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On a theorem of Lebow

by W. Mlak (Kraków)

Abstract. Let H be a complex Hilbert space, T a bounded linear operator in H with spectrum in the closed unit disc. It is shown that if for every $f, g \in H$ the function $((I-zT)^{-1}f, g) \in H^1$, then the spectral radius of T is less than one.

Let H be a complex Hilbert space with the inner product (f, g) $(f, g \in H)$. A linear, bounded operator $T: H \to H$ is called *polynomially bounded* if for some finite, positive K and every polynomial $p(z) = \sum_{k/0}^{n} a_k z^k$ the inequality

$$\|p(T)\| \leqslant K \sup_{|z|\leqslant 1} |p(z)| \tag{1}$$

holds true. If T satisfies (1), then there are complex measures $\mu(f, g)$ on $C = \{z \mid |z| = 1\}$ such that

$$(T^n f, g) = \int_C \bar{z}^n d\mu(f, g); \quad f, g \in H, \ n = 0, 1, 2, \ldots$$

Let m be the normalized linear Lebesgue measure on C. It is known that $H = H^a + H^s$ (direct sum), where H^a and H^s are invariant subspaces of T and for $T_a = T|H^a$, $T_s = T|H^s$ we have for $n = 0, 1, \ldots$

$$(T_a^n f, g) = \int_C \bar{z}^n h_{f,g} dm \big(h_{f,g} \in L^1(m)\big) \tag{2}$$

for $f, g \in H^a$ and

$$(T_s^n f, g) = \int_C \bar{z}^n d\mu_{f,g}^s$$

for $f, g \in H^s$, where $\mu_{f,g}^s \perp m$.

One can prove that T_s is similar to a unitary singular operator [2]. If $T = T_a$, then T is called *absolutely continuous*. Lebow proved in [1] that if for every $f, g \in H$ there is a function $h_{f,g}$ satisfying (2) for $n = 0, 1, 2, \ldots$ and such that $h_{f,g} \in \bigcup_{p>1} L^p(m)$, then the spectral radius

$r(T)$ of T is less than $1-r(T)<1$. There was involved a certain result of Grothendieck and the M. Riesz theorem saying that the natural projection of $L^p(m)$ onto the Hardy space $H^p(m)$ is bounded if $p>1$. We offer here a theorem which implies among others the Lebow's result simply by using the above metioned M. Riesz theorem. Roughly speaking we show that $r(T)<1$ provided $h_{f,g}\in H^1(m)$.

To begin with we recall the Hardy–Littlewood inequality

$$\sum_{n|0}^{\infty}\frac{|a_n|}{n+1}\leqslant\pi\|u\|_1$$

valid for $u\in H^1(m)$; a_n is the n-th Fourier coefficient of u and $\|u\|_1 = \int\limits_C |u|\,dm$.

THEOREM. *Suppose $T\colon H\to H$ is a linear, bounded operator with spectrum included in the unit disc $\{z\,|\,|z|\leqslant 1\}$. We assume that for every $f,g\in H$ the function $u_{f,g}(z)=\big((I-zT)^{-1}f,g\big)$ is in the Hardy space $H^1(m)$, that is for every $f,g\in H$ there is $h_{f,g}\in H^1(m)$ such that $(T^nf,g)=\int\limits_C\bar z^n h_{f,g}\,dm$ for $n=0,1,2,\ldots$ Then $r(T)<1$.*

Proof. Suppose that $r(T)=1$. Let A be an arbitrary, commutative algebra of operators closed in the operator norm, which contains T and the identity operator. Since $r(T)=\lim\|T^n\|^{1/n}$, the spectral radius of T relative to A equals 1. It follows that there is $z_0\in Sp_A(T)$, $|z_0|=1$. Define $S=\bar z_0T$. Since $Sp_A(S)=\bar z_0Sp_A(T)$, then $1\in Sp_A(S)$. On the other hand, since $|z_0|=1$ and m is invariant under rotations and $u_{f,g}\in H^1(m)$, then for $0\leqslant r<1$

$$\sup_r\int\limits_C\big|\big((I-rzS)^{-1}f,g\big)\big|\,dm_z\leqslant\|u_{f,g}\|_1.$$

It follows from the Hardy–Littlewood inequality that

$$\left|\left(\left(\sum_{n|0}^{m}\frac{S^n}{n+1}\right)f,g\right)\right|\leqslant\sum_{n|0}^{\infty}\frac{|(S^nf,g)|}{n+1}\leqslant\pi\|u_{f,g}\|_1$$

because (S^nf,g) is the n-th Fourier coefficient of $\big((I-zS)^{-1}f,g\big)$. We conclude that the sequence of operators

$$R_m=\sum_{n|0}^{m}\frac{S^n}{n+1}$$

is weakly bounded. Consequently, R_m is bounded, i.e., $M=\sup\limits_m\|R_m\|<\infty$. Since $1\in Sp_A(S)$ there is a character h of A such that $h(S)=1$. Then

$$|h(R_m)|\leqslant\|R_m\|\leqslant M$$

for all m. But

$$h(R_m) = \sum_{n|0}^{m} \frac{h(S^n)}{n+1} = \sum_{n|0}^{m} \frac{1}{n+1}$$

which implies that

$$\sum_{n|0}^{m} \frac{1}{n+1} \leqslant M < +\infty$$

which is impossible. That proves the claim.

Notice that the assumptions of our theorem imply that T is an absolutely continuous polynomially bounded operator. The Lebow's theorem can be derived from our theorem as follows: If $(T^n f, g) = \int_C \bar{z}^n h_{f,g}\, dm$, $n = 0, 1, 2, \ldots$ for $f, g \in H$ and $h_{f,g} \in \bigcup_{p>1} L^p(m)$, then $h_{f,g} \in L^p(m)$ for some p depending on f, g and by M. Riesz theorem $(T^n f, g)$ is a sequence of Fourier coefficients of a function in $H^p(m)$. Since $p > 1$, $H^p(m) \subset H^1(m)$ which proves that the function $\langle (I - zT)^{-1} f, g\rangle = \sum_{n|0}^{n} (T^n f, g) z^n$ is in $H^1(m)$ and our theorem applies.

References

[1] A. Lebow, *Spectral radius of an absolutely continuous operator*, Proc. Amer. Math. Soc. 36 (1972), p. 511–514.

[2] W. Mlak, *Algebraic polynomially bounded operators*, Ann. Polon. Math. 29 (1974), p. 133–139.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
KRAKÓW, POLAND

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