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Some results on the ultimate behavior of solutions of Volterra functional-differential equations

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The aim of this paper is the investigation of ultimate behavior of bounded solutions of certain systems of functional-differential equations involving abstract Volterra operators. More exactly, we shall be concerned with systems of the form

(1) $$\dot{x}(t)+(Lx)(t)+f(x(t)) = g(t), \quad t \in \boldsymbol{R}_+,$$

or

(2) $$\dot{x}(t)+(L\dot{x})(t)+f(x(t)) = g(t), \quad t \in \boldsymbol{R}_+,$$

where x, f, and g are n-dimensional vector-valued maps, and L is an abstract Volterra operator, which we will assume to be linear throughout this paper (the nonlinear case can be also dealt with, but the discussion is not an immediate extension of the linear case). For the definition of abstract Volterra operators, we refer the reader to the books [4], [8].

In the existing literature on equations involving abstract Volterra operators, there are two distinct methods used in discussing the asymptotic behavior of their solutions. The first method is basically based on monotonicity assumptions that allow to compare the behavior of solutions of the Volterra equations with those of convenient ordinary differential systems. This method has been initiated by Moser [7], and in case of nonconvolution integral operators has been used by the author [3], and by Cassago Jr. and Corduneanu [1]. This method applies without difficulty to the case of equations involving abstract Volterra operators, as we shall emphasize in the sequel. A second method is based on admissibility techniques, as illustrated in Corduneanu [5], and has its origin in Massera–Schäffer admissibility theory for ordinary differential equations (see [6]). In both directions, there is still a good amount of problems to be investigated.

The conditions under which our results will be derived are related to the first method of investigation mentioned above, and the problem of the asymptotic behavior of solutions to (1) or (2) will be reduced to the similar problem for the more common system of ordinary differential equations

$$\dot{x}(t)+f((x(t)) = 0, \quad t \in \boldsymbol{R}_+ . \tag{3}$$

Before we consider the asymptotic behavior of solutions to the systems (1) or (2), it is appropriate to make a few considerations on the existence of solutions, verifying an initial condition of the form

$$x(0) = x^0 \in \boldsymbol{R}^n. \tag{4}$$

As shown in our book [4], equation (1), with the initial condition (4) can be reduced to a nonlinear Volterra integral equation of the form

$$x(t)+\int_0^t k(t, s)x(s)ds+\int_0^t f(x(s))ds = x^0+\int_0^t g(s)ds, \tag{5}$$

for which local existence can be obtained by means of standard methods. The kernel $k(t, s)$ results from a representation theorem [4], stating that

$$\int_0^t (Lx)(s)ds = \int_0^t k(t, s)x(s)ds.$$

It does enjoy some regularity properties, which help investigating (5) in regard to the local existence problem.

In case of equation (2), if we assume $k(t, s)$ to be in L^∞_{loc}, a resolvent kernel $\tilde{k}(t, s)$ does exist [8], and (2) is equivalent to

$$\dot{x}(t) = -f(x(t))+g(t)+\int_0^t \tilde{k}(t, s)[-f(x(s))+g(s)]\,ds, \tag{6}$$

which can be also investigated in regard to the local existence by standard methods.

The hypotheses we shall assume on (1) or (2) will be of such nature, that global existence of solutions, as well as their boundedness (on $\boldsymbol{R}_+$), will be assured.

Let us consider now equation (1), under the following hypotheses:

(a) The linear operator L is of Volterra type, and is continuous from $C(\boldsymbol{R}_+, \boldsymbol{R}^n)$ — the space of continuous maps from $\boldsymbol{R}_+$ into $\boldsymbol{R}^n$, with the topology of uniform convergence on any compact — into $L_{\mathrm{loc}}(\boldsymbol{R}_+, \boldsymbol{R}^n)$, and it takes the bounded functions on $\boldsymbol{R}_+$ to functions belonging to $M_0(\boldsymbol{R}_+, \boldsymbol{R}^n)$: $LBC(\boldsymbol{R}_+, \boldsymbol{R}^n) \to M_0(\boldsymbol{R}_+, \boldsymbol{R}^n)$, i.e.

$$\int_t^{t+1} |(Lx)(s)|ds \to 0 \quad \text{as } t \to \infty. \tag{7}$$

(b) $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ is continuous, and such that

$$\int_0^t \langle (Lx)(s)+f(x(s)),\ x(s)\rangle\, ds \geqslant 0, \qquad t\in \boldsymbol{R}_+, \tag{8}$$

for any $x\in C(\boldsymbol{R}_+, \boldsymbol{R}^n)$.

(c) $g\in L^1(\boldsymbol{R}_+, \boldsymbol{R}^n)$.

The following result can be stated in regard to equation (1):

THEOREM 1. *Assume that conditions* (a), (b), (c) *are satisfied for equation* (1). *Then any solution of* (1) *is defined on* $\boldsymbol{R}_+$, *it is bounded there, and its limit set coincides with that of a convenient solution of the ordinary differential equation* (3).

Proof. Let $x = x(t)$ be a solution of (1), such that $x(0) = x^0 \in \boldsymbol{R}^n$. This solution is defined on some interval $[0, T)$, $T\leqslant \infty$ (possibly, for small T only).

From (1), by scalar multiplication with $x(t)$, $t\in[0, T)$, we obtain

$$\tfrac{1}{2}\tfrac{d}{dt}|x(t)|^2+\langle (Lx)(t)+f(x(t)),\ x(t)\rangle = \langle g(t),\ x(t)\rangle, \tag{9}$$

which implies (on the same interval)

$$\tfrac{1}{2}|x(t)|^2+\int_0^t \langle (Lx)(s)+f(x(s)),\ x(s)\rangle\, ds = \tfrac{1}{2}|x^0|^2+\int_0^t \langle g(s),\ x(s)\rangle\, ds. \tag{10}$$

Taking condition (b) into account, we obtain from (10) the inequality

$$|x(t)|^2 \leqslant |x^0|^2+2\int_0^t |\langle g(s),\ x(s)\rangle| ds, \tag{11}$$

valid on $[0, T)$. If we now let $X(t) = \sup|x(s)|$, $0\leqslant s\leqslant t < T$, then (11) yields

$$X^2(t) \leqslant |x^0|^2+2X(t)\int_0^t |g(s)|ds \tag{12}$$

on $[0, T)$. By (c), (12) leads to

$$X^2(t) \leqslant |x^0|^2+2X(t)\int_0^\infty |g(s)|ds, \tag{13}$$

also on $[0, T)$. But (13) implies

$$X(t) \leqslant \int_0^\infty |g(s)|ds+\Big\{|x^0|^2+\Big(\int_0^\infty |g(s)|ds\Big)^2\Big\}^{1/2} \tag{14}$$

on $[0, T)$, from which we obtain

$$|x(t)| \leqslant \int_0^\infty |g(s)|ds+\Big\{|x^0|^2+\Big(\int_0^\infty |g(s)|ds\Big)^2\Big\}^{1/2} \tag{15}$$

on $[0, T)$. Since the right-hand side in (15) is a constant (with respect to t), one sees that $x(t)$ must remain bounded on its maximal interval of existence. This means that $x(t)$ can be extended on the whole semi-axis $\boldsymbol{R}_+$, and it remains bounded there. Hence, the limit set of $x(t)$ is nonempty.

In order to prove that the limit set of $x(t)$ coincides with that of a convenient solution to (3), one has to proceed according to a well-known scheme first used by Moser [7]. See also [1], [2], [3]. An important property to be established for $x(t)$ is the compactness of the family $\{x(t+h);\ h\in \boldsymbol{R}_+\}$, in the space $C(\boldsymbol{R}_+, \boldsymbol{R}^n)$. First, $\{x(t+h);\ h\in \boldsymbol{R}_+\}$ is uniformly bounded on $\boldsymbol{R}_+$ because $x(t)$ is bounded there. Second, this family of functions is equicontinuous on each compact interval of $\boldsymbol{R}_+$. Indeed, the equicontinuity is a consequence of the uniform continuity of $x(t)$ on $\boldsymbol{R}_+$.

From (1), taking into account the boundedness of $x(t)$ on $\boldsymbol{R}_+$, we see that on $\boldsymbol{R}_+$ we have $x'(t)\in L^\infty \oplus M_0$. We claim that this implies the uniform continuity of $x(t)$ on $\boldsymbol{R}_+$. To prove this, let $x' = v+w$, with $v\in L^\infty(\boldsymbol{R}_+, \boldsymbol{R}^n)$ and $w\in M_0(\boldsymbol{R}_+, \boldsymbol{R}^n)$. There results

$$(16)\qquad |x(t)-x(u)| = \left|\int_u^t x'(s)\,ds\right| \leqslant K|t-u| + \left|\int_u^t |w(s)|\,ds\right|,$$

and since

$$\left|\int_u^t |w(s)|\,ds\right| \leqslant 2\sup \int_\tau^{\tau+1} |w(s)|\,ds, \qquad \tau = \max\{u, t\},\ |t-u|<1,$$

we see that (due to the fact $w\in M_0$)

$$(17)\qquad \left|\int_u^t |w(s)|\,ds\right| < \tfrac{1}{2}\varepsilon \quad \text{for } u, t \geqslant T(\varepsilon),\ |t-u|<1.$$

But $w\in L^1([0,\ T], \boldsymbol{R}^n)$, and consequently

$$(18)\qquad \left|\int_u^t |w(s)|\,ds\right| < \tfrac{1}{2}\varepsilon \quad \text{for } u, t\in[0, T],\ |t-u| < \delta(\varepsilon).$$

From (17) and (18) we derive

$$(19)\qquad \left|\int_u^t |w(s)|\,ds\right| < \tfrac{1}{2}\varepsilon \quad \text{for } |t-u| < \delta(\varepsilon) < 1,$$

for arbitrary $u,\ t\in \boldsymbol{R}_+$. From (16) we easily obtain for $u,\ t\in \boldsymbol{R}_+$

$$(20)\qquad |x(t)-x(u)| < \varepsilon \quad \text{when } |t-u| < \min\{\delta(\varepsilon), \varepsilon/2k\},$$

which shows the uniform continuity of $x(t)$ on $\boldsymbol{R}_+$.

From now on, the proof of Theorem 1 continues exactly as shown in [1], [3].

Remark. There are many possibilities in satisfying condition (8) in (b) of Theorem 1. For instance, one may assume that both L and f satisfy the conditions

$$\int_0^t \langle (Lx)(s), x(s) \rangle ds \geqslant 0, \quad \int_0^t \langle f(x(s)), x(s) \rangle ds \geqslant 0$$

for any continuous $x(t)$, $t \in \boldsymbol{R}_+$. The last condition is practically the same as $\langle f(x), x \rangle \geqslant 0$ for any $x \in \boldsymbol{R}^n$. The first condition states, basically, the *monotonicity* of L regarded as an operator on the space $L^2_{loc}(\boldsymbol{R}_+, \boldsymbol{R}^n)$.

Of course, condition (8) in (b) is satisfied when

$$\int_0^t \langle (Lx)(s) - \lambda x(s), x(s) \rangle ds \geqslant 0, \quad \lambda > 0,$$

which means that L is strictly monotone, while f is subject to the less restrictive assumption

$$\int_0^t \langle f(x(s)), x(s) \rangle ds \geqslant -\lambda \int_0^t |x(s)|^2 ds.$$

For instance, such a condition takes place (in the scalar case) when $f(x) = -\lambda x + \alpha x^3$, with $\alpha > 0$.

Let us consider now the integrodifferential equation (2), in which L stands again for a Volterra abstract operator acting on the space $L^2_{loc}(\boldsymbol{R}_+, \boldsymbol{R}^n)$. The following result, similar to Theorem 1, holds true.

THEOREM 2. *Consider the system* (2) *in which L stands for an abstract Volterra linear operator, acting continuously on* $L^2_{loc}(\boldsymbol{R}_+, \boldsymbol{R}^n)$, *and such that*

$$\int_0^t \langle (Lx)(s) + \eta x(s), x(s) \rangle ds \geqslant 0, \quad t \in \boldsymbol{R}_+, \tag{21}$$

for any $x \in L^2_{loc}(\boldsymbol{R}_+, \boldsymbol{R}^n)$, *with* η *real and* $\eta < 1$. *Moreover, L is assumed to take the space* $L^2(\boldsymbol{R}_+, \boldsymbol{R}^n)$ *into* $M_0(\boldsymbol{R}_+, \boldsymbol{R}^n)$.

Furthermore, let $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ *be such that*

$$f(x) = \nabla U(x), \quad U \in C^{(1)}(\boldsymbol{R}^n, \boldsymbol{R}), \tag{22}$$

where

$$U(x) \to \infty \quad \text{as } |x| \to \infty. \tag{23}$$

If $g \in L^2(\boldsymbol{R}_+, \boldsymbol{R}^n)$, *then the system* (2) *has all its solutions defined on* $\boldsymbol{R}_+$, *they are bounded there, and the limit set of each solution of* (2) *coincides with the limit set of a convenient solution of the ordinary differential system* (3).

Proof. Let $x = x(t)$ be a solution of (2), with the initial condition $x(0) = x^0 \in \boldsymbol{R}^n$. Such a solution does exist locally, say on some interval $[0, T)$, $T \leqslant \infty$. We shall prove that $x(t)$ can be actually continued to $\boldsymbol{R}_+$, and remains bounded there.

Indeed, let us multiply scalarly both members of (2) by $\dot{x}(t)$, and integrate over $[0, t)$, $t < T$. We obtain

$$(24)\qquad \int_0^t \langle \dot{x}(s)+(L\dot{x})(s),\ \dot{x}(s)\rangle\, ds + U(x(t)) = U(x^0) + \int_0^t \langle g(s),\ \dot{x}(s)\rangle\, ds,$$

if we take into account

$$\int_0^t \langle f(x(s)),\ \dot{x}(s)\rangle\, ds = U(x(t)) - U(x^0),$$

which is a consequently of (22). Let $\delta = 1-\eta > 0$, and notice that (21), (24) yield the inequality

$$(25)\qquad \delta\int_0^t |\dot{x}(s)|^2 ds + U(x(t)) \leqslant U(x^0) + \int_0^t \langle g(s), \dot{x}(s)\rangle\, ds$$

on the same interval $[0, t)$. Since

$$\int_0^t \langle g(s),\ \dot{x}(s)\rangle\, ds \leqslant \frac{2}{\delta}\int_0^t |g(s)|^2 ds + \frac{\delta}{2}\int_0^t |\dot{x}(s)|^2 ds,$$

we obtain from (25) the inequality

$$(26)\qquad \tfrac{1}{2}\delta\int_0^t |\dot{x}(s)|^2 ds + U(x(t)) \leqslant U(x^0) + \int_0^\infty |g(s)|^2 ds.$$

On behalf of assumption (23) on $U(x)$, (26) implies the boundedness of $x(t)$ on $[0, T)$, as well as the fact that $\dot{x}(t)$ belongs to $L^2([0, T), \boldsymbol{R}^n)$. Hence $x(t)$ is uniformly continuous on $[0, T)$, and $\lim x(t)$ as $t \uparrow T$ must exist and be finite. That means the continuability of $x(t)$ is assured beyond T, which shows that T cannot be finite. This ends the proof of the assertion that any (local) solution of (2) can be extended to a saturated solution defined on $\boldsymbol{R}_+$, and that this solution is bounded (because the right-hand side in (26) is finite).

Again, in order to prove the second part in the theorem, we need the compactness of the family of functions $\{x(t+h);\ h \in \boldsymbol{R}_+\}$ with $x(t)$ a solution of (2) on $\boldsymbol{R}_+$. This compactness must take place in the space $C(\boldsymbol{R}_+, \boldsymbol{R}^n)$, and it is a simple consequence of the uniform continuity of any solution of (2) on $\boldsymbol{R}_+$. Also, it is useful to notice that $L\dot{x} \in M_0(\boldsymbol{R}_+, \boldsymbol{R}^n)$, because $\dot{x} \in L^2(\boldsymbol{R}_+, \boldsymbol{R}^n)$, a property which assures that

$$\int_t^{t+h} (L\dot{x})(s)ds \to 0 \qquad \text{as } t \to \infty$$

for every fixed $h > 0$. We refer the reader to [1], [3] for the remainder of the proof.

In concluding this paper, we will notice that the results established in Theorems 1 and 2 can be applied in investigating the asymptotic behavior of solutions of various classes of functional differential equations, such as

$$\dot{x}(t) + \int_{-\infty}^t k(t, s)x(s)ds + f(x(t)) = g(t), \qquad t \in \boldsymbol{R}_+.$$

Examples of such nature can be found in [1], [3].

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