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# *SOME REMARKS ON MICHAEL'S QUESTION CONCERNING CARTESIAN PRODUCTS OF LINDELÖF SPACES*

BY

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E. Michael asked whether the class $\mathscr{L}$ of all spaces, whose Cartesian product with every hereditarily Lindelöf space is Lindelöf, is closed with respect to countable Cartesian products. In [1] it is proved that there is a large class $\mathscr{C}$ containing scattered Lindelöf spaces such that if $X \in \mathscr{C}$, then $X^{\aleph_0} \in \mathscr{L}$.

A particular case of Michael's hypothesis is the following very natural question: Is it true that if $M$ is a separable metric space and $X$ is a Lindelöf $P$-space, i.e. every $G_\delta$-set is open in $X$, then the product $M \times X^{\aleph_0}$ is Lindelöf? I conjecture that the answer to this question is negative but I can prove the following positive result:

THEOREM. *If $M$ is a separable metric space such that there are a complete, in the sense of Čech, separable space $M'$ and an embedding $e$: $M \to M'$ such that $M' \setminus e(M)$ does not contain uncountable compact subsets and $X_n$ is a Lindelöf $P$-space for $n = 1, 2, \ldots$, then the Cartesian product $M \times \mathop{\mathsf{P}}\limits_{n=1}^{\infty} X_n$ is Lindelöf.*

Proof. Suppose $\mathscr{F} = \{F_s\colon s \in S\}$ is a family of closed subsets of $M \times \mathop{\mathsf{P}}\limits_{n=1}^{\infty} X_n$ which has the countable intersection property, i.e. for every countable subset $S_0$ of $S$ the intersection $\bigcap \{F_s\colon s \in S_0\}$ is not empty. We shall show that the set $\bigcap \{F_s\colon s \in S\}$ is not empty. Without loss of generality we can assume that $M$ is a subset of $M'$. We shall consider two cases:

(a) There is $m \in M$ such that the family

$$\{F_s\colon s \in S\} \cup \Big\{\{m\} \times \mathop{\mathsf{P}}\limits_{n=1}^{\infty} X_n\Big\}$$

has the countable intersection property.

(b) Case (a) does not hold.

Proof of (a). This case is Noble's result (see [3], Corollary 4.2). We shall give its proof for the sake of completeness. Our proof is a little simpler than the Noble's one.

Let us assume that for $n \leqslant i$ we have defined $x_n \in X_n$ in a way such that

$(*_n)$ the family

$$\{F_s\colon s \in S\} \cup \{\{m\} \times \mathop{\mathsf{P}}_{n' \leqslant n} B_{n'} \times \mathop{\mathsf{P}}_{n'=n+1}^{\infty} X_{n'}\colon B_{n'} \in \mathscr{B}_{n'},\ \mathscr{B}_{n'} \text{ is a base at } x_{n'} \text{ for } n' \leqslant n\}$$

has the countable intersection property.

Then there is $x_{i+1} \in X_{i+1}$ such that $x_1, \ldots, x_{i+1}$ satisfy $(*_{i+1})$. Suppose to the contrary that such a point does not exist. Then for every $z$ in $X_{i+1}$ we shall find a neighbourhood $U_z$, a countable subset $S_z$ in $S$, and $B_{nz} \in \mathscr{B}_n$ for $n \leqslant i$ such that

$$\bigcap \{F_s\colon s \in S_z\} \cap \{m\} \times \mathop{\mathsf{P}}_{n=1}^{i} B_{nz} \times U_z \times \mathop{\mathsf{P}}_{n=i+2}^{\infty} X_n = \emptyset.$$

The space $X_{i+1}$ is Lindelöf, so we can find a countable subcover $\{U_{z_j}\colon j = 1, 2, \ldots\}$ of $\{U_z\colon z \in X_{i+1}\}$. Notice that the set

$$B_n = \bigcap_{j=1}^{\infty} B_{nz_j}$$

is a neighbourhood of $x_n$ for $n = 1, 2, \ldots, i$ because $X_n$ is a $P$-space. Hence the equality

$$\bigcap \{F_s\colon s \in \bigcup_{j=1}^{\infty} S_{z_j}\} \cap (\{m\} \times \mathop{\mathsf{P}}_{n=1}^{i} B_n \times \mathop{\mathsf{P}}_{n=i+1}^{\infty} X_n)$$
$$= \bigcap \{F_s\colon s \in \bigcup_{j=1}^{\infty} S_{z_j}\} \cap \big(\{m\} \times \mathop{\mathsf{P}}_{n=1}^{i} B_n \times (\bigcup \{U_{z_j}\colon j = 1, 2, \ldots\}) \times \mathop{\mathsf{P}}_{n=i+2}^{\infty} X_n\big)$$
$$= \emptyset$$

contradicts $(*_i)$.

For $n = 1, 2, \ldots$ it follows from $(*_n)$ that the point $(m, x_1, \ldots, x_n, \ldots)$ belongs to $\bigcap \{F_s\colon s \in S\}$.

Proof of (b). Let $d$ be a complete metric on $M'$. From the Lindelöf property of $M$ and the countable intersection property of $\mathscr{F}$ it follows that there is $m_1 \in M$ such that the family $\mathscr{F} \cup \{B(m_1, 1) \times \mathop{\mathsf{P}}_{n=1}^{\infty} X_n\}$ has the countable intersection property. Here the symbol $B(m_1, j)$ stands for

$$B(m_1, j) = \{m \in M'\colon d(m_1, m) \leqslant j\}.$$

Put $\mathscr{F}_0 = \{B(m_1, 1)\}$ and let us assume that $\mathscr{F}_j$ is defined for $j = 0, 1, \ldots, i$ in a way such that the following conditions are satisfied:

(1) if $j > 0$, then

$$\mathscr{F}_j = \Big\{\big(B(m_t, 1/2^{n(t)}), (x_{t1}, \ldots, x_{tj})\big):$$
$$t \in T_j,\ T_j \text{ is a finite set},\ m_t \in M,\ (x_{t1}, \ldots, x_{tj}) \in \mathop{\mathsf{P}}_{n=1}^{j} X_n,\ n(t) \geqslant j\Big\};$$

(2) if $j \leqslant i-1$ and $F = \big(B(m_t, 1/2^{n(t)}), (x_{t1}, \ldots, x_{tj})\big) \in \mathscr{F}_j$, then we can attach to $F$ two elements $F(0)$ and $F(1)$ of $\mathscr{F}_{j+1}$ in a way such that

$$F(k) = \Big(B(m_{t(k)}, 1/2^{n(t(k))}), \big(x_{t1}, \ldots, x_{tj}, y(t(k))\big)\Big) \quad \text{for } k = 0, 1,$$
$$\bigcup \{B(m_{t(k)}, 1/2^{n(t(k))}): k = 0, 1\} \subset B(m_t, 1/2^{n(t)}),$$
$$\bigcap_{k=0}^{1} B(m_{t(k)}, 1/2^{n(k)}) = \emptyset, \quad \text{and} \quad \mathscr{F}_{j+1} = \{F(k): F \in \mathscr{F}_j,\ k = 0, 1\};$$

(3) if $F = \big(B(m_t, 1/2^{n(t)}), (x_{t1}, \ldots, x_{tj})\big) \in \mathscr{F}_j$, then the family

$$\mathscr{F} \cup \Big\{B(m_t, 1/2^{n(t)}) \times \mathop{\mathsf{P}}_{n=1}^{j} B_n \times \mathop{\mathsf{P}}_{n=j+2}^{\infty} X_n:$$
$$B_n \text{ is an element of the base } \mathscr{B}(x_{tn}) \text{ at the point } x_{tn}\Big\}$$

has the countable intersection property.

Let $F = \big(B(m_t, 1/2^{n(t)}), (x_{t1}, \ldots, x_{ti})\big)$ be an element of $\mathscr{F}_i$. We claim that

(4) there are two different elements $m_{t(0)}, m_{t(1)}$ of $B(m_t, 1/2^{n(t)}) \cap M$ and $y(t(0)), y(t(1)) \in X_{i+1}$ such that if $U_j$ is an arbitrary neighbourhood of $m_{t(j)}$, then the family

$$\mathscr{F} \cup \Big\{U_j \times \mathop{\mathsf{P}}_{n=1}^{i+1} B_n \times \mathop{\mathsf{P}}_{n=i+2}^{\infty} X_n: B_n \in \mathscr{B}(x_{tn}) \text{ for } n \leqslant i \text{ and}$$
$$B_{i+1} \text{ belongs to the base } \mathscr{B}\big(y(t(j))\big) \text{ at the point } y(t(j))\Big\}$$

has the countable intersection property for $j = 0, 1$.

Notice that we do not require $y(t(0)) \neq y(t(1))$.

Suppose to the contrary that this is not the case. Then there is $m \in B(m_t, 1/2^{n(t)}) \cap M$ such that if $m' \in B(m_t, 1/2^{n(t)}) \cap M$ and $m' \neq m$, then for every $x \in X_{i+1}$ there exist neighbourhoods $V_{(x,m')}$ and $U_{(x,m')}$ of the points $m'$ and $x$, respectively, a countable subset $S(x, m')$, and $B_{(x,m')n} \in B(x_{tn})$ such that

$$\bigcap \{F_s: s \in S(x, m')\} \cap V_{(x,m')} \times \mathop{\mathsf{P}}_{n=1}^{i} B_{(x,m')n} \times U_{(x,m')} \times \mathop{\mathsf{P}}_{n=i+2}^{\infty} X_n = \emptyset.$$

Since the space $B(m_t, 1/2^{n(t)})\setminus\{m\}$ is Lindelöf and $X_1, X_2, \ldots$ are $P$-spaces, we can assume that for every $m' \in B(m_t, 1/2^{n(t)})\setminus\{m\}$ we have $S(x, m') = S_x$, $U_{(x,m')} = U_x$, and $B_{(m',x)n} = B_{xn}$ for $x \in X_{i+1}$ and $n = 1, 2, \ldots, i$. Let $\{U_{z_j}: j = 1, 2, \ldots\}$ be a countable subcover of $\{U_x: x \in X_{i+1}\}$. If we put

$$B_n = \bigcap_{j=1}^{\infty} B_{z_j n} \quad \text{and} \quad S_0 = \bigcup \{S_{z_j}: j = 1, 2, \ldots\}$$

for $n = 1, 2, \ldots, i$, then

$$\bigcap\{F_s: s \in S_0\} \cap \big(B(m_t, 1/2^{n(t)}) \times \mathop{\mathsf{P}}_{n=1}^{i} B_n \times \mathop{\mathsf{P}}_{n=i+1}^{\infty} X_n\big)$$

$$= \bigcap\{F_s: s \in S_0\} \cap \Big(\big(\{m\} \times \mathop{\mathsf{P}}_{n=1}^{i} B_n \times \mathop{\mathsf{P}}_{n=i+1}^{\infty} X_n\big) \cup \big(B(m_t, 1/2^{n(t)})\setminus\{m\}\big) \times$$

$$\times \mathop{\mathsf{P}}_{n=1}^{i} B_n \times \big(\bigcup\{U_{z_j}: j = 1, 2, \ldots\}\big) \times \mathop{\mathsf{P}}_{n=i+2}^{\infty} X_n\Big)$$

$$= \bigcap\{F_s: s \in S_0\} \cap \big(\{m\} \times \mathop{\mathsf{P}}_{n=1}^{i} B_n \times \mathop{\mathsf{P}}_{n=i+1}^{\infty} X_n\big).$$

The last equality and (3) contradict the assumption of case (b). Hence we have proved that there exist $m_{t(0)}, m_{t(1)}, y(t(0)), y(t(1))$ satisfying (4).

In order to complete the definition of $F(j)$ it is enough to put $(n(t))(j)$ for $j = 0, 1$ such that

$$\bigcap\{B(m_{t(j)}, 1/2^{(n(t))(j)}): j = 0, 1\} = \emptyset$$

and

$$\bigcup\{B(m_{t(j)}, 1/2^{(n(t))(j)}): j = 0, 1\} \subset B(m_t, 1/2^{n(t)}).$$

The family $\mathscr{F}_{i+1}$ is defined according to (2).

If we put

$$Z = \bigcap_{j=1}^{\infty}\big\{\bigcup B(m_t, 1/2^{n(t)}): t \in T_j\big\},$$

then from (1)-(3) we infer that $Z$ is an uncountable compact subset in $M'$, and so there is $z \in Z \cap M$. From (1)-(3) it follows that

(5) if we have $i > j \geqslant 1$, $F_1 = \big(B(m_{t_1}, 1/2^{n(t_1)}), (x_{t_1 1}, \ldots, x_{t_1 j})\big) \in \mathscr{F}_j$, $F_2 = \big(B(m_{t_2}, 1/2^{n(t_2)}), (x_{t_2 1}, \ldots, x_{t_2 i})\big) \in \mathscr{F}_i$, and $z \in F_1 \cap F_2$, then

$$x_{t_1 1} = x_{t_2 1}, \ldots, x_{t_1 j} = x_{t_2 j}.$$

Using (5) we can find a sequence

$$F_j = \big(B(m_j, 1/2^{n(j)}), (x_1, \ldots, x_j)\big) \in \mathscr{F}_j$$

such that

$$z \in \bigcap_{j=1}^{\infty} B(m_j, 1/2^{n(j)}).$$

We claim that $(z, (x_1, \ldots, x_j, \ldots)) \in \bigcap\{F_s\colon s \in S\}$. Indeed, if the set

$$U = V \times \mathop{\mathsf{P}}_{n=1}^{i} B_n \times \mathop{\mathsf{P}}_{n=i+1}^{\infty} X_n$$

is a neighbourhood of $(z, (x_1, \ldots, x_j, \ldots))$, then there is $j > i$ such that $B(m_j, 1/2^{n(j)}) \subset V$. Hence it follows from (3) that $U \cap F \neq \emptyset$ for every $F \in \mathcal{F}$.

**Remark 1.** One can show, modifying the proof of the Theorem, that if $M_i$ is a separable metric space satisfying the assumption of the Theorem and $X_i$ a Lindelöf $P$-space for $i = 1, 2, \ldots$, then $\mathop{\mathsf{P}}_{i=1}^{\infty} (M_i \times X_i)$ is Lindelöf.

**Remark 2.** Notice that $M$ need not have to be a complete space. Indeed, if $E$ is an uncountable complete, metric space and $F$ an uncountable subspace of $E$ which does not contain uncountable compact subspaces (see [2], Theorem 1, p. 514), then $M = E \setminus F$ satisfies the assumption of the Theorem and is not complete.

**Remark 3.** One can notice that if $M$ satisfies the assumption of the Theorem and $H$ is a closed or open subset of $M$, then $H$ has the Baire category property, i.e. $H$ does not admit a partition $H = \bigcup\{H_i\colon i = 1, 2, \ldots\}$, where $H_i$ is a nowhere dense subset of $H$. The opposite implication does not hold. One can ask whether the Cartesian product $M \times \mathop{\mathsf{P}}_{n=1} X_n$ is Lindelöf provided that $M$ is a separable metric space such that every closed or open subset of $M$ has the Baire category property and $X_n$ is a Lindelöf $P$-space for $n = 1, 2, \ldots$ Assuming the continuum hypothesis one can show that if $A$ is an uncountable subset of $M \times \mathop{\mathsf{P}}_{n=1}^{\infty} X_n$ and the set $p(A)$, where $p$ is the projection onto $M$, has the Baire category property and is dense in itself, then there is $x \in M \times \mathop{\mathsf{P}}_{n=1}^{\infty} X_n$ such that for every neighbourhood $U$ of $x$ the set $A \cap U$ is uncountable.

In [1], as we said before, it was proved that if $M$ is a separable metric space and $X_n$ is a Lindelöf scattered space for $n = 1, 2, \ldots$, i.e. $X_n$ does not contain a subset dense in itself, then the Cartesian product $M \times \mathop{\mathsf{P}}_{n=1}^{\infty} X_n$ is Lindelöf. Hence it is natural to ask whether every Lindelöf $P$-space $X$ admits a partition $X = \bigcup\{X_n\colon n = 1, 2, \ldots\}$, where $X_n$ is

Lindelöf and scattered for $n = 1, 2, \ldots$ The same question was asked by R. Telgarsky in relation with the game theory.

Example. *There is a Lindelöf P-space $X$ which does not admit a partition*

$$X = \bigcup_{n=1}^{\infty} X_n,$$

*where $X_n$ is Lindelöf scattered for $n = 1, 2, \ldots$*

The space $X$ appeared in [4], where R. Pol proved that it has the Lindelöf property. The fact that $X$ does not admit a suitable partition was observed independently by R. Pol and myself.

Proof. Put $Y = D^{\omega_1}$, where $D = \{0, 1\}$ and $\omega_1$ is the first uncountable ordinal number. The topology on $Y$ is induced by $G_\delta$-subsets of $D^{\omega_1}$ considered in the Tychonoff topology. The space $X$ will be a subspace of $Y$. Let us attach to every limit ordinal number $a < \omega_1$ a sequence $\big(a(n)\big)_{n=1}^{\infty}$ converging to $a$ in the order topology. Then

$$\begin{aligned} X = \{x \in D^{\omega_1}\colon\ & x^{-1}(1) \text{ is finite}\} \cup \\ & \cup \{x_a \in D^{\omega_1}\colon\ a \text{ is a limit ordinal number and } x_a^{-1}(1)\} \\ = \{a(n)\colon\ & n = 1, 2, \ldots\}. \end{aligned}$$

Notice that $x \in X$ is isolated if and only if the set $x^{-1}(1)$ is infinite.

For the sake of completeness we shall give the sketch of the proof that $X$ is Lindelöf. Let $\mathcal{U}$ be an open cover of $X$ and $x_0$ an arbitrary point of $X$. Then there exist $a_0 < \omega_1$ and a countable subfamily $\mathcal{U}_0$ of $U$ such that

$$\{x \in X\colon\ x(\beta) = x_0(\beta) \text{ for } \beta \leqslant a_0\} \subset \bigcup \mathcal{U}_0.$$

Let us assume that $a_n < \omega_1$ and a countable family $\mathcal{U}_n \subset U$ are defined. Then the set $A_n = \{x \in X\colon\ x^{-1}(1) \subset a_n + 1\}$ is countable, and so there are $a_n \leqslant a_{n+1} < \omega_1$ and a countable family $\mathcal{U}_{n+1} \subset U$ such that

$$\bigcup_{y \in A_n} \{x \in X\colon\ x(\beta) = y(\beta) \text{ for } \beta \leqslant a_{n+1}\} \subset \bigcup \mathcal{U}_{n+1}.$$

Now it suffices to notice that the family $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ covers $X \setminus \{x_a\}$ if $a = \sup\{a_n\colon\ n = 1, 2, \ldots\}$ is a limit ordinal number or $\bigcup_{n=1}^{\infty} \bigcup \mathcal{U}_n = X$.

We shall show that $X$ does not admit a partition $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n$ is a Lindelöf scattered space. Let us assume to the contrary that $X = \bigcup_{n=1}^{\infty} X_n$. There is $n_0$ such that $\{a < \omega_1\colon\ x_a \in X_{n_0}\}$ is stationary (recall that $S \subset \omega_1$ is *stationary* if for every uncountable and closed subset $B$ of $\omega_1$ the intersection $B \cap S$ is not empty). Now, let $X_n^{(a)}$ be the set

$\bigcap\{X_n^{(\beta)}: \beta < \alpha\}$ if $\alpha$ is a limit ordinal and let $X_n^{(\alpha)}$ be the set of accumulation points of $X_n^{(\beta)}$ if $\alpha = \beta+1$. The space $X_{n_0}$ is scattered and Lindelöf, so there are $\beta_0$ and $z_0$ such that $z_0 \in X_{n_0}^{(\beta_0)} \setminus X_{n_0}^{(\beta_0+1)}$ and a neighbourhood $U$ of $z_0$ in $X_{n_0}$ such that

(6) the set $\{\alpha < \omega_1: x_\alpha \in U\}$ is stationary and for every $x \neq z_0$ and $x \in U$ there is a neighbourhood $U_x$ of $x$ in $U$ such that $\{\alpha < \omega_1: x_\alpha \in U_x\}$ is not stationary.

Without loss of generality we can assume that

$$U = \{x \in X_{n_0}: x(\alpha) = z_0(\alpha) \text{ for } \alpha \leqslant \beta\},$$

where $\beta$ is such that $z_0^{-1}(1) \subset \beta$. Using the pressing-down lemma one can find $\beta < \beta_1 < \omega_1$ such that $\{\alpha < \omega_1: x_\alpha \in U \text{ and } x_\alpha(\beta_1) = 1\}$ is stationary, which contradicts (6).

Remark 4. We have proved that every subspace of $X$ of the form

$$X(B) = \{x \in X: x^{-1}(1) \text{ is finite}\} \cup \{x_\alpha: \alpha \in B\},$$

$B$ being a stationary subset of $\omega_1$, does not admit a partition $\bigcup_{n=1}^{\infty} X_n$, where $X_n$ is scattered and Lindelöf. It is easy to notice that the opposite implication holds; namely, if $B$ is not stationary, then $X(B)$ admits a suitable partition.

Remark 5. One can show that if there are a separable metric space $M$ and a $P$-space $X$ of weight less than or equal to $\aleph_1$ such that the Cartesian product $M \times X^{\aleph_0}$ is not Lindelöf, then there is an uncountable subset $B$ of the Cantor set $C$ without uncountable compact subsets and such that $C \setminus B = \bigcup\{A_\alpha: \alpha < \omega_1\}$, where $A_\alpha$ is an analytic set. Let us notice that Solovay [5] proved that if a measurable number exists, then every complement of an analytic set is countable or contains an uncountable compact subset.

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