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N-STEP NILPOTENT LIE GROUPS WITH FLAT KIRILLOV ORBITS

BY

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Moore and Wolf showed in [5] that irreducible unitary representations of a nilpotent Lie group which are square-integrable modulo the center correspond to Kirillov orbits which are flat – that is, translates of subspaces of the linear dual of the Lie algebra. They allude also to the existence of nilpotent Lie groups having arbitrarily long lower central series for which the orbits of maximal dimension are flat. (See also [2].) In this note*, we will provide an example, for each positive integer n, of an n-step $(2n+1)$-dimensional nilpotent Lie algebra N_n with rational structure constants for which *all* coadjoint orbits are flat – even those of lower dimension. Then we will prove a simple Fejer theorem for all compact nilmanifolds corresponding to Lie algebras with exclusively flat coadjoint orbits.

In the following example, $[X_i X_j] = 0 = [Y_i Y_j]$ for all $i, j = 1, \ldots, n$, and Z is central. All non-zero bracket products will be generated by the following list:

$$[X_1 Y_1] = Y_2,\ [X_1 Y_2] = Y_3,\ [X_1 Y_3] = Y_4, \ldots, [X_1 Y_n] = Z,$$

$$[X_2 Y_1] = Z,$$

$$[X_3 Y_2] = Z,\ [X_3 Y_1] = Y_n,$$

$$[X_4 Y_3] = Z,\ [X_4 Y_2] = Y_n,\ [X_4 Y_1] = Y_{n-1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$[X_n Y_{n-1}] = Z,\ [X_n Y_{n-2}] = Y_n,\ [X_n Y_{n-3}] = Y_{n-1}, \ldots, [X_n Y_1] = Y_3.$$

Clearly, the lower central series has n steps. To check Jacobi's identity, let $Z = Y_{n+1}$, for convenience, and let $Y_{n+1+p} = 0$ for $p > 0$. Then the identity $[X_k [X_1 Y_j]] + [X_1 [Y_j X_k]] = 0$ follows from

$$(*) \qquad [X_k Y_l] = \begin{cases} 0, & l \geqslant k, \\ Y_{n+l-k+2}, & l < k, \end{cases} \qquad \text{for } k = 2, 3, \ldots, n.$$

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To check Jacobi's identity for X_j, X_k, and Y_l, where j and k are both greater than 1, we use (*) to show that $[X_j[X_k Y_l]] = [X_k[X_j Y_l]]$.

To see that all Kirillov orbits are flat, suppose first that $\Lambda(Z) = \lambda \neq 0$. We can see then that $O_N(\Lambda) = \Lambda + Z^{\perp}$ as follows. The action of Y_n saturates the orbit in the X_1'-direction, while the action of X_1 saturates the orbit in the Y_n'-direction. The action of Y_1 saturates the orbit in the X_2'-direction, while the action of X_2 saturates the orbit in the Y_1'-direction. We reason similarly for the pairs (X_3, Y_2), $(X_4, Y_3), \ldots, (X_n, Y_{n-1})$.

Now suppose $\Lambda(Z) = 0$, but $\Lambda(Y_n) = \lambda \neq 0$. Now the new pairs (X_1, Y_{n-1}), (X_3, Y_1), $(X_4, Y_2), \ldots, (X_n, Y_{n-2})$ show that $O_N(\Lambda) = \Lambda + \langle Y_n, Z\rangle_{\text{span}}^{\perp}$. Similar reasoning shows that every orbit, even those not in standard position, is flat. It is interesting to note that although it is initially only to satisfy Jacobi's identity that we have, for example, $[X_3 Y_1] = Y_n$, these same relations are exactly what we need to flatten those orbits which are not of maximum dimension.

Since this Lie algebra $\dot{N}_n$ has rational structure constants, $N = \exp(\dot{N}_n)$ has discrete subgroups Γ such that $\Gamma\backslash N$ is compact [4]. If π is an irreducible unitary representation of N occurring in the direct sum decomposition of $L^2(\Gamma\backslash N)$, then the π-primary component $P_\pi(f)$ of $f \in L^2(\Gamma\backslash N)$ is simply the orthogonal projection of f onto the π-primary summand. Since the orbit of π is flat, if f is continuous, so is $P_\pi(f)$ (see [1]).

THEOREM. *Let $\Gamma\backslash N$ be any compact nilmanifold for which N has exclusively flat Kirillov orbits. Then every continuous function on $\Gamma\backslash N$ is the uniform limit of a sequence of finite linear combinations of its own primary components.*

Proof(¹). We will proceed by induction on the dimension of N. Let Z denote the center of N, let dz denote normalized Haar measure on the torus $\Gamma \cap Z\backslash Z$, and let F_n denote the n-th Fejer kernel on that central torus [6]. By the classical Fejer theorem, if $\varepsilon > 0$, there is an n for which

$$\|f - f * F_n(z)\, dz\|_{L^\infty(\Gamma\backslash N)} < \varepsilon.$$

But $f * F_n(z)\, dz$ is a finite linear combination of terms $f_1, \ldots, f_n$ of the Fourier decomposition of f over $Z \cap \Gamma\backslash Z$. And, if $\dim Z \geqslant 2$, then each f_i lives on a lower-dimensional manifold. Thus each f_i is a uniform limit of finite linear combinations of its primary components, which in turn are primary components of f. But if $\dim Z = 1$, then the f_i's are already primary components of f because of the flat orbits (see [1]).

We note that a much more general approximation theorem of this sort has been obtained recently by Hulanicki and Jenkins [3], but their theorem does not use the actual primary components of f to approximate f.

(¹) The author's original proof of this theorem was much longer. It is a pleasure to thank the referee of the earlier manuscript for the shorter proof.

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