ANNALES
POLONICI MATHEMATICI
XVII (1966)

On expansions of Meijer's functions I

The object of the paper and auxiliary results

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§ 1. The object of the paper. Let m, n, p, q denote arbitrary integers such that $q \geqslant 1$, $0 \leqslant n \leqslant p \leqslant q$ and $0 \leqslant m \leqslant q$. Let z denote an arbitrary complex number not equal to zero, and assume additionally for $p = q$ that $|z| < 1$. Further let $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ be arbitrary parameters satisfying the relations $a_j - b_h \neq 1, 2, \ldots$ $(j = 1, \ldots, n;\ h = 1, \ldots, m)$. *Meijer's function* is defined (see [5], I, p. 369-373) by the formula

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) \frac{1}{2\pi i}\int_\gamma = \frac{\prod\limits_{j=1}^{n} \Gamma(1-a_j+s) \prod\limits_{j=1}^{m} \Gamma(b_j-s)}{\prod\limits_{j=n+1}^{p} \Gamma(a_j-s) \prod\limits_{j=m+1}^{q} \Gamma(1-b_j+s)}\, z^s ds,$$

where γ is a contour running from $+\infty - i\tau$ to $+\infty + i\tau$ $(\tau > 0)$ with the poles $b_j, b_j+1, \ldots$ $(j = 1, \ldots, m)$ inside and the poles $a_j - 1, a_j - 2, \ldots$ $(j = 1, \ldots, n)$ outside γ. If r is a non-negative integer and μ satisfies the relations $\mu - a_j \neq 0, \pm 1, \ldots$ $(j = 1, \ldots, n)$, this definition may be completed by the formula

$$G_{p+1,q+1}^{m,n+1}\left(z\,\middle|\,\begin{matrix} \mu, a_1, \ldots, a_p \\ b_1, \ldots, b_q, \mu+r \end{matrix}\right)$$
$$= \frac{1}{2\pi i}\int_\gamma \frac{\Gamma(1-\mu+s) \prod\limits_{j=1}^{n} \Gamma(1-a_j+s) \prod\limits_{j=1}^{m} \Gamma(b_j-s)}{\prod\limits_{j=n+1}^{p} \Gamma(a_j-s) \prod\limits_{j=m+1}^{q} \Gamma(1-b_j+s)\Gamma(1-\mu-r+s)}\, z^s ds$$

without the assumption $\mu - b_h \neq 1, 2, \ldots$ for $h = 1, \ldots, m$ (the points $\mu - 1, \mu - 2, \ldots$ need not lie outside γ). The function $G_{p,q}^{m,n}$ can be defined outside the unit circle also for $p = q$ by

$$G_{p,p}^{m,n}\left(z\,\middle|\,\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}\right) = G_{p,p}^{n,m}\left(1/z\,\middle|\,\begin{matrix} 1-b_1, \ldots, 1-b_p \\ 1-a_1, \ldots, 1-a_p \end{matrix}\right)$$

provided that $m+n \geqslant p+1$, $|\arg z| < (m+n-p)\pi$ and $\arg(1/z) = -\arg z$.

Formulae involving Meijer's function may be used as key formulae from which many formulae with Bessel functions, Legendre functions and other higher transcendental functions follow by specializing parameters (see e.g. [2], II, pp. 434-444). The special functions here discussed are widely applied particularly in physics and especially in modern physics (cf. [6], esp. chapters 4, 5, 6, 10, 11 and 12).

Let ${}_kF_l$ denote the generalized hypergeometric series (for the definition, see e.g. [2], II, p. 429). C. S. Meijer proved in the series of his papers [5] that for a function $G^{m,n}_{p,q}$, under certain assumptions, there occur the following developments

$$(1)\quad \Big[1/\prod_{j=1}^{l}\Gamma(a_j)\Big]G^{m+l,n}_{p+l,q+l}\left(\eta\omega\,\middle|\,\begin{matrix}c_1,\dots,c_p,b_1,\dots,b_l\\ a_1,\dots,a_l,d_1,\dots,d_q\end{matrix}\right)$$
$$=\Big[1/\prod_{j=1}^{l}\Gamma(b_j)\Big]\sum_{r=0}^{\infty}(1/r!)\,{}_{l+1}F_l\left(\begin{matrix}-r,a_1,\dots,a_l;\\ b_1,\dots,b_l;1/\eta\end{matrix}\right)G^{m,n+1}_{p+1,q+1}\left(\omega\,\middle|\,\begin{matrix}1-r,c_1,\dots,c_p\\ d_1,\dots,d_q,1\end{matrix}\right),$$

$$(2)\quad \Big[1/\prod_{j=1}^{l}\Gamma(a_j)\Big]G^{m+l,n}_{p+l+1,q+l}\left(\eta\omega\,\middle|\,\begin{matrix}c_1,\dots,c_p,b_1,\dots,b_{l+1}\\ a_1,\dots,a_l,d_1,\dots,d_q\end{matrix}\right)$$
$$=\Big[1/\prod_{j=1}^{l+1}\Gamma(b_j)\Big]\sum_{r=0}^{\infty}(1/r!)\,{}_{l+1}F_{l+1}\left(\begin{matrix}-r,a_1,\dots,a_l;\\ b_1,\dots,b_{l+1};1/\eta\end{matrix}\right)G^{m,n+1}_{p+1,q+1}\left(\omega\,\middle|\,\begin{matrix}1-r,c_1,\dots,c_p\\ d_1,\dots,d_q,1\end{matrix}\right),$$

$$(3)\quad \Big[1/\prod_{j=1}^{l}\Gamma(1-c_j)\Big]G^{m,n+l}_{p+1,p+l+1}\left(\eta\omega\,\middle|\,\begin{matrix}c_1,\dots,c_l,b_1,\dots,b_p\\ a_1,\dots,a_p,d_1,\dots,d_{l+1}\end{matrix}\right)$$
$$=\Big[1/\prod_{j=1}^{l+1}\Gamma(1-d_j)\Big]\sum_{r=0}^{\infty}(1/r!)\,{}_{l+1}F_{l+1}\left(\begin{matrix}-r,1-c_1,\dots,1-c_l;\\ 1-d_1,\dots,1-d_{l+1};\omega\end{matrix}\right)\times$$
$$\times G^{n,m+1}_{p+1,p+1}\left(1/\eta\,\middle|\,\begin{matrix}1-r,1-a_1,\dots,1-a_p\\ 1-b_1,\dots,1-b_p,1\end{matrix}\right),$$

$$(4)\quad \Big[1/\prod_{j=1}^{l}\Gamma(1-c_j)\Big]G^{m,n+l}_{p+l,q+l}\left(\eta\omega\,\middle|\,\begin{matrix}c_1,\dots,c_l,b_1,\dots,b_p\\ a_1,\dots,a_q,d_1,\dots,d_l\end{matrix}\right)$$
$$=\Big[1/\prod_{j=1}^{l}\Gamma(1-d_j)\Big]\sum_{r=0}^{\infty}(1/r!)(-1)^r\,{}_{l+1}F_l\left(\begin{matrix}-r,1-c_1,\dots,1-c_l;\\ 1-d_1,\dots,1-d_l;\omega\end{matrix}\right)\times$$
$$\times G^{m,n+1}_{p+1,q+1}\left(\eta\,\middle|\,\begin{matrix}0,b_1,\dots,b_p\\ a_1,\dots,a_q,r\end{matrix}\right),$$

$$(5)\quad \Big[1/\prod_{j=1}^{l}\Gamma(1-c_j)\Big]G^{m,n+l}_{p+l,q+l+1}\left(\eta\omega\,\middle|\,\begin{matrix}c_1,\dots,c_l,b_1,\dots,b_p\\ a_1,\dots,a_q,d_1,\dots,d_{l+1}\end{matrix}\right)$$
$$=\Big[1/\prod_{j=1}^{l+1}\Gamma(1-d_j)\Big]\sum_{r=0}^{\infty}(1/r!)(-1)^r\,{}_{l+1}F_{l+1}\left(\begin{matrix}-r,1-c_1,\dots,1-c_l;\\ 1-d_1,\dots,1-d_{l+1};\omega\end{matrix}\right)\times$$
$$\times G^{m,n+1}_{p+1,q+1}\left(\eta\,\middle|\,\begin{matrix}0,b_1,\dots,b_p\\ a_1,\dots,a_q,r\end{matrix}\right),$$

$$(6)\quad \Big[1/\prod_{j=1}^{l}\Gamma(1-a_j)\Big]G^{m,n+l}_{p+l+1,p+l+1}\left(\eta\omega\,\middle|\,\begin{matrix}a_1,\ldots,a_l,d_1,\ldots,d_{p+1}\\ c_1,\ldots,c_p,b_1;\ldots,b_{l+1}\end{matrix}\right)$$

$$=\Big[1/\prod_{j=1}^{l+1}\Gamma(1-a_j)\Big]\sum_{r=0}^{\infty}(1/r!)_{l+1}F_{l+1}\left(\begin{matrix}-r,1-a_1,\ldots,1-a_l;\\ 1-b_1,\ldots,1-b_{l+1};\eta\end{matrix}\right)\times$$

$$\times G^{n,m+1}_{p+1,p+2}\left(1/\omega\,\middle|\,\begin{matrix}1-r,1-c_1,\ldots,1-c_p\\ 1-d_1,\ldots,1-d_{p+1},1\end{matrix}\right),$$

(see [5], formulae (79), (154), (157), (51), (147), (149) respectively). The purpose of this paper is to generalize, if it is possible, the formulae (1), ..., (6) in such a way as to permit the generalized hypergeometric series, appearing as coefficients, to be of an arbitrary order k, l.

The first part of this paper contains the necessary auxiliary results, the second part — a generalization of the formulae (1), (2), (3), (6), and the third — a generalization of the formulae (4), (5). The method used in the second part of this paper will be called the *method of the exponential factor*. The method used in the third part of this paper is in fact a variant of the previous one; the main idea is to find a formula analogous to (7) below in such a way as to keep the righthand side unchanged, while on the left instead of the function $G^{m,n}_{p,q}$ there appears the function $G^{n,m}_{q,p}$, obviously under different assumptions (the necessary formula, numbered (32), is given and proved in the first part); the corresponding paragraph is entitled as *problem of the changed parameters*. The paper ends up with an analysis of the known particular cases of the results obtained.

The methods applied are completely different from those used by Meijer in his series of works [5], which in fact are not used in this paper, since the formulae (201) from paper [5], IX (p. 244) and 20.5(5) from the tables [2], II (p. 419) applied here may be calculated independently of the series [5]. A considerable drawback of this paper is that the conditions of applicability of the formulae obtained are given in a form which is not very handy for practical use; this is caused by difficulties concerning the problem of the behaviour of the functions ${}_kF_l$ and $G^{m,n}_{p,q}$ for large values of the parameters. This behaviour is extremely complicated and there are only fragmentary and incomplete results known in this field (cf. [5], IV, p. 192, footnote [50]), [5], V, p. 356, footnote [64]) and [5], VII, p. 86, footnote [78])). Direct estimations, fragmentary and also cumbersome, are not included here.

The author hopes to apply the methods presented in this paper also to other problems.

The theorems proved in this paper were presented to the Conference on Analytic Functions in Kraków on September 1, 1962 (see [3]).

I should like to express my thanks to Dr T. Świątkowski for his kindness in reading the manuscript and making some useful improvements in the text.

§ 2. Auxiliary results. The starting point of my considerations is the following formula on the integral from the product of two Meijer's functions (cf. [4], p. 83-85):

$$(7)\qquad (1/\eta)\int_0^\infty G_{p,q}^{m,n}\left(x/\eta\,\middle|\,\begin{matrix}-a_1,\ldots,-a_p\\ -b_1,\ldots,-b_q\end{matrix}\right)G_{\sigma,\tau}^{\mu,\nu}\left(\omega x\,\middle|\,\begin{matrix}c_1,\ldots,c_\sigma\\ d_1,\ldots,d_\tau\end{matrix}\right)dx$$
$$= G_{q+\sigma,p+\tau}^{n+\mu,m+\nu}\left(\eta\omega\,\middle|\,\begin{matrix}b_1,\ldots,b_m,c_1,\ldots,c_\sigma,b_{m+1},\ldots,b_q\\ a_1,\ldots,a_n,d_1,\ldots,d_\tau,a_{n+1},\ldots,a_p\end{matrix}\right).$$

The sets of validity of this formula are given in the following

LEMMA 1 (due to C. S. Meijer). *Formula* (7) *is valid if* $m,n,p,q,\mu,\nu,\sigma,\tau$ *are integers and if at the same time one of the following five cases occurs*:

case (I):

$$(8)\qquad \eta\neq 0,\quad \omega\neq 0,\quad |\arg\eta|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\quad |\arg\omega|<(\mu+\nu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi,$$

$$(9)\qquad 0\leqslant n\leqslant p<q<p+\tau-\sigma\ (^1),\quad \tfrac{1}{2}p+\tfrac{1}{2}q-n<m\leqslant q,$$

$$(10)\qquad 0\leqslant\nu\leqslant\sigma,\quad \tfrac{1}{2}\sigma+\tfrac{1}{2}\tau-\nu<\mu\leqslant\tau,$$

$$(11)\qquad \mathrm{re}(d_h-b_j)>-1\quad (j=1,\ldots,m;\ h=1,\ldots,\mu),$$

$$(12)\qquad \mathrm{re}(c_h-a_j)<1\quad (j=1,\ldots,n;\ h=1,\ldots,\nu),$$

$$(13)\qquad b_h-a_j\neq 1,2,\ldots\quad (j=1,\ldots,n;\ h=1,\ldots,m),$$

$$(14)\qquad c_j-d_h\neq 1,2,\ldots\quad (j=1,\ldots,\nu;\ h=1,\ldots,\mu);$$

case (II):

$$(15)\qquad \eta\neq 0,\quad \omega\neq 0,\quad |\arg\eta|\leqslant(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\quad |\arg\omega|<(\mu+\nu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi,$$

$$(16)\qquad 0\leqslant n\leqslant p<q<p+\tau-\sigma,\quad \tfrac{1}{2}p+\tfrac{1}{2}q-n\leqslant m\leqslant q,$$

the conditions (10), (11), (12), (13), (14) *are fulfilled and*

$$(17)\qquad \sum_{h=1}^{q}\mathrm{re}\,b_h-\sum_{h=1}^{p}\mathrm{re}\,a_h+\tfrac{1}{2}(q-p+1)>(q-p)\max_{j=1,\ldots,\nu}\mathrm{re}\,c_j;$$

case (III):

$$(18)\qquad \eta\neq 0,\quad \omega\neq 0,\quad |\arg\eta|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\quad |\arg\omega|\leqslant(\mu+\nu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi,$$

$$(19)\qquad 0\leqslant\nu\leqslant\sigma,\quad \tfrac{1}{2}\sigma+\tfrac{1}{2}\tau-\nu\leqslant\mu\leqslant\tau,$$

(1) In the quoted paper [4] in case (I) it is additionally assumed that $n\neq 0$. This assumption, as is seen from (V), may be rejected.

the conditions (9), (11), (12), (13), (14) *are fulfilled and*

$$\sum_{h=1}^{\tau} \operatorname{re} d_h - \sum_{h=1}^{\sigma} \operatorname{re} c_h - \tfrac{1}{2}(\tau-\sigma+1) < (\tau-\sigma) \min_{j=1,\dots,n} \operatorname{re} a_j\,; \tag{20}$$

case (IV):

$$\begin{gathered} \eta \neq 0\,, \qquad \omega \neq 0\,, \\ |\arg\eta| \leqslant (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi\,, \qquad |\arg\omega| \leqslant (\mu+\nu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi\,, \end{gathered} \tag{21}$$

the conditions (16), (19), (11), (12), (13), (14), (17), (20) *are fulfilled and*

$$\frac{1}{q-p}\Big(\tfrac{1}{2}+\sum_{h=1}^{q} \operatorname{re} b_h - \sum_{h=1}^{p} \operatorname{re} a_h\Big) > \frac{1}{\tau-\sigma}\Big(\tfrac{1}{2}+\sum_{h=1}^{\tau} \operatorname{re} d_h - \sum_{h=1}^{\sigma} \operatorname{re} c_h\Big)\,; \tag{22}$$

case (V):

$$\eta \neq 0\,, \qquad \omega \neq 0\,, \qquad |\arg\eta| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi\,, \qquad n = 0\,, \tag{23}$$

$$0 \leqslant p < q < p+\tau-\sigma\,, \qquad \tfrac{1}{2}p+\tfrac{1}{2}q < m \leqslant q\,, \tag{24}$$

$$0 \leqslant \nu \leqslant \sigma\,, \qquad 1 \leqslant \mu \leqslant \tau \tag{25}$$

and the conditions (11), (14) *are fulfilled* (²).

Note now that for arbitrary complex numbers $x \neq 0$, $\varrho, \beta \neq 0$ we have

$$x^{-\varrho} e^{-\beta x} = \beta^{\varrho} G^{1,0}_{0,1}(\beta x|-\varrho)\,. \tag{26}$$

If $\operatorname{re} r \geqslant 0$, then from Lemma 1 we immediately obtain the formulae

$$\begin{aligned} (1/\eta)\int_0^{\infty} (x/\eta)^{-b_h} \exp(-x/t)(x/t)^r G^{\mu,\nu}_{\sigma,\tau}\Big(\omega x\,\Big|\,\begin{matrix} c_1, \dots, c_\sigma \\ d_1, \dots, d_\tau \end{matrix}\Big) dx \\ = (t/\eta)^{1-b_h} G^{\mu,\nu+1}_{\sigma+1,\tau}\Big(\omega t\,\Big|\,\begin{matrix} b_h-r, c_1, \dots, c_\sigma \\ d_1, \dots, d_\tau \end{matrix}\Big) \qquad (h = 1, \dots, m)\,, \end{aligned} \tag{27}$$

which are valid in view of (V) if the relations

$$\begin{gathered} t \neq 0\,, \qquad |\arg t| < \tfrac{1}{2}\pi\,, \\ \omega \neq 0\,, \qquad 0 \leqslant \nu \leqslant \sigma \leqslant \tau-2\,, \qquad 1 \leqslant \mu \leqslant \tau \end{gathered} \tag{28}$$

(²) In the quoted paper [4] it is moreover assumed that

$$\begin{aligned} &b_h - b_j \neq 0, \pm 1, \dots && (j = 1, \dots, m;\ h = 1, \dots, m;\ j \neq h)\,, \\ &a_h - a_j \neq 0, \pm 1, \dots && (j = 1, \dots, n;\ h = 1, \dots, n;\ j \neq h)\,, \\ &d_h - d_j \neq 0, \pm 1, \dots && (j = 1, \dots, \mu;\ h = 1, \dots, \mu;\ j \neq h)\,, \\ &a_j - d_h \neq 0, \pm 1, \dots && (j = 1, \dots, n;\ h = 1, \dots, \mu)\,. \end{aligned}$$

These conditions may be omitted if we use as the definition of the Meijer's function the formula quoted here instead of formula (1) of paper [4]. The proof, as given in [4], remains almost unchanged. This remark was kindly communicated to me by Professor Meijer.

and (11), (14) are satisfied; the formulae will be of use in further considerations. Comparing the result obtained with the assumptions of the formula 20.5(5) from tables [2], II (p. 419) we state that if $\sigma+\tau < 2(\mu+\nu)$, $|\arg\omega| < (\mu+\nu-\frac{1}{2}\sigma-\frac{1}{2}\tau)\pi$, then the condition

$$\tau \neq \sigma+1$$

is superfluous. This is equivalent to the statement that if $\mathrm{re}\, r \geqslant 0$ and the relations (28),

$$\eta \neq 0\,, \quad 0 \leqslant n \leqslant p \leqslant q-2\,, \quad 1 \leqslant m \leqslant q$$

and (11), (13) are satisfied, then the formulae

$$(29) \quad (1/\omega)\int_0^\infty (\omega x)^{d_h}\exp(-tx)(tx)^r G^{m,n}_{p,q}\left(x/\eta \,\middle|\, \begin{matrix} -a_1, \ldots, -a_p \\ -b_1, \ldots, -b_q \end{matrix}\right)dx$$
$$= (\omega/t)^{1+d_h} G^{m,n+1}_{p+1,q}\left(1/\eta t \,\middle|\, \begin{matrix} -d_h-r, -a_1, \ldots, -a_p \\ -b_1, \ldots, -b_q, \end{matrix}\right) \qquad (h = 1, \ldots, \mu)$$

hold, where the condition

$$(30) \qquad q \neq p+1$$

is superfluous provided $p+q < 2(m+n)$, $|\arg\eta| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$.

Now, as has been announced previously, we shall find a formula analogous to (7) in such a way that the right-hand side will remain unchanged, while at the left instead of the function $G^{m,n}_{p,q}$ the function $G^{n,m}_{q,p}$ will appear, obviously under different assumptions. Accordingly we remark that starting from the right-hand side of formula (7) we arrive at the function $G^{m,n}_{p,q}$ under the integral by applying formula (29) from [4], I (p. 86); this formula results from the properties of the Mellin transform. Replacing the formula in question by an analogous one

$$(31) \quad \int_0^\infty G^{n,m}_{q,p}\left(\eta x \,\middle|\, \begin{matrix} b_1, \ldots, b_q \\ a_1, \ldots, a_p \end{matrix}\right) x^{-s}\frac{dx}{x} = \frac{\prod_{j=1}^{m}\Gamma(1-b_j+s)\prod_{j=1}^{n}\Gamma(a_j-s)}{\prod_{j=m+1}^{q}\Gamma(b_j-s)\prod_{j=n+1}^{p}\Gamma(1-a_j+s)}\,\eta^s,$$

we obtain, in view of relations (30) and (31) from [4], I (p. 86) the required formula

$$(32) \quad \int_0^\infty G^{n,m}_{q,p}\left(\eta x \,\middle|\, \begin{matrix} b_1, \ldots, b_q \\ a_1, \ldots, a_p \end{matrix}\right) G^{\mu,\nu}_{\sigma,\tau}\left(\omega/x \,\middle|\, \begin{matrix} c_1, \ldots, c_\sigma \\ d_1, \ldots, d_\tau \end{matrix}\right)\frac{dx}{x}$$
$$= G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}\left(\eta\omega \,\middle|\, \begin{matrix} b_1, \ldots, b_m, c_1, \ldots, c_\sigma, b_{m+1}, \ldots, b_q \\ a_1, \ldots, a_n, d_1, \ldots, d_\tau, a_{n+1}, \ldots, a_p \end{matrix}\right).$$

In order to make the above precise we shall prove the following

LEMMA 2. *Formula* (32) *is valid if* $m, n, p, q, \mu, \nu, \sigma, \tau$ *are integers and if at the same time one of the following four cases occurs*:

case (VI):

$$\text{(8)}\qquad \begin{gathered} \eta \neq 0\,, \qquad \omega \neq 0\,, \\ |\arg\eta| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi\,, \qquad |\arg\omega| < (\mu+\nu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi\,, \end{gathered}$$

$$\text{(33)}\qquad 0 \leqslant m \leqslant q < p < p+\tau-\sigma\,, \qquad \tfrac{1}{2}p+\tfrac{1}{2}q-m < n \leqslant p$$

and the conditions (10), (11), (12), (13), (14) *are fulfilled;*

case (VII):

$$\text{(15)}\qquad \begin{gathered} \eta \neq 0\,, \qquad \omega \neq 0\,, \\ |\arg\eta| \leqslant (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi\,, \qquad |\arg\omega| < (\mu+\nu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi\,, \end{gathered}$$

$$\text{(34)}\qquad 0 \leqslant m \leqslant q < p < p+\tau-\sigma\,, \qquad \tfrac{1}{2}p+\tfrac{1}{2}q-m \leqslant n \leqslant p\,,$$

the conditions (10), (11), (12), (13), (14) *are fulfilled and*

$$\text{(35)}\qquad \sum_{h=1}^{p} \operatorname{re} a_h - \sum_{h=1}^{q} \operatorname{re} b_h - \tfrac{1}{2}(p-q+1) < (p-q) \min_{j=1,\ldots,\mu} \operatorname{re} d_j\,;$$

case (VIII):

$$\text{(18)}\qquad \begin{gathered} \eta \neq 0\,, \qquad \omega \neq 0\,, \\ |\arg\eta| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi\,, \qquad |\arg\omega| \leqslant (\mu+\nu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi\,, \end{gathered}$$

the conditions (33), (19), (11), (12), (13), (14) *are fulfilled and*

$$\text{(36)}\qquad \sum_{h=1}^{\sigma} \operatorname{re} c_h - \sum_{h=1}^{\tau} \operatorname{re} d_h - \tfrac{1}{2}(\sigma-\tau-1) > (\sigma-\tau) \min_{j=1,\ldots,n} \operatorname{re} a_j\,;$$

case (IX):

$$\text{(21)}\qquad \begin{gathered} \eta \neq 0\,, \qquad \omega \neq 0\,, \\ |\arg\eta| \leqslant (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi\,, \qquad |\arg\omega| \leqslant (\mu+\nu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi \end{gathered}$$

and the conditions (34), (19), (11), (12), (13), (14), (35), (36) *are fulfilled.*

Proof. The proof is analogous to that of Theorem 1 from [4], I (p. 83-91) (3).

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(3) A case analogous to (V) has not been obtained, because for $m = 0$, $|\arg\omega| \geqslant (\mu+\nu-\frac{1}{2}\sigma-\frac{1}{2}\tau)\pi$ an argumentation similar to that in [4], I (pp. 88-89) gives no guarantee of the convergency of the integral (32) in the neighbourhood of the point 0.

It is known (see [5], I, formula (18), p. 371, or [4], I, formula (1), p. 82) that the function $G_{q,p}^{n,m}$ can be expressed in the form

$$(37)\qquad G_{q,p}^{n,m}\left(\eta x\left|\begin{matrix} b_1, \ldots, b_q \\ a_1, \ldots, a_p\end{matrix}\right.\right) = \sum_{h=1}^{n}\Big\{\Big[\prod_{\substack{j=1\\ j\neq h}}^{n}\Gamma(a_j-a_h)\prod_{j=1}^{m}\Gamma(1+a_h-b_j)\Big]\times$$
$$\times\Big[\prod_{j=n+1}^{p}\Gamma(1+a_h-a_j)\prod_{j=m+1}^{q}\Gamma(b_j-a_h)\Big]^{-1}(\eta x)^{a_h}\times$$
$$\times\, {}_qF_{p-1}\left(\begin{matrix}1+a_h-b_1, \ldots, 1+a_h-b_q; \\ 1+a_h-a_1, \ldots * \ldots, 1+a_h-a_p; (-1)^{q-m-n}\eta x\end{matrix}\right)\Big\},$$

where the asterisk * denotes that the number $1+a_h-a_h$ is to be omitted in the sequence $1+a_h-a_1, \ldots, 1+a_h-a_p$ if the conditions

$$(38)\qquad \begin{aligned} &a_h-a_j \neq 0, \pm 1, \ldots \quad (j = 1, \ldots, n;\ h = 1, \ldots, n;\ j\neq h), \\ &a_j-a_h \neq 1, 2, \ldots \quad (j = n+1, \ldots, p;\ h = 1, \ldots, n)\end{aligned}$$

are fulfilled. In the case where some of the numbers a_j-a_h ($j = n+1, \ldots, p$; $h = 1, \ldots, n$) are natural, the respective coefficients

$$\Big[\prod_{\substack{j=1\\ j\neq h}}^{n}\Gamma(a_j-a_h)\prod_{j=1}^{m}\Gamma(1+a_h-b_j)\Big]\Big/\Big[\prod_{j=n+1}^{p}\Gamma(1+a_h-a_j)\prod_{j=m+1}^{q}\Gamma(b_j-a_h)\Big]$$

are to be replaced by the limit of the products

$$\prod_{j=n+1}^{p}\Gamma(1+a_h^*-a_j)\times$$
$$\times\Big\{\Big[\prod_{\substack{j=1\\ j\neq h}}^{n}\Gamma(a_j-a_h^*)\prod_{j=1}^{m}\Gamma(1+a_h^*-b_j)\Big]\Big/\Big[\prod_{j=n+1}^{p}\Gamma(1+a_h^*-a_j)\prod_{j=m+1}^{q}\Gamma(b_j-a_h^*)\Big]\Big\}$$

as $a_h^*\to a_h$ and the respective functions ${}_qF_{p-1}(1+a_h-b_1, \ldots)$ by the limit of the quotients ${}_qF_{p-1}(1+a_h^*-b_1, \ldots)$ and $\Gamma(1+a_h^*-a_{n+1})\Gamma(1+a_h^*-a_{n+2})\times \ldots \times\Gamma(1+a_h^*-a_p)$ as $a_h^*\to a_h$. An analogous result holds for the function $G_{\sigma,\tau}^{\mu,\nu}$ if the conditions

$$(39)\qquad d_h-d_j \neq 0, \pm 1, \ldots \quad (j = 1, \ldots, \mu;\ h = 1, \ldots, \mu;\ j\neq h)$$

are fulfilled.

Moreover, it is known (see [4], I, formulae (32), (34) and (33), pp. 87-88) that the function $G_{q,p}^{n,m}$ has for $m > 0$, $|\arg\eta| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ an asymptotic expansion

$$G_{q,p}^{n,m}\left(\eta x\left|\begin{matrix} b_1, \ldots, b_q \\ a_1, \ldots, a_p\end{matrix}\right.\right) \sim \sum_{j=1}^{m}(\eta x)^{b_j-1}[C_{j,0}+C_{j,1}(\eta x)^{-1}+C_{j,2}(\eta x)^{-2}+\ldots]$$

and also

$$G_{q,p}^{n,0}(\eta x) = \sum_{h=0}^{p-n} C'_h G_{q,p}^{p,0}(\eta x e^{(p-n-2h)\pi i}),$$

where the function $G_{q,p}^{p,0}$ has for $q < p-1$, $|\arg\eta| < (p-q+1)\pi$ and for $q = p-1$, $|\arg\eta| < \frac{3}{2}\pi$ an asymptotic expansion

$$G_{q,p}^{p,0}\left(\eta x \middle| \begin{matrix} b_1, \dots, b_q \\ a_1, \dots, a_p \end{matrix}\right) \sim (\eta x)^{\vartheta} \exp\left((q-p)(\eta x)^{1/(p-q)}\right) \times$$
$$\times [C''_0 + C''_1(\eta x)^{-1/(p-q)} + C''_2(\eta x)^{-2/(p-q)} + \dots].$$

In the above formulae $C_{j,h}$ $(j = 1, \dots, m;\ h = 1, 2, \dots)$, C'_h $(h = 0, \dots \dots, p-n)$, ϑ, C''_h $(h = 1, 2, \dots)$ are constants and

$$\vartheta = (p-q)^{-1}\left[\tfrac{1}{2}(q-p+1) + \sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h\right].$$

An analogous result holds for the function $G_{\sigma,\tau}^{\mu,\nu}$.

From the definition of Meijer's functions and from the above we infer that the functions $G_{q,p}^{n,m}$, $G_{\sigma,\tau}^{\mu,\nu}$, $G_{q+\sigma,p+\tau}^{n+\mu,m+\nu}$ in (32) are defined and at the same time the functions $G_{q,p}^{n,m}$, $G_{\sigma,\tau}^{\mu,\nu}$ can be expressed in the form of generalized hypergeometric series given by formula (37) and have the asymptotic expansions quoted above if the conditions (8), (33), (10), (13), (14), (38), (39) and

$$(40) \qquad b_j - d_h \neq 1, 2, \dots \quad (j = 1, \dots, m;\ h = 1, \dots, \mu),$$

$$(41) \qquad c_j - a_h \neq 1, 2, \dots \quad (j = 1, \dots, \nu;\ h = 1, \dots, n)$$

are fulfilled. The integral (32) is convergent in the neighbourhood of the point 0 if the inequalities (12) are valid, and in the neighbourhood of the point ∞ if the inequalities (11) are satisfied; this results from expressing Meijer's functions in the form of generalized hypergeometric series and from the asymptotic expansions discussed above. Before beginning the calculations notice that formula (32) may be applied (cf. [4], pp. 85-86) with all the conditions mentioned above provided the supplementary conditions

$$(42) \qquad \operatorname{re}(b_h - a_j) < 1 \quad (j = 1, \dots, n;\ h = 1, \dots, m)$$

are fulfilled.

Beginning the calculations, we use at first on the left-hand side of formula (32), which we shall denote by I, the definition of $G_{\sigma,\tau}^{\mu,\nu}$; then we obtain

$$I = \frac{1}{2\pi i}\int_0^{\infty}\int_{\gamma} \frac{\prod_{j=1}^{\nu}\Gamma(1-c_j+s)\prod_{j=1}^{\mu}\Gamma(d_j-s)}{\prod_{j=\nu+1}^{\sigma}\Gamma(c_j-s)\prod_{j=\mu+1}^{\tau}\Gamma(1-d_j+s)}\, \omega^s G_{q,p}^{n,m}\left(\eta x \middle| \begin{matrix} b_1, \dots, b_q \\ a_1, \dots, a_p \end{matrix}\right) x^{-s} ds \frac{dx}{x},$$

where γ' is a contour running from $\infty - i\tau'$ to $\infty + i\tau'$ ($\tau' > 0$) with the poles $d_j, d_j+1, \ldots$ ($j = 1, \ldots, \mu$) inside and the poles $c_j-1, c_j-2, \ldots$ ($j = 1, \ldots, \nu$) outside γ'.

In consequence, applying the well-known theorem concerning the inversion of a repeated infinite integral (see e.g. [1], § 177, p. 504) and using formula (31) we obtain

$$I = \frac{1}{2\pi i} \int_{\gamma'} \frac{\prod_{j=1}^{m} \Gamma(1-b_j+s) \prod_{j=1}^{\nu} \Gamma(1-c_j+s) \prod_{j=1}^{n} \Gamma(a_j-s) \prod_{j=1}^{\mu} \Gamma(d_j-s)\,(\eta\omega)^s ds}{\prod_{j=m+1}^{q} \Gamma(b_j-s) \prod_{j=\nu+1}^{\sigma} \Gamma(c_j-s) \prod_{j=n+1}^{p} \Gamma(1-a_j+s) \prod_{j=\mu+1}^{\tau} \Gamma(1-d_j+s)}.$$

In view of (11), (12) the contour γ' runs from $\infty - i\tau'$ to $\infty + i\tau'$ ($\tau' > 0$) with all the poles $a_j, a_j+1, \ldots$ ($j = 1, \ldots, n$), $d_j, d_j+1, \ldots$ ($j = 1, \ldots, \mu$) inside and all the poles $b_j-1, b_j-2, \ldots$ ($j = 1, \ldots, m$), $c_j-1, c_j-2, \ldots$ ($j = 1, \ldots, \nu$) outside γ'; then

$$I = G_{q+\sigma,p+\tau}^{n+\mu,m+\nu}\left(\eta\omega \,\middle|\, \begin{matrix} b_1, \ldots, b_m, c_1, \ldots, c_\sigma, b_{m+1}, \ldots, b_q \\ a_1, \ldots, a_n, d_1, \ldots, d_\tau, a_{n+1}, \ldots, a_p \end{matrix}\right),$$

and this proves formula (32) if the conditions (8), (33), (10), (13), (14), (38), (39), (40), (41), (12), (11), (42) are fulfilled. The conditions (40), (41) are superfluous, because they result from (11), (12) respectively, and the conditions (38), (39), (42) may be rejected by analytic continuation, and this completes the proof of Lemma 2 in case (VI).

In the case where $|\arg\eta| = (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, in view of the well-known formulae

$$G_{q,p}^{n,m}(\eta x) = kG_{q,p}^{p,0}(\eta x e^{(p-m-n)\pi i}) + \sum_{h=0}^{m-1} \gamma_h G_{q,p}^{p,m-h}(\eta x e^{(p-n-h-2)\pi i}) + \sum_{h=1}^{p-n} \varkappa_h G_{q,p}^{p,m}(\eta x e^{(p-n-2h)\pi i}),$$

$$G_{q,p}^{n,m}(\eta x) = lG_{q,p}^{p,0}(\eta x e^{(m+n-p)\pi i}) + \sum_{h=0}^{m-1} \delta_h G_{q,p}^{p,m-h}(\eta x e^{(n+h+2-p)\pi i}) + \sum_{h=1}^{p-n} \lambda_h G_{q,p}^{p,m}(\eta x e^{(n+2h-p)\pi i}),$$

where k, γ_h ($h = 0, \ldots, m-1$), $\varkappa_h$ ($h = 1, \ldots, p-n$), l, δ_h ($h = 0, \ldots, m-1$), λ_h ($h = 1, \ldots, p-n$) are constants (see [4], I, formulae (39), (40), p. 90) and in view of the asymptotic expansions quoted above together with formula (37), condition (12) of convergency of the integral (32) in the neighbourhood of the point 0 remains unchanged, but for the point ∞ the inequalities (11) must be completed by (35). Further, applying the well-known formulae

$$G_{q,p}^{n,m}\left(\eta x\left|\begin{matrix}b_1,\dots,b_q\\a_1,\dots,a_p\end{matrix}\right.\right)=-(1/2\pi i)\Big\{e^{-\pi i a_{n+1}}G_{q,p}^{n+1,m}\left(\eta xe^{\pi i}\left|\begin{matrix}b_1,\dots,b_q\\a_1,\dots,a_p\end{matrix}\right.\right)-$$
$$-e^{\pi i a_{n+1}}G_{q,p}^{n+1,m}\left(\eta xe^{-\pi i}\left|\begin{matrix}b_1,\dots,b_q\\a_1,\dots,a_p\end{matrix}\right.\right)\Big\},$$

$$G_{q,p}^{n,m}\left(\eta x\left|\begin{matrix}b_1,\dots,b_q\\a_1,\dots,a_p\end{matrix}\right.\right)=-(1/2\pi i)\Big\{e^{-\pi i b_{m+1}}G_{q,p}^{n,m+1}\left(\eta xe^{\pi i}\left|\begin{matrix}b_1,\dots,b_q\\a_1,\dots,a_p\end{matrix}\right.\right)-$$
$$-e^{\pi i b_{m+1}}G_{q,p}^{n,m+1}\left(\eta xe^{-\pi i}\left|\begin{matrix}b_1,\dots,b_q\\a_1,\dots,a_p\end{matrix}\right.\right)\Big\}$$

(see [4], I, formulae (41), (42), p. 90) we state the possibility of replacing (33) by (34). Similarly we prove that if $|\arg\omega|=(\mu+\nu-\frac{1}{2}\sigma-\frac{1}{2}\tau)\pi$, then the integral (32) is convergent in the neighbourhood of the point 0 provided the inequalities (12) are completed by (36), but the condition (11) of convergency in the neighbourhood of the point ∞ remains unchanged, and that (10) can be replaced by (19). Finally we easily verify that if conditions (34), (19) are simultaneously fulfilled and $|\arg\eta|=(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $|\arg\omega|=(\mu+\nu-\frac{1}{2}\sigma-\frac{1}{2}\tau)\pi$, then the integral (32) is convergent provided not only relations (11), (12) but also (35), (36) are valid; this ends the proof.

The lemma proved above in case (VIII) implies immediately for $\operatorname{re} r\geqslant 0$ the formulae

$$(43)\quad \int_0^\infty(\eta x)^{a_h}\exp(-tx)(tx)^r G_{\sigma,\tau}^{\mu,\nu}\left(\omega/x\left|\begin{matrix}c_1,\dots,c_\sigma\\d_1,\dots,d_\tau\end{matrix}\right.\right)\frac{dx}{x}$$
$$=(\eta/t)^{a_h}G_{\sigma,\tau+1}^{\mu+1,\nu}\left(\omega t\left|\begin{matrix}c_1,\dots,c_\sigma\\a_h+r,d_1,\dots,d_\tau\end{matrix}\right.\right)\quad(h=1,\dots,n)$$

which are valid if the relations (28),

$$\omega\neq 0,\quad |\arg\omega|\leqslant(\mu+\nu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi,$$
$$0\leqslant\nu\leqslant\sigma\leqslant\tau-1,\quad 1\leqslant\mu\leqslant\tau,\quad \sigma+\tau\leqslant 2(\mu+\nu)$$

and (12), (14) are satisfied. Similarly for $\operatorname{re} r\geqslant 0$ we obtain the formulae

$$(44)\quad \int_0^\infty(\omega/x)^{d_h}\exp(-t/x)(t/x)^r G_{q,p}^{n,m}\left(\eta x\left|\begin{matrix}b_1,\dots,b_q\\a_1,\dots,a_p\end{matrix}\right.\right)\frac{dx}{x}$$
$$=(\omega/t)^{d_h}G_{q,p+1}^{n+1,m}\left(\eta t\left|\begin{matrix}b_1,\dots,b_q\\d_h+r,a_1,\dots,a_p\end{matrix}\right.\right)\quad(h=1,\dots,\mu)$$

provided the relations (28),

$$\eta\neq 0,\ |\arg\eta|\leqslant(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,$$
$$0\leqslant m\leqslant q\leqslant p-1,\quad 1\leqslant n\leqslant p,\quad p+q\leqslant 2(m+n)$$

and (11), (13) are satisfied.

In connection with (27), (29), (43), (44), which will be applied further for $r = 0, 1, \ldots$ in the generalization of (1), ..., (6), it is to be noted that in theorems 1A, 1B, 2A, 2B below we intend to express the function $G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}(\eta\omega)$ as a series by $G^{\mu,\nu+1}_{\sigma+1,\tau}(\omega t)$, $G^{m,n+1}_{p+1,q}(1/\eta t)$, $G^{\mu+1,\nu}_{\sigma,\tau+1}(\omega t)$, $G^{n+1,m}_{q,p+1}(\eta t)$ respectively. Here we make general remarks concerning the choice in the corresponding expansions of the branches of the above-mentioned functions (cf. [5], I, Remark 1, p. 376 and Remark 3, p. 377) (4).

Remark 1. If the value of $\arg(\omega t)$ for the function $G^{\mu,\nu+1}_{\sigma+1,\tau}$ is chosen, then the value of $\arg(\eta\omega)$ in $G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}$ is determined by the conditions $\arg(\eta\omega) = \arg(\eta/t) + \arg(\omega t)$ and $|\arg(-1)^{p-m-n+1}(\eta/t)| < \frac{1}{2}\pi$.

Remark 2. If the value of $\arg(1/\eta t)$ for the function $G^{m,n+1}_{p+1,q}$ is chosen, then the value of $\arg(\eta\omega)$ in $G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}$ is determined by the conditions $\arg(\eta\omega) = \arg(\omega/t) + \arg(\eta t)$ and $|\arg(-1)^{\sigma-\mu-\nu+1}(\omega/t)| < \frac{1}{2}\pi$, where in the case $q = p+1$ the value of $G^{m,n+1}_{p+1,p+1}(1/\eta t)$ is derived by analytic continuation along the straight line segment with the beginning at the point $|\eta t|/(2\eta t)$.

Remark 3. If the value of $\arg(\omega t)$ for the function $G^{\mu+1,\nu}_{\sigma,\tau+1}$ is chosen, then the value of $\arg(\eta\omega)$ in $G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}$ is determined by the conditions $\arg(\eta\omega) = \arg(\eta/t) + \arg(\omega t)$ and $|\arg(-1)^{q-m-n+1}(\eta/t)| < \frac{1}{2}\pi$.

Remark 4. If the value of $\arg(\eta t)$ for the function $G^{n+1,m}_{q,p+1}$ is chosen, then the value of $\arg(\eta\omega)$ in $G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}$ is determined by the conditions $\arg(\eta\omega) = \arg(\omega/t) + \arg(\eta t)$ and $|\arg(-1)^{\sigma-\mu-\nu+1}(\omega/t)| < \frac{1}{2}\pi$.

At the end we shall give two tests to be used for integrating term by term the series considered further (cf. [1], § 176B, p. 500, and § 175A, p. 495).

TEST 1. *If Φ and f_r ($r = 0, 1, \ldots$) are complex (or, in particular, real) functions of the real variable x defined in the interval $\beta \leqslant x < \infty$ and if at the same time*

1° *the series*

$$\sum_{r=0}^{\infty} f_r(x)$$

converges uniformly in any fixed interval $\beta \leqslant x \leqslant x_0$, where x_0 is arbitrary,

2° *the function Φ is continuous in the interval $\beta \leqslant x < \infty$,*

3° *either the integral*

$$\int_{\beta}^{\infty} |\Phi(x)| \sum_{r=0}^{\infty} |f_r(x)|\, dx$$

(4) We assume here $\sigma < \tau - 1$ in the case of expansion with respect to $G^{\mu,\nu+1}_{\sigma+1,\tau}$ or $G^{\mu+1,\nu}_{\sigma,\tau+1}$, $p < q$ in the case of $G^{m,n+1}_{p+1,q}$, and $q < p-1$ in the case of $G^{n+1,m}_{q,p+1}$.

or the series

$$\sum_{r=0}^{\infty} \int_{\beta}^{\infty} |\Phi(x)| |f_r(x)| dx$$

is convergent,

then

$$\int_{\beta}^{\infty} \Phi(x) \sum_{r=0}^{\infty} f_r(x) dx = \sum_{r=0}^{\infty} \int_{\beta}^{\infty} \Phi(x) f_r(x) dx .$$

TEST 2. *If Φ is a complex function of the real variable x defined in the finite interval $a < x < \beta$ and if f_r $(r = 0, 1, \ldots)$ are complex functions of the real variable x defined in the interval $a \leqslant x \leqslant \beta$ and if at the same time*

1° *the series*

$$\sum_{r=0}^{\infty} f_r(x)$$

converges uniformly in the interval $a \leqslant x \leqslant \beta$,

2° *the integral*

$$\int_{a}^{\beta} |\Phi(x)| dx$$

is convergent,

then

$$\int_{a}^{\beta} \Phi(x) \sum_{r=0}^{\infty} f_r(x) dx = \sum_{r=0}^{\infty} \int_{a}^{\beta} \Phi(x) f_r(x) dx .$$

References

[1] T. J. I'a Bromwich, *An introduction to the theory of infinite series*, II ed. London 1959.

[2] A. Erdélyi, (Editor), *Tables of integral transforms I-II*, New York 1954.

[3] J. Ławrynowicz, *On the series of Meijer's functions*, Coll. Math. 11 (1964), pp. 286-287.

[4] C. S. Meijer, *Neue Integraldarstellungen für Whittakersche Funktionen I-V* Proc. Kon. Ned. Akad. v. Wetensch. 44 (1941), pp. 81-92, pp. 186-194, pp. 298-307, pp. 442-451, pp. 590-598.

[5] — *Expansion theorems for the G-function I-XI*, Proc. Kon. Ned. Akad. v. Wetensch., Series A, 55 (1952), pp. 369-379 and pp. 483-487; 56 (1953), pp. 43-49, pp. 187-193, pp. 349-357; 57 (1954), pp. 77-82, pp. 83-91, pp. 273-279; 58 (1955), pp. 243-251, pp. 309-314, and 59 (1956), pp. 70-82.

[6] P. M. Morse and H. Feshbach, *Methods of theoretical physics*, New York 1953.

Reçu par la Rédaction le 7. 5. 1964