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On some integral transforms involving Jacobi functions

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1. A result which has recently been used by C. J. Tranter for obtaining a solution of some dual integral equations and for obtaining an infinite integral for dual Fourier-Bessel series (see [3], [4]) is a special case of the Weber-Schafheitlin integral. We shall make use of this result along with two other results for obtaining inversion formulae for certain integral transforms involving Jacobi's polynomials as their kernels. The results in question, which are easily derived from the more general ones given by Watson ([5]), are:

(i) if m is zero or a positive integer, $\nu > -1-m$, and $k > 0$, then

$$(1.1)\qquad \int_0^\infty x^{1-k} J_{\nu+2m+k}(x) J_\nu(rx)\,dx$$
$$= \begin{cases} \dfrac{\Gamma(1+\nu+m)}{2^{k-1}\Gamma(\nu+1)\Gamma(m+k)} r^\nu (1-r^2)^{k-1} \mathfrak{J}_m(k+\nu, \nu+1, r^2), & 0 < r < 1, \\ 0, \; r > 1, \end{cases}$$

$\mathfrak{J}_m(k+\nu, \nu+1, r^2)$ being Jacobi polynomials ([2]),

(ii) if m is zero or a positive integer, $\nu > -m-k$, and $k < 1$, then

$$(1.2)\qquad \int_0^\infty x^k J_{\nu+2m+k}(x) J_{\nu-1}(rx)\,dx$$
$$= \begin{cases} \dfrac{\Gamma(\nu+m+k)}{2^{-k}\Gamma(\nu)\Gamma(m+1)} r^{\nu-1} \mathfrak{J}_m(\nu+k, \nu, r^2), & 0 < r < 1, \\ \dfrac{\Gamma(\nu+m+k) r^{\nu-1}}{2^{-k}\Gamma(\nu+2m+k+1)\Gamma(-k-m)} H_m(\nu+k, \nu, r^2), & r > 1, \end{cases}$$

$r^{-2m-2\nu}(r^2-1)^{-k} {}_2F_1[m+1, m+\nu; \nu+2m+k+1; 1/r^2] = H_m(\nu+k, \nu, r^2)$ being Jacobi functions of the second kind,

(iii) if m is zero or a positive integer, $\nu > -m-k$, and $k < 1$, then

$$(1.3)\qquad \int_0^\infty x^k J_{\nu+2m+k-1}(x) J_\nu(rx)\,dx$$

$$= \begin{cases} \dfrac{\Gamma(\nu+m+k)}{2^{-k}\Gamma(\nu+1)\Gamma(m)}\, r^\nu \mathfrak{J}_m(k+\nu+1,\nu+1,r^2), & 0<r<1, \\ \dfrac{\Gamma(\nu+m+k)\, r^\nu H_m(k+\nu+1,\nu+1,r^2)}{2^{-k}\Gamma(\nu+2m+k)\Gamma(1-m-k)}, & r>1, \end{cases}$$

ν and k will be assumed to be real throughout this note.

2. In this paper we prove the following two theorems:

THEOREM 1. *Let*

$$(2.1)\qquad \varphi(x) = \int_x^\infty u^{1-2\nu-2k}(u^2-x^2)^{k-1}\mathfrak{J}_m(\nu+k,\nu+1,x^2/u^2)\psi(u)\,du\,;$$

then

$$(2.2)\qquad \psi(x)$$

$$= \frac{2\Gamma(1+\nu+m)\Gamma(\nu+m+k)}{\Gamma(\nu)\Gamma(\nu+1)\Gamma(m+1)\Gamma(m+k)}\Bigg[\int_0^x \mathfrak{J}_m(\nu+k,\nu,u^2/x^2)\,d[u^{2\nu}\varphi(u)]+$$

$$+\frac{\Gamma(\nu)\Gamma(m+1)}{\Gamma(\nu+2m+k+1)\Gamma(-k-m)}\int_x^\infty H_m(\nu+k,\nu,u^2/x^2)\,d[u^{2\nu}\varphi(u)]\Bigg],$$

provided

(i) $\nu > -1$, $0 < k < 1$ *and* m *is a positive integer,*

(ii) $\int_0^\infty t^{1-\nu-k}|\psi(t)|\,dt$ *and* $\int_0^\infty t^{1-\nu}\left|\dfrac{d}{dt}\{t^{2\nu}\varphi(t)\}\right|dt$ *are convergent,*

(iii) $\dfrac{d}{dt}[t^{2\nu}\varphi(t)]$ *is continuous.*

THEOREM 2. *Let*

$$(2.3)\qquad \varphi(x) = \int_0^x u^{1+2\nu}(x^2-u^2)^{k-1}\mathfrak{J}_m(k+\nu,\nu+1,u^2/x^2)\psi(u)\,du\,;$$

then

$$(2.4)\qquad \psi(x)$$

$$= \frac{2\Gamma(m+\nu+k)\Gamma(m+\nu+1)}{\{\Gamma(\nu+1)\}^2\Gamma(m)\Gamma(m+k)}\Bigg[\int_x^\infty \mathfrak{J}_{m-1}(\nu+k+1,\nu+1,x^2/u^2)\,d[u^{2m}\varphi(u)]+$$

$$+\frac{\Gamma(m)\Gamma(\nu+1)}{\Gamma(\nu+2m+k+1)\Gamma(1-k-m)}\int_0^x H_{m-1}(\nu+k+1,\nu+1,x^2/u^2)\,d[u^{2m}\varphi(u)]\Bigg],$$

provided

(i) $\nu > -1$, $0 < k < 1$ *and* m *is a positive integer,*

(ii) $\int_0^\infty t^{1-\nu-2m-k}\left|\frac{d}{dt}\{t^{2m}\varphi(t)\}\right|dt$ *and* $\int_0^\infty t^{\nu+1}|\psi(t)|\,dt$ *are convergent,*

(iii) $\frac{d}{dt}[t^{2m}\varphi(t)]$ *is continuous.*

3. The proofs of these theorems are contained in the following lemmas.

LEMMA 1. *Let*

$$x^\nu\varphi(x) = A_m\int_0^\infty yJ_\nu(xy)f(y)\,dy\,, \tag{3.1}$$

and

$$x^{-\nu-k}\psi(x) = \int_0^\infty yJ_{\nu+2m+k}(xy)y^k f(y)\,dy\,; \tag{3.2}$$

then

$$\varphi(x) = \int_x^\infty u^{1-2k-2\nu}(u^2-x^2)^{k-1}\mathfrak{J}_m(\nu+k,\nu+1,x^2/u^2)\psi(u)\,du\,, \tag{3.3}$$

provided

(i) $\nu > -1$, $k > 0$ *and* m *is zero or a positive integer,*

(ii) $\int_0^1 t^{\nu+2m+2k+1}|f(t)|\,dt$ *and* $\int_1^\infty t^{k+1/2}|f(t)|\,dt$ *are convergent,*

(iii) $\int_0^\infty t^{1-\nu-k}|\psi(t)|\,dt$ *is convergent, where* $A_m = \dfrac{2^{k-1}\Gamma(\nu+1)\Gamma(m+k)}{\Gamma(1+\nu+m)}$.

Proof. If $\nu > -1$, and the conditions (i) and (ii) are satisfied, then by Hankel's inversion theorem ([1]), we have

$$y^k f(y) = \int_0^\infty uJ_{\nu+2m+k}(uy)u^{-\nu-k}\psi(u)\,du\,.$$

Substituting the value of $f(y)$ in (3.1), we have

$$\begin{aligned}
x^\nu\varphi(x) &= A_m\int_0^\infty y^{1-k}J_\nu(xy)\,dy\int_0^\infty uJ_{\nu+2m+k}(uy)u^{-\nu-k}\psi(u)\,du\\
&= A_m\int_0^\infty u^{1-\nu-k}\psi(u)\,du\int_0^\infty y^{1-k}J_\nu(xy)J_{\nu+2m+k}(uy)\,dy\\
&= A_m\int_0^\infty u^{-1-\nu}\psi(u)\,du\int_0^\infty t^{1-k}J_\nu(xt/u)J_{\nu+2m+k}(t)\,dt\,.
\end{aligned}$$

(3.3) is obtained by using (1.1). The change of the order of integration is justified under the conditions mentioned in the lemma.

LEMMA 2. *Let* $\varphi(x)$ *and* $\psi(x)$ *be defined as in* (3.1) *and* (3.2); *then*

(3.4) $$\psi(x) = \frac{2\Gamma(m+\nu+1)\Gamma(\nu+m+k)}{\Gamma(\nu)\Gamma(\nu+1)\Gamma(m+1)\Gamma(m+k)}\left[\int_0^x \mathfrak{J}_m(\nu+k,\nu,u^2/x^2)\,d[u^{2\nu}\varphi(u)]+\right.$$
$$\left.+\frac{\Gamma(\nu)\Gamma(m+1)}{\Gamma(\nu+2m+k+1)\Gamma(-k-m)}\int_x^\infty H_m(\nu+k,\nu,u^2/x^2)\,d[u^{2\nu}\varphi(u)]\right],$$

provided

(i) $\nu > -1$, $0 < k < 1$ *and* m *is a positive integer,*

(ii) $\int_0^1 t^{\nu+1}|f(t)|\,dt$ *and* $\int_1^\infty |t^{3/2}f(t)|\,dt$ *are convergent,*

(iii) $\int_0^\infty t^{1-\nu}\left|\frac{d}{dt}\{t^{2\nu}\varphi(t)\}\right|dt$ *is convergent,*

(iv) $\frac{d}{dt}\{t^{2\nu}\varphi(t)\}$ *is continuous.*

Proof. From (3.1) we have

$$\chi_1(x) = \frac{1}{x^\nu}\cdot\frac{d}{dx}\{x^{2\nu}\varphi(x)\} = A_m\int_0^\infty yJ_{\nu-1}(xy)yf(y)\,dy\,.$$

If the conditions under (ii) are satisfied, then by Hankel's inversion theorem we have

$$yf(y) = \frac{1}{A_m}\int_0^\infty uJ_{\nu-1}(uy)\chi_1(u)\,du\,.$$

Substituting the value of $yf(y)$ in (3.2), we have

$$\psi(x) = \frac{x^{\nu+k}}{A_m}\int_0^\infty y^kJ_{\nu+2m+k}(xy)\,dy\int_0^\infty u\chi_1(u)J_{\nu-1}(uy)\,du$$
$$= \frac{x^{\nu-1}}{A_m}\int_0^\infty u\chi_1(u)\,du\int_0^\infty t^kJ_{\nu+2m+k}(t)J_{\nu-1}\left(\frac{u}{x}t\right)dt\,.$$

(3.4) is obtained by using (1.2). The change of the order of integration is justified under the conditions mentioned in the lemma.

LEMMA 3. *Let*

(3.5) $$x^{-\nu-k}\varphi(x) = A'_m\int_0^\infty yJ_{\nu+2m+k}(xy)f(y)\,dy\,,$$

and

(3.6) $$x^\nu\psi(x) = \int_0^\infty yJ_\nu(xy)y^kf(y)\,dy\,;$$

then

$$\varphi(x) = \int_0^x u^{2\nu+1}(x^2-u^2)^{k-1}\mathfrak{J}_m(k+\nu, \nu+1, u^2/x^2)\psi(u)\,du\,, \tag{3.7}$$

provided

(i) $\nu > -1$, $k > 0$ *and* m *is zero or a positive integer,*

(ii) $\int_0^1 t^{\nu+k+1}|f(t)|\,dt$ *and* $\int_1^\infty t^{k+1/2}|f(t)|\,dt$ *are convergent,*

(iii) $\int_0^\infty t^{\nu+1}|\psi(t)|\,dt$ *is convergent, where* $A'_m = \dfrac{2^{k-1}\Gamma(\nu+1)\Gamma(m+k)}{\Gamma(m+\nu+1)}$.

LEMMA 4. *Let* $\varphi(x)$ *and* $\psi(x)$ *be defined as in* (3.5) *and* (3.6); *then*

$$\begin{aligned}\psi(x) &= \frac{2\Gamma(m+\nu+1)\Gamma(m+k+\nu)}{\{\Gamma(\nu+1)\}^2\Gamma(m)\Gamma(m+k)}\Bigg[\int_x^\infty \frac{\mathfrak{J}_{m-1}(\nu+k+1, \nu+1, x^2/u^2)}{u^{2(m+\nu+k)}}\,d\{u^{2m}\varphi(u)\} + \\ &+ \frac{\Gamma(\nu+1)\Gamma(m)}{\Gamma(\nu+2m+k+1)\Gamma(1-k-m)}\int_0^x \frac{H_{m-1}(\nu+k+1, \nu+1, x^2/u^2)}{u^{2(m+k+\nu)}}\,d\{u^{2m}\varphi(u)\}\Bigg],\end{aligned} \tag{3.8}$$

provided

(i) $\nu > -1$, $0 < k < 1$ *and* m *is a positive integer,*

(ii) $\int_0^1 t^{\nu+2m+k+1}|f(t)|\,dt$ *and* $\int_1^\infty t^{3/2}|f(t)|\,dt$ *are convergent,*

(iii) $\int_0^\infty t^{1-\nu-2m-k}\left|\dfrac{d}{dt}\{u^{2m}\psi(t)\}\right|dt$ *is convergent,*

(iv) $\dfrac{d}{dt}\{t^{2m}\psi(t)\}$ *is continuous.*

The proofs of these lemmas are similar to the proofs of the Lemmas 1 and 2. Thus our theorems are proved.

References

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