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On the convergence of the iterates of the Frobenius–Perron operator associated with a Markov map defined on an interval. The lower-function approach

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Abstract. The problem of the convergence of $\{P^j_\varphi g\}_{j=1}^\infty$ is considered for a certain class of expanding transformations φ of an interval (bounded or not) into itself with countably many monotonicity intervals (P_φ is the Frobenius–Perron operator associated with φ). An approach to the problem is presented which relies on the lower-function argument.

Introduction. We study the problem of the convergence of the sequence $\{P^j_\varphi g\}_{j=1}^\infty$ (P_φ is the Frobenius–Perron operator) for a certain class of expanding transformations φ of an interval (bounded or not) into itself with countably many monotonicity intervals. The main result of our study is contained in the convergence theorem in Section 1.

The above problem and some related ones have been studied by several authors during the last few decades (see e.g. [1], [3]–[7] and [10]–[12]). Their approaches are based on compactness arguments; moreover, they study the problem separately for transformations defined on bounded and on unbounded intervals.

The main aim of this paper is to present a new approach to the problem in question. Our approach does not rely on compactness arguments but on the lower-function argument introduced in [8]. This technique turns out to be well suited to control possible tendencies of the mass to escape to infinity under the action of the transformations in the case of an unbounded interval. It also enables us to bind together the cases of bounded and of unbounded interval.

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1. Preliminaries and the statement of the main result. Let I be an interval and m be the Lebesgue measure on I. Denote by $L^1 = L^1(I, m)$ the space of all real-valued Lebesgue-integrable functions f with norm $\|f\| = \int |f| dm$. Then the Frobenius–Perron operator (F–P operator in short) P_τ (see [9]) of a nonsingular transformation $\tau \colon I \to I$ acts on L^1. (We shall often omit the subscript τ in P_τ.)

We denote by G the set of all probability densities of L^1, i.e., $G = \{g \in L^1 \colon g \geqslant 0 \text{ and } \|g\| = 1\}$, by $D^n f$ the n-th derivative and by $f|W$ the restriction of a function f to a set W. We shall also use the following abbreviation: $\mathrm{Reg}(g) = \sup\{|Dg(x)|/g(x) \colon x \in I,\ g(x) > 0, \text{ and } Dg(x) \text{ exists}\}$.

We are concerned with piecewise monotonic transformations $\varphi \colon I \to I$ given by

DEFINITION 1.1. A piecewise monotonic transformation φ from an interval I (bounded or not) into itself is said to be a *Markov map* iff there exists an at most countable family $\pi = \{I_k \colon k \in K\}$ of open intervals which is a partition of I (mod m) such that:

(1.M1) for each I_k, the function $\varphi_k = \varphi | I_k$ is strictly monotonic, and $D\varphi_k$ is locally Lipschitzean function which possesses one-sided limits at the endpoints of I_k;

(1.M2) π is a Markov partition for φ, i.e., for each I_j, $I_k \in \pi$, if $\varphi(I_j) \cap I_k \neq \emptyset$, then $I_k \subset \varphi(I_j)$;

(1.M3) φ is an irreducible transformation, i.e., $\bigcup_{j=0}^{\infty} \varphi^j(I_k) = I$ for all $I_k \in \pi$.

Throughout the paper we shall assume that φ is a Markov map satisfying:

(1.H1) $C_1 = \inf\{|D\varphi(x)| \colon x \in \bigcup_{k \in K} I_k\} > 1$ (expanding condition);

(1.H2) $C_2 = \sup\{|D^2 \varphi_k(x)|/(D\varphi_k(x))^2 \colon x \in I_k,\ k \in K\} < \infty$ (Rényi's condition);

(1.H3) $C_3 = \inf\{m(\varphi(I_k)) \colon k \in K\} > 0$;

(1.H4) $C_4 = \sup\{m(I_k) \colon k \in K\} < \infty$.

To state the last condition we have to introduce a certain family of densities generated by φ^r (the r-fold composition of φ with itself) in the following way:

Let $I_{k(r)} \neq \emptyset$ be a monotonicity interval of φ^r where $k(r) = (k_1, \ldots, k_r) \in K^r$, and let $\varphi_{k(r)}^{-1}$ be the inverse function of $\varphi_{k(r)} = \varphi^r | I_{k(r)}$. Then we put

$$\sigma_{k(r)} = |D\varphi_{k(r)}^{-1}|. \tag{1.1}$$

By convention, $\varphi^1 = \varphi$, consequently $I_{k(1)} = I_k$, $\varphi_{k(1)} = \varphi_k$ and $\sigma_{k(1)} = \sigma_k$.

Now we denote by $G(\varphi^r)$ the family consisting of all densities of the form

$$g_{s(r)} = \sum_{k(r) \in K(r)} w_{k(r)} \sigma_{k(r)} \int_{I_{k(r)}} w_{s(r)} \sigma_{s(r)} dm, \tag{1.2}$$

where $K(r) = \{k(r) = (k_1, \ldots, k_r) \in K^r \colon I_{k(r)} \neq \emptyset\}$, $s(r) = (s_1, \ldots, s_r) \in K(r)$ and $w_{k(r)} = m(I_{k(r)})^{-1}$ is the normalizing factor for $\sigma_{k(r)}$, i.e., $\|w_{k(r)} \sigma_{k(r)}\| = 1$.

The last condition guarantees that some $G(\varphi^r)$ has a nontrivial lower bound:

(1.H5) There exist $r_0 \geqslant 1$ and a function $u_{r_0} \geqslant 0$ such that $\|u_{r_0}\| > 0$ and $g_{s(r_0)} \geqslant u_{r_0}$ for all $g_{s(r_0)} \in G(\varphi^{r_0})$.

Remarks. (1.1) Counterexample 1 in [2] shows that (1.H5) is indispensable. However, it is always satisfied provided φ is defined on a bounded I and $\varphi^{\bar{r}}(I_{\bar{k}}) = I$ for some $\bar{r} \geqslant 1$ and $I_{\bar{k}}$. This follows from the boundedness of the family $\{w_k \sigma_k \colon k \in K\}$ of densities under conditions (1.H1)–(1.H3).

(1.2) When φ is defined on an unbounded I, then (1.H5) turns out to be essentially less restrictive than the following counterparts of it: (10) in [7], (ix) in [5], and (3.1) in [1]. In fact, (10) $\Rightarrow$ (ix) $\Rightarrow$ (3.1), and (3.1) $\Rightarrow$ (1.H5). Furthermore, this chain of implications is one way (see Example 2.1).

Let φ be a Markov map with respect to $\pi = \{I_k \colon k \in K\}$. We denote by $G(1)$ the set of all densities $g \in G$ satisfying the following two conditions:

(1) for each I_k, $g \mid I_k$ is a locally Lipschitzean function which possesses one-sided limits at the endpoints of I_k; and

(2) $\mathrm{Reg}(g) < \infty$.

We are now ready to state our main result.

CONVERGENCE THEOREM. *Let* φ *satisfy* (1.H1)–(1.H5). *Then there is precisely one* $g_0 \in G$ *such that*:

(1) $g_0 = \lim_{j\to\infty} P_\varphi^j g$ *(in* L^1*) for all* $g \in G$. *In consequence,* $P_\varphi g_0 = g_0$ *and* φ *is an exact endomorphism.*

(2) *For each* I_k *and all* $g \in G(1)$, *the sequence* $\{P_\varphi^j g \mid I_k\}_{j=1}^{\infty}$ *converges in the supremum norm to* $g_0 \mid I_k$.

Remark. (1.3) Let conditions (1.M1) and (1.H2) be replaced, respectively, by the following two conditions:

(1.M^{n}1) For each I_k, the function $\varphi_k = \varphi \mid I_k$ satisfies: (1) φ_k is strictly monotonic, (2) $\varphi_k \in C^n_{\mathrm{loc\,Lip}}(I_k)$ $(n \geqslant 1)$, and (3) $D^i \varphi_k$ $(i = 0, 1, \ldots, n)$ possesses one-sided limits at the endpoints of I_k $(D^0 \varphi_k = \varphi_k)$.

(1.H^{n}2) $C_{2,i} = \sup\{\sup_x |D^{i+1}\varphi_k^{-1}(x)|/|D\varphi_k^{-1}(x)| \colon k \in K\} < \infty$

for $i = 1, 2, \ldots, n$.

The symbol $C^n_{\mathrm{loc\,Lip}}(W)$ in (1.M^n1) denotes the set of all functions f (defined on W) with continuous derivatives $D^i f$ $(i = 0, 1, \ldots, n)$ and locally Lipschitzean $D^n f$.

Denote by $G(n)$ the set of all densities $g \in G$ satisfying the conditions: (1) for each $I_k \in \pi$, $g \mid I_k \in C^{n-1}_{\mathrm{loc\,Lip}}(I_k)$ and $D^i(g \mid I_k)$ $(i = 0, 1, \ldots, n-1)$ has one-sided limits at the endpoints of I_k, and (2) $\sup_{x \in I} |D^i g(x)|/g(x) < \infty$ for $i = 1, 2, \ldots, n$.

Under the replacements described above, the following differentiable counterpart of assertion (2) of the convergence theorem holds: (2″) For each I_k and all $g \in G(n)$, the sequence $\{D^i(P^j_\varphi g \mid I_k)\}_{j=1}^{\infty}$ converges in the supremum norm to $D^i(g_0 \mid I_k)$ for $i = 0, 1, \ldots, n-1$. In consequence, $D^{n-1}(g_0 \mid I_k)$ is a Lipschitzean function.

We conclude this section with the following:

COROLLARY 1.1. (1) *There exist a density* $\tilde{g}_0$ *of the form*

$$\tilde{g}_0 = \sum_{k(r_0)\in K(r_0)} w_{k(r_0)} \sigma_{k(r_0)} \int_{I_{k(r_0)}} g_0 \, dm,$$

and some constants $C_5 > 0$ *and* $C_6 > 0$ *such that* $C_5 \tilde{g}_0 \leqslant g_0 \leqslant C_6 \tilde{g}_0$ (*here the index* r_0 *has the same meaning as in* (1.H5)).

(2) *There exists a constant* $C_7 > 0$ *such that for each* I_k,

$$|(g_0 \mid I_k)(x) - (g_0 \mid I_k)(y)| \leqslant C_7 |x-y| \quad \textit{for all } x, y \in I_k.$$

2. Proof of Convergence Theorem. We first prove the first assertion of the convergence theorem. The idea of the proof is to show, under (1.H1)–(1.H4), that any lower bound of $G(\varphi^r)$ (r is arbitrary but fixed) is a lower-function for P_φ. More precisely, set

$$\tilde{u}_r = \inf\{g_{s(r)}\colon\ g_{s(r)} \in G(\varphi^r)\}$$

where $g_{s(r)}$ is defined by (1.2). Then there exists a constant $C_8 > 0$ such that

(2.1) $\quad P^{j+2r} g \geqslant C_8 \tilde{u}_r \quad$ for each $g \in G(1)$, $g > 0$, and all $j \geqslant j_1(g)$.

Then assertion (1) of the convergence theorem follows from (2.1), (1.H5), Theorem 2 in [8] and the denseness of the set $\{g \in G(1)\colon g > 0\}$.

The proof of (2.1) will be done in four steps.

Step 2.1. *Let* $g \in G(1)$ *and set* $g_k = g \circ \varphi_k^{-1} \sigma_k$ *where* σ_k *is defined by* (1.1). *Then for each* $k \in K$,

$$\operatorname{Reg}(g_k) \leqslant C_2 + C_1^{-1} \operatorname{Reg}(g); \tag{2.2}$$

$$|g_k(x)| \leqslant (\operatorname{Reg}(g_k) + C_3^{-1}) \|g 1_{I_k}\| \quad \textit{for all } x \in J_k = \varphi(I_k), \tag{2.3}$$

where $C_3^{-1} = 0$ *if* $C_3 = \infty$;

$$|g_k(x) - g_k(y)| \leqslant \operatorname{Reg}(g_k)\big(\sup_{z\in J_k} |g_k(z)|\big)|x-y| \quad \textit{for all } x,\ y \in J_k. \tag{2.4}$$

Proof. (2.2) follows from (1.H1)–(1.H2) and the inequality

$$|Dg_k|/g_k \leqslant |D\sigma_k|/\sigma_k + \sigma_k\big(|(Dg)\circ \varphi_k^{-1}|/(g \circ \varphi_k^{-1})\big).$$

To prove (2.3) take $x_k \in J_k$ such that $g_k(x) \geqslant g_k(x_k)$ for all $x \in J_k$. Then we have

$$|g_k(x)| \leqslant \Big|\int_x^{x_k} Dg_k \, dm\Big| + m(J_k)^{-1} \int_{J_k} g_k \, dm.$$

The desired inequality follows from this and (1.H3).

Finally, (2.4) follows from the inequalities:

$$|g_k(x)-g_k(y)| \leqslant \Big|\int_x^y Dg_k\,dm\Big| \quad \text{for all } x,\, y\in J_k,$$

$$|Dg_k| \leqslant |g_k|\,\mathrm{Reg}(g_k). \ \blacksquare$$

In the second step, we show that the inequalities analogous to those from step 2.1 are valid for Pg.

Step 2.2. *For each* $g\in G(1)$,

$$\mathrm{Reg}(Pg) \leqslant C_2+C_1^{-1}\,\mathrm{Reg}(g); \tag{2.5}$$

$$\sup_{x\in I}|Pg(x)| \leqslant (C_2+C_3^{-1})+C_1^{-1}\,\mathrm{Reg}(g); \tag{2.6}$$

and for each $I_k\in\pi$, *and all* $x,\ y\in I_k$

$$\big|Pg|I_k(x)-Pg|I_k(y)\big| \leqslant C_9(g)|x-y|, \tag{2.7}$$

where $C_9(g) = \sup_{k\in K}\mathrm{Reg}(g_k)\big(\sup_{k\in K}\mathrm{Reg}(g_k)+C_3^{-1}\big)$.

Proof. To prove (2.5), we first note that the F–P operator of φ can be expressed by

$$Pg = \sum_{k\in K} g_k, \quad \text{where } g_k = g\circ\varphi_k^{-1}\sigma_k. \tag{2.8}$$

Now, since g_k is a function of bounded variation, it follows from the Jordan decomposition and Fubini's theorem that the sum on the right-hand side of (2.8) is termwise differentiable a.e. (m) and its derivative is equal, a.e. (m), to DPg. Thus (2.5) follows from (2.2) of step 2.1.

(2.6) follows from (2.8) and (2.2)–(2.3) of step 2.1.

Finally, (2.7) follows from (2.8), (1.M2) and (2.3)–(2.4) of step 2.1. ■

Step 2.3. *For each* $g\in G(1)$, $g>0$, *there exists* $j_1=j_1(g)$ *such that for each* I_k *and all* $x,\ y\in I_k$,

$$C_5P^jg(x) \leqslant P^jg(y) \leqslant C_6P^jg(x) \quad \text{for all } j\geqslant j_1,$$

where the constants $C_5>0$ *and* $C_6>0$ *depend solely on* φ.

Proof. We note first that from (2.5) of step 2.2 it follows that

$$\mathrm{Reg}(P^jg) \leqslant C_{10} \quad \text{for all } j\geqslant j_1(g), \tag{2.9}$$

where C_{10} is an arbitrarily fixed constant such that $C_{10} > C_1C_2/(C_1-1)$.

Next, notice that (1.M3) implies $\varphi(I)=I$, which, in turn, implies $P_\varphi^jg(x)>0$ for all $x\in I$ and $j\geqslant 1$.

Finally, set $f_j=\ln(P^jg)$ for all j. Then, using (2.7) of step 2.2, (2.9) and (1.H4), we obtain

$$|f_j(x)-f_j(y)| \leqslant \Big|\int_x^y Df_j\,dm\Big| \leqslant C_4C_{10}$$

for each I_k, all $x, y \in I_k$, and all $j \geqslant j_1(g)$. This implies the inequality of step 2.3 with $C_5 = \exp(-C_4 C_{10})$ and $C_6 = \exp(C_4 C_{10})$. ■

Step 2.4. *For each $g \in G(1)$, $g > 0$, there exists $j_1 = j_1(g)$ such that*

$$C_5 F_r(P^j g) \leqslant P^{j+r} g \leqslant C_6 F_r(P^j g) \quad \textit{for all } j \geqslant j_1 \textit{ and each } r \geqslant 1,$$

where F_r is the operator given by

$$F_r(g) = \sum_{k(r)\in K(r)} w_{k(r)} \sigma_{k(r)} \int_{I_{k(r)}} g\, dm,$$

and $I_{k(r)}$, $w_{k(r)}$ and $\sigma_{k(r)}$ have the same meaning as in (1.1)–(1.2).

Proof. From step 2.3 we obtain

$$(2.10) \qquad C_5 (P^j g)_{k(r)}(x) \sigma_{k(r)}(x) \leqslant (P^j g)_{k(r)}(y) \sigma_{k(r)}(x) \leqslant C_6 (P^j g)_{k(r)}(x) \sigma_{k(r)}(x)$$

for each $J_{k(r)} = \varphi_{k(r)}(I_{k(r)}) \neq \emptyset$, all $x, y \in J_{k(r)}$ and $j \geqslant j_1$; here $(P^j g)_{k(r)} = (P^j g) \circ \varphi_{k(r)}^{-1}$ and $\varphi_{k(r)} = \varphi^r \mid I_{k(r)}$.

Note that from the piecewise monotonicity of φ^r and the equality $P_{\varphi^r} = P_\varphi^r$ it follows that

$$(2.11) \qquad P_\varphi^r g = \sum_{k(r)\in K(r)} g \circ \varphi_{k(r)}^{-1} \sigma_{k(r)}.$$

Integrating (2.10) with respect to x on $J_{k(r)}$, multiplying by $w_{k(r)} \sigma_{k(r)}$ and summing the resulting inequalities over all $k(r) = (k_1, \ldots, k_r) \in K(r)$, we obtain, upon using (2.11), the desired inequalities. ■

From step 2.4 and the equality

$$\sum_{k(r)\in K(r)} \| 1_{I_{k(r)}} P_\varphi^j g \| = 1$$

it follows that (2.1) holds with $C_8 = C_5^2$. As has been mentioned at the beginning of this section, this implies assertion (1) of the convergence theorem.

For (2), note first that from (2.6) of step 2.2 and (2.9) it follows that

$$(2.12) \qquad \sup_{x\in I} |P^j g(x)| \leqslant C_{11} \quad \text{for all } j \geqslant j_1(g);$$

and from (2.7) of step 2.2, (2.2) and (2.9) it follows, for each I_k and all $x, y \in I_k$, that

$$(2.13) \qquad \left| P^j g \,| I_k(x) - P^j g |\, I_k(y) \right| \leqslant C_7 |x-y| \quad \text{for all } j \geqslant j_2(g).$$

Inequalities (2.12)–(2.13) imply that the sequence $\{P_\varphi^j g \mid I_k\}_{j \geqslant j_3}$ is bounded and equicontinuous. Thus (2) follows from the lemma of Ascoli–Arzelà and (1). The proof of the convergence theorem is complete.

The two assertions of Corollary 1.1 follow directly from step 2.4, (2.13) and the convergence theorem.

We conclude this section with a simple example of a transformation φ, defined on the real line, for which $\{P_\varphi^j g\}$ converges by the convergence theorem but for which the previously known results give no information.

EXAMPLE 2.1. Let φ_0 be an arbitrary, twice differentiable function from the interval $I_0 = (-1, 1)$ onto the whole real line $\boldsymbol{R}$ such that $|D\varphi_0| \geqslant C_1 > 1$ and $|D^2\varphi_0|/(D\varphi_0)^2 \leqslant C_2 < \infty$. Let $\varphi_{2k}(x) = \varphi_0(x-4k)$ if $4k-1 < x < 4k+1$ and let $\varphi_{2k+1}(x) = \varphi_0(x-2(2k+1))+2(2k+1)$ if $4k+1 < x < 4k+3$ for $k = 0, 1, -1, 2, -2, \ldots$ Then, for each $x \in \bigcup_{k=-\infty}^{\infty} I_k$, put $\varphi(x) = \varphi_k(x)$ iff $x \in I_k = (2k-1, 2k+1)$.

Since $\sigma_{2k+1}(x) = \sigma_0(x-2(2k+1))$, φ does not satisfy (3.1) in [1]. In particular, it does not satisfy (10) in [7] and (ix) in [5], either. Nevertheless, φ satisfies (1.H5) because

$$g_s \geqslant \tfrac{1}{4}\sigma_0 \int_A \sigma_1 \, dm,$$

where g_s is defined by (1.2) and A is the union of all I_{2k}'s.

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