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ANTIATOMICITY OF THE LATTICE OF BOREL STRUCTURES AND EXTENSIONS OF TWO-VALUED MEASURES

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1. In this paper we show that $L_{\mathscr{B}}$, the lattice of Borel substructures of a Borel structure $\mathscr{B}$, is antiatomic iff for any two-valued measure on a substructure of $\mathscr{B}$ there is a two-valued measure on $\mathscr{B}$ extending it. This and the second characterization of antiatomicity of $L_{\mathscr{B}}$ in terms of σ-isomorphism and a Lindelöf property provide an answer to the problem raised by K. P. S. Bhaskara Rao and B. V. Rao.

2. Preliminaries. A (*Borel*) *structure* ($\mathscr{B}$) on a set X is a σ-algebra of subsets of X; a pair $(X, \mathscr{B})$ is called a *Borel space*. $L_{\mathscr{B}}$ stands for the lattice of all Borel substructures of $\mathscr{B}$. The infimum of a family of structures is their intersection and the supremum is the structure generated by their union. $\mathscr{B}$ is the unit element and $\{\emptyset, X\}$ is the null element. $L_{\mathscr{B}}$ is a complete lattice. $L_{\mathscr{B}}$ is always atomic; atoms are of the form $\{\emptyset, X, B, X-B\}$ where $\emptyset \neq B \neq X$, and any $\mathscr{A} \in L_{\mathscr{B}}$ not null is a supremum of atoms. Following K. P. S. Bhaskara Rao and B. V. Rao ([1]), an $\mathscr{A} \in L_{\mathscr{B}}$, $\mathscr{A} \neq \mathscr{B}$, is an *antiatom* if $\mathscr{B}$ is the only element in $L_{\mathscr{B}}$ greater than $\mathscr{A}$. The idea to consider antiatoms appeared in [3] (where the term "ultrastructure" was used instead of "antiatom"). $L_{\mathscr{B}}$ is called *antiatomic* if every non-unit element of $L_{\mathscr{B}}$ is an infimum of antiatoms.

THEOREM 1 ([2], Theorem 1). *$\mathscr{A} \in L_{\mathscr{B}}$ is an antiatom iff there are two distinct* 0-1 *measures μ, ν on $\mathscr{B}$ such that*

$$\mathscr{A} = \{B \in \mathscr{B}: \mu(B) = \nu(B)\}$$

(a *measure* is a σ-additive real-valued nonnegative function and a 0-1 *measure* is a measure taking exactly two values 0, 1).

Thus $L_{\mathscr{B}}$ is antiatomic iff for every $\mathscr{A} \in L_{\mathscr{B}}$ and $Z \in \mathscr{B} - \mathscr{A}$ there are two 0-1 measures agreeing on $\mathscr{A}$ but differing on Z. In particular, if $\mathscr{B}$ contains a countably generated Borel structure with uncountably many atoms then $L_{\mathscr{B}}$ is not antiatomic ([1], Proposition 36). As K. P. S. Bhaskara Rao and B. V.

Rao wrote in [2] and repeated in [1], P12, "a neat characterization of $\mathscr{B}$ for which $L_{\mathscr{B}}$ is antiatomic is not known". Whether the two characterizations given below (Corollaries 3 and 4) are neat is the matter of taste but the author finds them interesting.

Two Borel structures $\mathscr{A}$ and $\mathscr{B}$ are called *isomorphic* if there is a one-to-one function h from $\mathscr{A}$ onto $\mathscr{B}$ which preserves finite unions and complementation, any such function is called an *isomorphism* ([5], p. 13). Because Borel structure is by definition a σ-field any isomorphism between Borel structures is σ-isomorphism, i.e. also preserves countable sums.

A Borel structure $\mathscr{B}$ on a set X *separates x and y*, $x, y \in X$, if there is a set $B \in \mathscr{B}$ such that $\{x, y\} \cap B = \{x\}$. A Borel space $(X, \mathscr{B})$ *has the Lindelöf property* if for any family $\mathscr{F} \subseteq \mathscr{B}$, $\bigcup\{F\colon F \in \mathscr{F}\} = X$ there is a countable subfamily $\mathscr{F}_1 \subseteq \mathscr{F}$, $\bigcup\{F\colon F \in \mathscr{F}_1\} = X$. A Borel space is *super Blackwell* if any two substructures separating the same points coincide.

We say that a 0-1 measure μ defined on the structure $\mathscr{B}$ is *given by a point* if there is $x \in X$ such that $\mu(B) = 1$ if $x \in B$ and $\mu(B) = 0$ if $x \notin B$ for any $B \in \mathscr{B}$.

3. Lindelöf property.

PROPOSITION 1. *A Borel space $(X, \mathscr{B})$ has the Lindelöf property iff any 0-1 measure defined on a substructure of $\mathscr{B}$ is given by a point.*

Proof. Suppose there is a substructure $\mathscr{A}$ of $\mathscr{B}$ and a 0-1 measure μ on $\mathscr{A}$ not given by a point. Thus the family $\{A \in \mathscr{A}\colon \mu(A) = 0\}$ is a covering of X which is closed under countable unions and has no countable subcover. So $(X, \mathscr{B})$ does not have the Lindelöf property.

On the other hand if $(X, \mathscr{B})$ does not have the Lindelöf property, then let $\mathscr{F} = \{B_s\colon s \in S\} \subseteq \mathscr{B}$ be a covering of X without a countable subcover. Let $\mathscr{A}$ be a family of sets elements of which may be covered by a countable subfamily of $\mathscr{F}$ and complements of such sets. $\mathscr{A}$ is a Borel structure. Let μ be a measure on $\mathscr{A}$ defined by: $\mu(A) = 0$ if A is a subset of the union of countably many sets from $\mathscr{F}$ and $\mu(A) = 1$ otherwise. μ is not given by a point. This completes the proof.

COROLLARY 1. *A Borel space $(X, \mathscr{B})$ has the Lindelöf property iff any 0-1 measure defined on a substructure of $\mathscr{B}$ can be extended to a 0-1 measure on $\mathscr{B}$ and any 0-1 measure on $\mathscr{B}$ is given by a point.*

THEOREM 2. *A Borel space $(X, \mathscr{B})$ is super Blackwell iff $(X, \mathscr{B})$ has the Lindelöf property.*

Proof. Suppose $(X, \mathscr{B})$ does not have the Lindelöf property. There is a family $\mathscr{F} \subseteq \mathscr{B}$ which covers X but no countable subfamily of $\mathscr{F}$ does. Let $\mathscr{A}$ be a Borel structure on X generated by $\mathscr{F}$. As in the proof of Proposition 1 let μ be a 0-1 measure on $\mathscr{A}$ not given by a point. Take some point $x_0 \in X$ and denote by ν a 0-1 measure on $\mathscr{A}$ given by the point x_0. For any $x \in X$ fix a set $A(x) \in \mathscr{A}$ containing x such that $\mu(A(x)) = 0$. For any two points x, y separated by $\mathscr{A}$ fix a set $A(x, y) \in \mathscr{A}$ separating them, that is

$\{x, y\} \cap A(x, y) = \{x\}$. Put $\mathscr{C} = \{A \in \mathscr{A}: \mu(A) = \nu(A)\}$. $\mathscr{C}$ is a Borel structure strictly smaller than $\mathscr{A}$ ($A(x_0)$ is not in $\mathscr{C}$). If two points $x, y \in X$ are separated by $\mathscr{A}$ and, say, x, x_0 are also separated by $\mathscr{A}$, then the set $A(x) \cap A(x, y) \cap A(x, x_0)$ belongs to $\mathscr{C}$ and separates x, y. We have obtained two different structures in $L_{\mathscr{B}}$ which separate the same points; therefore $(X, \mathscr{B})$ is not super Blackwell.

Assume now that $(X, \mathscr{B})$ has the Lindelöf property. Let $\mathscr{A}, \mathscr{C} \in L_{\mathscr{B}}$ separate the same points. Fix an arbitrary set $A \in \mathscr{A}$. For any two points x, y, $x \in A$, $y \in X - A$, choose a set $C(x, y) \in \mathscr{C}$ such that $C(x, y) \cap \{x, y\} = \{x\}$. For each $y \in X - A$ a family $\{C(x, y): x \in A\} \cup \{X - A\}$ covers X. By assumption there is a countable set $\{x_n: n = 1, \ldots\} \subseteq A$ for which the set $C(y) = \bigcup \{C(x_n, y): n = 1, \ldots\}$ belongs to $\mathscr{C}$, contains A and by our choice does not contain y. A family $\{X - C(y): y \in X - A\} \cup \{A\}$ covers X; by the Lindelöf property there is a countable set $\{y_n: n = 1, \ldots\} \subseteq X - A$ such that $\bigcup \{X - C(y_n): n = 1, \ldots\} \supseteq X - A$. But $X - C(y_n) \subseteq X - A$. Then the above union is equal to $X - A$, and $A = \bigcap \{C(y_n): n = 1, \ldots\}$ belongs to $\mathscr{C}$. This shows that $\mathscr{A} \subseteq \mathscr{C}$. By symmetry of assumptions about $\mathscr{A}$ and $\mathscr{C}$, also $\mathscr{C} \subseteq \mathscr{A}$. We have shown that $(X, \mathscr{B})$ is super Blackwell.

Theorem 1 and Theorem 2 yield

COROLLARY 2. *Let $(X, \mathscr{B})$ be a Borel space every* 0-1 *measure on which is given by a point. The following conditions are equivalent*:

(i) *$L_{\mathscr{B}}$ is antiatomic,*

(ii) *$(X, \mathscr{B})$ is super Blackwell,*

(iii) *$(X, \mathscr{B})$ has the Lindelöf property.*

4. Antiatomicity and extensions of 0-1 measures. To obtain promised characterizations of antiatomicity we need

PROPOSITION 2. *Any Borel structure is σ-isomorphic to a Borel structure every* 0-1 *measure on which is given by a point.*

Proof. Let $(X, \mathscr{B})$ be a Borel space. Let Z be the set of all 0-1 measures on $\mathscr{B}$. For any $B \in \mathscr{B}$ let $h(B)$ be the set of all 0-1 measures which take value 1 on $\mathscr{B}$. It is straightforward to verify that the family $h(\mathscr{B}) = \{h(B): B \in \mathscr{B}\}$ is a Borel structure on Z, and h is a σ-isomorphism from $\mathscr{B}$ onto $h(\mathscr{B})$ (for details see [3], Theorem 8.1, preservation of countable unions relies on the fact that Z consists of countably additive set functions). Using the σ-isomorphism h it is easily seen that on $(Z, h(\mathscr{B}))$ any 0-1 measure is given by a point. So the proposition is proved.

It is obvious that antiatomicity is preserved by σ-isomorphism. The property that any 0-1 measure defined on a substructure can be extended to the whole structure is also preserved by σ-isomorphism. Combining these facts with Proposition 2, Corollary 1 and Corollary 2 we immediately obtain

COROLLARY 3. *$L_{\mathscr{B}}$ is antiatomic iff $\mathscr{B}$ is σ-isomorphic to a Borel structure with the Lindelöf property.*

COROLLARY 4. *$L_{\mathscr{B}}$ is antiatomic iff any* 0-1 *measure defined on a substructure of $\mathscr{B}$ can be extended to a* 0-1 *measure on $\mathscr{B}$.*

PROPOSITION 3. *Let $(X, \mathscr{B})$ be a Borel space. Any* 0-1 *measure defined on a substructure of $\mathscr{B}$ can be extended to a* 0-1 *measure on $\mathscr{B}$ iff* (a) *any countably generated substructure of $\mathscr{B}$ has countably many atoms and* (b) *any measure defined on a substructure of $\mathscr{B}$ can be extended to a measure on $\mathscr{B}$.*

Proof. If $\mathscr{B}$ has the property (a) then any measure on $\mathscr{B}$ (and obviously on any substructure of $\mathscr{B}$) is a countable sum of two-valued measures.

If $\mathscr{B}$ has properties (a) and (b) and μ is any 0-1 measure on a substructure of $\mathscr{B}$ then there is a measure ν on $\mathscr{B}$ which extends μ. By the remark at the beginning of this proof $\nu = \nu_1 + \nu_2 + \ldots$ where ν_i is a two-valued measure for all i (this sum may be finite). $\tilde{\mu}$ defined by $\tilde{\mu}(B) = \nu_1(B)/\nu_1(X)$ is a 0-1 measure extending μ.

If any 0-1 measure on a substructure can be extended to a 0-1 measure on $\mathscr{B}$ then (e.g., by Corollary 4 and the second remark after Theorem 1) $\mathscr{B}$ has property (a). By the remark at the beginning of the proof, any measure μ on a substructure of $\mathscr{B}$ is a countable sum of two-valued measures all of which can be extended to a measure on $\mathscr{B}$, so μ also has an extension onto $\mathscr{B}$. This ends the proof.

Proposition 3 is not true without condition (a). More precisely, D. H. Fremlin has shown (private communication of June 1981) that it is not a theorem of ZFC: whether it is true or false depends on additional axioms (for example Continuum Hypothesis implies that the σ-field of Borel sets on the Sierpiński set satisfies (b) but is not antiatomic). It is not known to the author whether Proposition 3 is true without condition (b). **(P 1302)**

5. Example. K. P. S. Bhaskara Rao and B. V. Rao have shown that for any set X and the Borel structure of countable subsets of X and their complements, say $\mathscr{A}$, $L_{\mathscr{A}}$ is antiatomic (it is easy to see this via Corollary 4). We want to give an example of an atomless Borel structure $\mathscr{B}$ with $L_{\mathscr{B}}$ antiatomic. $\mathscr{B}$ is *atomless* if any nonempty set in $\mathscr{B}$ is a disjoint union of two nonempty sets belonging to $\mathscr{B}$.

PROPOSITION 4. *On any uncountable set I there is an atomless structure separating all points of I which has the Lindelöf property.*

Proof. Consider the set $X(I) \subseteq \{0, 1\}^I$ of points whose all but finitely many coordinates are 0. Later on we will identify a point $x \in X(I)$ with a subset of I — the set of coordinates on which x is 1. In this terminology $X(I)$ consists of all finite subsets of I. Because I and $X(I)$ have the same cardinality it is sufficient to construct a Borel structure with desired properties on $X(I)$. Let $B(i)$ be the set of points in $X(I)$ the ith coordinate of which is 1, and $\mathscr{B}(I)$ the Borel structure on $X(I)$ generated by the family $\{B(i): i \in I\}$. $\mathscr{B}(I)$ is the Borel structure we are looking for. It separates all points.

The following property of $\mathscr{B}(I)$ is easily established: for any set $B \in \mathscr{B}(I)$

there exists a countable set $J(B) \subseteq I$ such that if two points $x, y \in X(I)$ have the ith coordinates equal for all $i \in J(B)$ then either $\{x, y\} \subseteq B$ or $\{x, y\} \cap B = \emptyset$ (the family of sets with this property is closed under complementation and countable unions, the sets $B(i)$ have this property). This implies that $\mathscr{B}(I)$ is atomless (if $B \in \mathscr{B}(I)$ is nonempty and $i \in I - J(B)$ then $B \cap B(i) \neq \emptyset \neq B - B(i)$).

Let $\mathscr{F} \subseteq \mathscr{B}(I)$ be a covering of $X(I)$. Take $F_0 \in \mathscr{F}$ containing the empty set (i.e. the point with each coordinate equal to 0). All finite subsets of $I - J(F_0)$ are in F_0. For all remaining (countably many) one-element subsets of I there is a countable family $F'_1 \subseteq \mathscr{F}$ covering them. Put $\mathscr{F}_1 = \mathscr{F}'_1 \cup \{F_0\}$. All finite subsets of I of the form $K \cup L$, where $K \subseteq \bigcup \{J(F)\colon F \in \mathscr{F}_1\}$, $\operatorname{card}(K) \leqslant 1$ and $L \subseteq I - \bigcup \{J(F)\colon F \in \mathscr{F}_1\}$, belong to $\bigcup \{F\colon F \in \mathscr{F}_1\}$. For all remaining (countably many) two-element subsets of I there is a countable family $\mathscr{F}'_2 \subseteq \mathscr{F}$ covering them. Put $\mathscr{F}_2 = \mathscr{F}'_2 \cup \mathscr{F}_1$. All finite subsets of I of the form $K \cup L$, where $K \subseteq \bigcup \{J(F)\colon F \in \mathscr{F}_2\}$, $\operatorname{card}(K) \leqslant 2$, $L \subseteq I - \bigcup \{J(F)\colon F \in \mathscr{F}_2\}$, are contained in $\bigcup \{F\colon F \in \mathscr{F}_2\}$. By analogy, step by step, we define an increasing sequence of countable families $\mathscr{F}_n \subseteq \mathscr{F}$, such that all n-element subsets of I belong to $\bigcup \{F\colon F \in \mathscr{F}_n\}$. $\bigcup \{\mathscr{F}_n\colon n = 1, \ldots\}$ is a countable family contained in $\mathscr{F}$ and by definition of $X(I)$ covers $X(I)$. Hence $(X(I), \mathscr{B}(I))$ has the Lindelöf property. The proof is complete.

Proposition 4 with the same Borel structure in the proof but without the Lindelöf property is stated in Theorem 4 of [4].

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