2 - 3A + 2 - B, \quad c_3 = 4 - 5A + B, \]

$\mathfrak{A}$ and $\mathfrak{B}$ are given by (3.34)-(3.35) and $r^*$ is the unique root of the polynomial (3.36) in the interval $(0, 1]$.

Let

(5.4) \[ B_1 = B_1(A) = -A^2 + 3A - 1 \quad \text{for } 0 \leq A < 1, \]

(5.5) \[ B_2 = B_2(A) = 5A - 4 \quad \text{for } \frac{3}{5} \leq A < 1, \]

(5.6) \[ G_1 = \{(A, B): (-1 < A \leq 0, -1 \leq B < A) \cup \}

\[ \cup (0 < A < 1, B_1 \leq B < A)\}, \]

(5.7) \[ G_2 = \{(A, B): (0 < A \leq \frac{3}{5}, -1 \leq B < B_1) \cup \]

\[ \cup (\frac{3}{5} < A < 1, B_2 \leq B < B_1)\}, \]

(5.8) \[ G_3 = \{(A, B): \frac{2}{5} < A < 1, -1 \leq B < B_2\}. \]

It can easily be verified that $u(r) > 0$ for $0 < r < 1$ if $(A, B) \in G_1$; $u(r)$ has one root $r_1$ in the interval $(0, 1)$ if $(A, B) \in G_2 \cup G_3$, $u(r) > 0$ for $0 < r < r_1$; hence $u(r) < 0$ for $r_1 < r < 1$ in this case; $v(r)$ has one root $r_2$ in the interval $(0, 1)$ if $(A, B) \in G_4 \cup G_2$ and at the same time $v(r) > 0$ for $0 < r < r_2$; thus $v(r) < 0$ for $r_2 < r < 1$; finally $v(r) < 0$ for $0 < r < 1$ when $(A, B) \in G_3$.

Hence, because of (5.1)-(5.3) and the fact that $u(r^*)$ and $v(r^*)$ must have the same sign, we obtain the following

**Lemma 8.** If: $1^o (A, B) \in G_1$ or $2^o (A, B) \in G_2$ and $r_1 \geq r^*$, then $\omega(r) > 0$ for $0 < r < r_2$, $\omega(r_2) = 0$ and $\omega(r) < 0$ for $r_2 < r < 1$. If: $3^o (A, B) \in G_3$ and $r_1 < r^*$ or $4^o (A, B) \in G_3$, then $\omega(r) > 0$ for $0 < r < r_1$, $\omega(r_1) = 0$ and $\omega(r) < 0$ for $r_1 < r < 1$. 

Proof. The first or the second assumption implies immediately the assertion. If the third condition is satisfied, then, because of \( u(r^*) < 0 \), we have \( v(r^*) < 0 \), thus the lemma is true in this case. Finally if \( (A, B) \epsilon G_3 \), then \( v(r^*) < 0 \); thus \( u(r^*) < 0 \) and \( r_1 < r^* \), which ends the proof.

**Lemma 9.** The root \( r_1 \) of the polynomial \( u(r) \) satisfies the condition \( r_1 < r^* \) if and only if

\[
y(A, B) = B^2 + k_1(A)B - k_2(A)B + k_3(A) < 0,
\]

where

\[
k_1(A) = 2A^2 - 11A + 2,
\]

\[
k_2(A) = A^4 + 12A^3 - 41A^2 + 12A + 1,
\]

\[
k_3(A) = 5A^5 + 10A^4 - 39A^3 + 10A^2 + 5A.
\]

Proof. If \( \Phi(s, t) \) defined by (3.6) attains its minimum equal to zero for \( r = r_1 \), then because of (3.6) we get for \( r = r_1 \)

\[
(E_1 + E_4)(c - q) + E_2 + E_5 + \frac{E_3 + E_6}{c - q} = 0.
\]

Thus

\[
E_3 + E_6 + (E_1 + E_4)(c - q)^2 + (E_2 + E_5)(c - q) = 0, \quad r = r_1.
\]

On the other hand, in view of (3.14) - (3.15) and because of the definition of \( r^* \), we obtain for \( r = r_1 \)

\[
E_3 + E_6 - (E_1 + E_4)(c - q)^2 = \frac{g(r)}{(1 - r^2)(1 - Br)^2}.
\]

Equalities (5.11) and (5.12) imply

\[
2(E_3 + E_6) + (E_2 + E_5)(c - q) = \frac{g(r)}{(1 - r^2)(1 - Br)^2}, \quad r = r_1.
\]

Hence, in view of (3.7) and (3.15),

\[
2(1 - A)(1 + Ar_1^2) + [(2AB - A - B)r_1^2 + A + B + 2] \frac{1 - Ar_1}{1 - Br_1} = \frac{(A - B)g(r_1)}{(1 - Br_1)^2}.
\]

Thus

\[
Ar_1^3 + (1 - 2A)r_1^2 - (A - 2)r_1 - 1 = \frac{g(r_1)}{1 - Br_1}.
\]

Since \( u(r_1) = 0 \), we have

\[
A^2r_1^2 + (3A - B)r_1 + 1 = 0.
\]

From (5.13) and (5.14) we obtain

\[
A^2r_1^2 + A(1 - A)^2r_1 - A(4A - B - 2) = A \frac{g(r_1)}{r_1(1 - Br_1)}.
\]
By (5.15) and (5.14) we get
\[ (A^3 - 2A^2 + 4A - B)r_1 - A(4A - B - 2) - 1 = A \frac{g(r_1)}{r_1(1 - Br_1)}. \]

The polynomial \( g(r) \) increases in the interval \((0, 1)\) and \( g(0) < 0, \) \( g(r^*) = 0. \) Thus \( r_1 < r^* \) if and only if \( g(r_1) < 0. \) Since the root \( r_1, 0 < r_1 < 1 \) of \( u(r) \) exists if and only if \((A, B) \in G_2 \cup G_3, \) we have \( A > 0. \)

Therefore \( r_1 < r^* \) if
\[ (A^3 - 2A^2 + 4A - B)r_1 < 4A^2 - AB - 2A + 1. \]

Putting
\[ r_0 = \frac{4A^2 - AB - 2A + 1}{A^3 - 2A^2 + 4A - B}, \]
we easily find that \( 0 < r_0 < 1. \) It follows immediately that \( r_1 < r^* \) if and only if \( u(r_0) < 0, \) i.e. when
\[ A^2r_0^2 - (3A - B)r_0 + 1 < 0. \]

Hence we obtain after some calculations inequality (5.9).

Let \( 0 < A < \frac{3}{2} \). For \( B = -1 \) we obtain
\[ y(A, -1) = (1 + A) \cdot \hat{y}(A), \]
where
\[ \hat{y}(A) = 5A^4 + 6A^3 - 33A^2 + 4A + 2. \]

Since the derivative \( \hat{y}'(A) \) decreases as \( A \) increases in the interval \((0, \frac{3}{2})\) and \( \hat{y}'(0) > 0, \hat{y}'(\frac{3}{2}) < 0, \hat{y}'(A) \) has a root \( \hat{A} \) in this interval. Hence, in view of (5.17), the polynomial (5.16) has exactly one root \( A_0 \) in the interval \((0, \frac{3}{2}).\)

**Lemma 10.** For every \( A \) of the interval \( A_0 < A < 1 \) the equation \( y(A, B) = 0 \) with the unknown \( B \) (cf. (5.9)) has exactly one solution \( B = B(A) \) in the interval \((-1, B_1(A)) \) for every \( A \in (A_0, \frac{3}{2}) \) and in the interval \((B_2(A), B_1(A)) \) for every \( A \in [\frac{3}{2}, 1], \) \( B_1(A) \) and \( B_2(A) \) being given by (5.4) and (5.5), respectively.

**Proof.** For \( A_0 < A < \frac{3}{2} \) we have \( y(A, -1) < 0. \) If \( 0 < A < 1, \) then
\[ y(A, B_1) = 2(1 - A)^2(A^4 + 2A^3 + 2A + 1) > 0. \]

Thus, for \( A_0 < A < \frac{3}{2} \) the equation \( y(A, B) = 0 \) has at least one solution in the interval \((-1, B_1).\)

Now, differentiating the function \( y(A, B) \) twice w. r. t. \( B, \) we obtain
\[ y'(A, B) = 3B^2 + 2k_1(A)B - k_2(A) \]
and
\[ y''(A, B) = 2[3B + k_1(A)] \]
(cf. (5.9) and (5.10)).
Since \( y''_{BB}(A, B) \) is negative for \( B < B_1 \), \( y'(A, B) \) decreases in the interval \((-1, B_1)\). Next we have

\[
(5.21) \quad y''_B(A, B_1) = -2[A^3 + (\sqrt{3} - 1)A + 1][A^3 - (\sqrt{3} + 1)A + 1].
\]

It can easily be verified that

\[
(5.22) \quad A^2 + 1 < (1 + \sqrt{3})A \quad \text{for } A_0 < A < 1.
\]

From (5.21) and (5.22) it follows that \( y(A, B) \) increases in the interval \(-1 < B < B_1 \) for \( A_0 < A < \frac{3}{4} \). Hence, the lemma is true in this case.

Let \( \frac{3}{4} \leq A < 1 \). Since

\[ y(A, B_2) = 4(A - 1)(A^3 - 3A^2 - A + 7) < 0, \]

by (5.18) the equation \( y(A, B) = 0 \) has at least one solution in the interval \((B_1, B_2)\). Next we have

\[
y'(A, B_2) = -A^4 + 8A^3 - 10A^2 - 24A + 31
\]
and

\[
y''_{BA}(A, B_2) = -4A^3 + 24A^2 - 20A - 24 < 0;
\]
thus, because of \( y'_B(1, B_2) > 0 \), the function \( y(A, B) \) increases in the interval \((B_2, B_1)\), which completes the proof.

**Corollary.** If \((A, B) \in G_2\), then \( y(A, B) < 0 \) if and only if \( B < B(A) \) and \( y(A, B) > 0 \) for \( B > B(A) \).

Basing ourselves on Lemmas 8–10, we obtain

**Theorem 6.** Let

\[
D_1 = \{(A, B): A_0 < A \leq 1, -1 < B < B(A)\},
\]

\[
D_2 = \{(A, B): (-1 < A \leq A_0, -1 < B < A) \cup (A_0 < A < 1, B(\leq A) \leq B < A)\},
\]

where \( B(\leq A) \) is the unique solution of the equation \( y(A, B) = 0 \) in the interval \((-1, B_1(\leq A))\) for \( A \in (A_0, \frac{3}{4}) \) and in the interval \((B_2(\leq A), B_1(\leq A))\) for \( A \in [\frac{3}{4}, 1)\), where \( A_0 \) is the unique root of the equation \( y(A, -1) = 0 \) in the interval \((0, \frac{3}{4})\) (cf. (5.9), (5.4), (5.5)).

Then the radius of convexity for the family \( S^*(A, B) \) is

\[
r \cdot c. S^*(A, B) = \begin{cases} r_1, & \text{if } (A, B) \in D_1, \\ r_2, & \text{if } (A, B) \in D_2, \end{cases}
\]

where

\[
r_1 = r_1(A, B) = 2[3A - B + \sqrt{(A - B)(5A - B)}]^{-1},
\]

\[
r_2 = r_2(A, B) = \sqrt[4]{(4 - 5A + B)[2A^2 - 3A + 2 - B + 2(1 - A)\sqrt{A^2 + 4A + 1 - 2B}]}^{-1}.
\]
The equality r. c. \( \{f\} = r_1 \) holds for the functions

\[
(5.27) \quad f^*(z) = \begin{cases} \varepsilon \cdot \exp \left[ \frac{A-B}{B} \log(1-B\varepsilon) \right] & \text{if } B \neq 0, \\ \varepsilon \cdot \exp(-A\varepsilon) & \text{if } B = 0 \end{cases}
\]

(cf. Theorem 4) and r. c. \( \{f\} = r_2 \) — for the function

\[
(5.28) \quad f^{**}(z) = \begin{cases} \varepsilon \cdot \exp \left( \frac{2(1+B)\varepsilon \zeta}{d(1-B)^2} \cdot \frac{1-d^2}{d^2} \cdot \log(1-d\varepsilon) \right) & \text{if } B = 0, A = A^*, \\ \varepsilon \cdot \exp \left( A \cdot \frac{d^2}{d^2} \cdot \frac{1-d^2}{d^2} \cdot \log(1-d\varepsilon) \right) & \text{if } B = 0, A \neq A^* \end{cases}
\]

where

\[
\log 1 = 0, \quad \varepsilon = e^{\frac{-i\varphi}{2}}, \quad 0 \leq \varphi \leq 2\pi, \quad A^* = (14 - 5\sqrt{3})11^{-1},
\]

\[
(5.29) \quad d = \frac{Ar_2^4 - 3(1-A)r_2^2 - 1}{(1-A)(1+r_2^2)r_2}, \quad \Delta = (1-B)^2d^2 + 4B,
\]

\[
\sqrt{\Delta} = \begin{cases} \sqrt{|d|} & \text{if } \Delta > 0, \\ i\sqrt{|d|} & \text{if } \Delta < 0 \end{cases}
\]

\[
z_0 = -\frac{(1-B)d\varepsilon}{2B}, \quad z_k = \frac{-(1-B)d + (-1)^k\sqrt{\Delta}}{2B}\varepsilon, \quad k = 1, 2,
\]

\[
W(z) = -B\varepsilon z^2 -(1-B)d\varepsilon z + 1.
\]

Proof. In view of Theorem 1 and Lemma 8

r. c. \( S^*(A, B) = \begin{cases} r_1 & \text{if } (A, B) \in G_2 \text{ or } (A, B) \in G_3, \; r_1 < r^*, \\ r_2 & \text{if } (A, B) \in G_1 \text{ or } (A, B) \in G_2, \; r_1 \geq r^* \end{cases} \]

where \( G_k \) \((k = 1, 2, 3)\) are given by (5.6)-(5.8), \( r_j \) \((j = 1, 2)\) are the roots of polynomials \( u(r) \) and \( v(r) \) (cf. (5.2), (5.3)), i.e. are the numbers (5.25) and (5.26), respectively, and finally \( r^* \) is the root of equation (3.36).

Because of Lemma 9 the condition \( r_1 < r^* \) is satisfied if inequality (5.9) is satisfied, and this is equivalent to \( B < B(A) \) (Lemma 10). Hence, we obtain (5.24). For \( B = B(A) \) we have \( r_1 = r_2 \).
Let \( f^*(z) \) be a function of \( S^*(A, B) \) such that

\[
\frac{zf'^*(z)}{f^*(z)} = P^*(z),
\]

where \( P^*(z) \) is given by (3.37). Then, from (5.30) we find

\[
\frac{f'^*(z)}{f^*(z)} - \frac{1}{z} = -\frac{(A - B)\varepsilon}{1 - Bz}.
\]

The functions of the variable \( z \) which appears on the right-hand side and the left-hand side of equation (5.31) are regular in the disc \( K \); hence the integrals of these functions exist along any regular curve \( F \subset K \) with the origin and the end-point at 0 and \( z \), respectively, where \( z \in K \). Thus we conclude that \( f^*(z) \) is of the form (5.27).

Evidently

\[
\text{re} \left(1 + \frac{zf'^{**}(z)}{f^{**}(z)}\right) \geq 0
\]

for \( |z| \leq r_1 \) with equality if and only if \( z = r_1 \). Thus \( f^*(z) \) is not convex in the disc \( |z| < r \) for \( r > r_1 \); i.e. r.e. \( \{f^*\} = r_1 \).

Next, let \( f^{**}(z) \) be a function of \( S^*(A, B) \) for which

\[
\frac{zf^{***}(z)}{f^{**}(z)} = P^{**}(z),
\]

where \( P^{**}(z) \) is given by (3.38).

Thus

\[
\frac{f^{***}(z)}{f^{**}(z)} - \frac{1}{z} = J(z)
\]

where

\[
J(z) = (A - B)\varepsilon \frac{d - ez}{W(z)}.
\]

We distinguish four cases.

1. \( B = 0, \; d = 0 \). Integrating (5.32) we obtain the first formula in (5.38). Since \( B = 0 \), we have \( X_2(r_2; A, 0) = 0 \). Thus

\[
(4A^2 - 5A)r_2^4 - 2(2A^2 - 3A + 2)r_2^4 + A - 5A = 0
\]

and in view of \( d = 0 \) we have

\[
A^2 + 3(1 - A)r_2^4 - 1 = 0
\]

(cf. (5.20)).

Eliminating \( r_2 \) from (5.34) and (5.35), we obtain \( A = A^* \). It can easily be verified that \( (A^*, 0) \in D_2 \).
Some extremal problems

2. \( B = 0, \ d \neq 0 \). We have \( W(z) = 1 - d z \), thus because of \( |d| \leq 1 \)
the function (5.33) is regular in \( K \). Integrating (5.32), we obtain the second
formula in (5.28).

3. \( B \neq 0, \ A = 0 \). In this case \( W(z) = -B \bar{z}^2 (z-z_0)^2 \), where \( B < 0 \)
and \( z_0 \neq 0 \). Next, we obtain \( |z_0| = \sqrt{-B^{-1}} \geq 1 \). Thus
\[
J(z) = -\frac{(A-B)z}{Bz_0^2} \frac{d - \bar{z} z}{\left(1 - \frac{z}{z_0}\right)^2}
\]
is a regular function in \( K \). Integrating (5.32) we obtain the third formula
in (5.28).

4. \( B \neq 0, A \neq 0 \). The polynomial \( W(z) \) can be represented in the form
\[
W(z) = \left(1 - \frac{z}{z_1}\right) \left(1 - \frac{z}{z_2}\right).
\]

We state that \( |z_k| \geq 1 \) for \( k = 1, 2 \). If \( A > 0 \) and \( B > 0 \), then \( e z_1 \leq -1 \).
Supposing the contrary, we would have \( A < 2B - (1-B)d \) and hence
\( (1-B)(1+d) < 0 \), which is impossible. Similarly we prove that \( e z_2 \geq 1 \).
As in the case just considered, we find that, for \( A > 0 \) and \( B < 0 \), \( e z_1 \geq 1 \)
and \( e z_2 \leq -1 \). If \( A < 0 \), then \( B < 0 \) and \( |z_k|^2 = -B^{-1} \geq 1 \). Thus \( |z_k| \geq 1 \)
for \( k = 1, 2 \) in every case. Hence, \( J(z) \) is regular in \( K \). Integrating (5.32),
we obtain the fourth formula in (5.27).

Evidently, in each of the four cases considered above, we have
\[
\text{re} \left(1 + \frac{z^{**}}{z^{*}}(z)\right) \geq 0
\]
for \( |z| \leq r_2 \), with equality if and only if \( z = r_2 \bar{z} \). Thus r. c. \( \{f^{**}\} = r_2 \),
and this completes the proof.

Applying Theorem 6 to the case where \( A = 1 - 2a \) and \( B = -1 \),
we obtain the result for the class \( S^* \) given by Zmorovič [9]. The problems
of the radius of convexity for \( S^* \) and \( S^*_1 \) have first been solved by Nevan-
linna [5] and Mac Gregor [3], respectively. If \( A = 1, B = 1/M - 1 \) or
\( A = \beta, B = -\beta \), then we obtain the corresponding theorems on r. c. \( S^*(M) \)[1] and r. c. \( S^{*(\theta)} [6] \), respectively. For the class \( S^*_0 \) we have
\[
\text{r. c. } S^*_0 = \begin{cases}
    r_1, & \text{if } \beta_0 < \beta \leq 1, \\
r_2, & \text{if } 0 < \beta \leq \beta_0,
\end{cases}
\]
where \( r_1 = r_1(\beta, 0), r_2 = r_2(\beta, 0) \) and
\[
\beta_0 = \frac{(3 - V5)(1 + V6)}{2V5}.
\]
References


[9] В. А. Зморович, *О границах выпуклости свяжных функций порядка а в круге \(|z| < 1\) и круговой области \(0 < |z| < 1\)*, Math. sb. 68 (1965), p. 518–526

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