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ON THE NUMBER OF NON-ISOMORPHIC STEINER TRIPLES

BY

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1. Any family consisting of 3-element subsets of an m-element set, with the property that each pair of elements of the set is contained in one and only one triple of the family, is called a *system of Steiner triples* in the m-element set and will be denoted by $C(2,3,m)$. As is known since long [2], a system of Steiner triples does exist if and only if

$$m \equiv 1 \text{ or } 3 \ (\mathrm{mod}\, 6).$$

Let $N(m)$ be the number of non-isomorphic systems of Steiner triples in an m-element set. The problem of precise evaluation of the number $N(m)$ seems to be difficult (cf. [4]), but some particular results are known. For instance, $N(7) = N(9) = 1$, $N(13) = 2$, and, with the help of a computer, $N(15) = 80$. For $m > 15$ there is only estimation $N(m) > 2$. However, if $m \equiv 9 \ (\mathrm{mod}\ 18)$, there is also an estimation using the number of divisors p of m such that $p \equiv 1 \ (\mathrm{mod}\, 6)$, see [3]. Recently, Doyen proved [1] that

$$N(m) > 2^{\log_3(m/17)} \quad \text{for } m \geqslant 15.$$

Let $N_k(m)$ be the number of non-isomorphic systems of Steiner triples in an m-element set which contains some $C(2,3,k)$. Doyen has shown [1] that

$$N_7(m) \geqslant 2^{\log_3(m/17)} \quad \text{for } m \geqslant 15. \tag{1}$$

The aim of the present paper is to show that

$$N_{(m-1)/2}(m) \geqslant \frac{\left(\left[\frac{m+1}{16}\right] + \frac{m-15}{8} - 1\right)!}{\left[\frac{m+1}{16} - 1\right]!\, \frac{m-15}{8}!} \quad \text{for } m \equiv 7 \text{ or } 15 \ (\mathrm{mod}\ 24). \tag{2}$$

Since each of the systems, which we shall construct to prove (2), contains besides $C(2, 3, (m-1)/2)$ also some $C(2, 3, 7)$, estimation (2) yields

$$(3)\quad N_7(m) \geqslant \frac{\left(\left[\frac{m+1}{16}\right]+\frac{m-15}{8}-1\right)!}{\left[\frac{m+1}{16}-1\right]!\,\frac{m-15}{8}!} \quad \text{for } m \equiv 7 \text{ or } 15 \pmod{24}$$

which gives an improvent of estimation (1) for $m \equiv 7$ or $15 \pmod{24}$. One can, however, show that for each such m there exist systems $C(2, 3, m)$ which contain $C(2, 3, (m-1)/2)$ but do not contain $C(2, 3, 7)$, and which contain $C(2, 3, 7)$ but do not contain $C(2, 3, (m-1)/2)$.

Estimation given in (2) tends to infinity faster than m in any finite power.

2. Now we shall formulate some auxiliary notions and prove some lemmas.

Let $T = \{1, 2, 3, \ldots, n, n+1\}$, where n is odd. By P we denote the family of all 2-element subsets of T, and by a system of pairs of the set T we mean any family $A = \{A_a\}$, where $A_a \subset P$ and $a = 1, 2, \ldots, n$, for which the following conditions are satisfied:

1° $\bigcup_{a=1}^{n} A_a = P$.

2° Each A_a consists of $(n+1)/2$ pairs, i.e., $|A_a| = (n+1)/2$ for $a = 1, \ldots, n$.

3° $A_a \cap A_\beta = \emptyset$ for $a \neq \beta$.

4° For each $i \in T$ and each $a = 1, \ldots, n$ there exists $j \in T$ such that $(i, j) \in A_a$.

In the sequel each A_a will be called a *column* of A. Reiss proved [2] that for each n odd there exists at least one system of pairs of T.

A system of pairs of T will be called *normal* if it satisfies three additional conditions:

5° $A_1 = \{(1, 2), (3, 4), (5, 6), \ldots, (n, n+1)\}$.

6° All elements of A_{2a}, where $a = 1, \ldots, (n-1)/2$, are of the form $(2i-1, 2j-1)$ or $(2i, 2j)$, and $(2i-1, 2j-1) \in A_{2a}$ iff $(2i, 2j) \in A_{2a}$.

7° All elements of A_{2a+1}, where $a = 1, \ldots, (n-1)/2$, are of the form $(2i-1, 2j)$ or $(2j-1, 2i)$. Moreover, if $(2i-1, 2j) \in A_{2a+1}$, then $(2j-1, 2i) \in A_{2a+1}$, and if $(2i, 2j) \in A_{2a}$, then $(2i-1, 2j) \in A_{2a+1}$.

LEMMA 1. *For each $n \equiv 3 \pmod 4$ there exists a normal system of pairs of T.*

Proof. Let $A_1 = \{(1, 2), (3, 4), \ldots, (n, n+1)\}$. Since $|A_1| = (n+1)/2 \equiv 0 \pmod 2$, one can construct a system of pairs of A_1. Let it be $\bar{A}$

$= \{A_{1,a}\}$, $a = 1, 2, \ldots, (n-1)/2$. We proceed to define columns of a normal system. A column A_{2a_0} consists of pairs $(2i_0-1, 2j_0-1)$ and $(2i_0, 2j_0)$, and a column A_{2a_0+1} of pairs $(2i_0-1, 2j_0)$ and $(2j_0-1, 2i_0)$, where pair $\big((2i_0-1, 2i_0), (2j_0-1, 2j_0)\big)$ is from A_{1,a_0}.

A pair two elements of which are either both odd or both even will be called *even*, otherwise — *odd*.

Let $A = \{A_a\}$ be a normal system. Divide A into five classes, A_1, C, D, E and F, as follows:

$$A_1 = \{(1, 2), (3, 4), \ldots, (n, n+1)\},$$

$$C = \{A_{2\beta}\}, \quad D = \{A_{2\gamma}\}, \quad E = \{A_{2\beta+1}\}, \quad F = \{A_{2\gamma+1}\},$$

where

$$\beta = \begin{cases} 1, \ldots, \dfrac{n-3}{4} & \text{if } n \equiv 7 \pmod 8, \\[2ex] 1, \ldots, \dfrac{n-7}{4} & \text{if } n \equiv 3 \pmod 8, \end{cases}$$

and

$$\gamma = \begin{cases} \dfrac{n+1}{4}, \ldots, \dfrac{n-1}{2} & \text{if } n \equiv 7 \pmod 8, \\[2ex] \dfrac{n-3}{4}, \ldots, \dfrac{n-1}{2} & \text{if } n \equiv 3 \pmod 8. \end{cases}$$

Put $T = T_1 \cup T_2$, where $T_1 = \{1, 2, \ldots, k\}$, $T_2 = \{k+1, \ldots, n+1\}$ and

$$k = \begin{cases} \dfrac{n+1}{2} & \text{if } n \equiv 7 \pmod 8, \\[2ex] \dfrac{n-3}{2} & \text{if } n \equiv 3 \pmod 8. \end{cases}$$

A system A of pairs of T will be denoted by A_* if it satisfies, in addition to 1°-7°, also the following condition:

8° If $(i, j) \in A_{*a} \in C \cup E$, then either both i and j are in T_1 or none of them is; and if $(i, j) \in A_{*a} \in D \cup F$ and $i \in T_1$, then $j \in T_2$.

LEMMA 2. *For each* $n \equiv 7 \pmod 8$ *there exists a system* A_* *of pairs of* T.

Proof. Since $n+1 \equiv 0 \pmod 8$, there is $|T_1| = |T_2| \equiv 0 \pmod 4$ and one can construct a normal system $\bar{A}$ of pairs of T_1 and a normal system $\bar{\bar{A}}$ of pairs of T_2. In virtue of 8°, if $(i, j) \in A_{*a} \in C \cup E$ and $i \in T_2$, then $j \in T_2$. Thus we can put $\bar{A}_a \cup \bar{\bar{A}}_a = A_{*a}$, where $A_{*a} \in A_1 \cup C \cup E$

for each $a = 1, \ldots, k-1$. And since, in virtue of 8° and of the equality $|T_1| = |T_2|$, if $(i, j) \in A_{*a} \in D \cup F$ and $i \in T_2$, then $j \in T_1$, one can take D as consisting from the following A_{*a}'s:

$$A_{*(n+1)/2} = \left\{\left(1, \frac{n+3}{2}\right), \left(2, \frac{n+5}{2}\right), \ldots, \left(\frac{n+1}{2}, n+1\right)\right\},$$

$$A_{*(n+5)/2} = \left\{\left(1, \frac{n+7}{2}\right), \left(2, \frac{n+9}{2}\right), \ldots, \left(\frac{n+1}{2}, \frac{n+5}{2}\right)\right\},$$

$$A_{*(n+9)/2} = \left\{\left(1, \frac{n+11}{2}\right), \left(2, \frac{n+13}{2}\right), \ldots, \left(\frac{n+1}{2}, \frac{n+9}{2}\right)\right\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_{*n-1} = \left\{(1, n), (2, n+1), \ldots, \left(\frac{n+1}{2}, n-1\right)\right\}.$$

Columns A_{*a} belonging to F are obtained from those of D by the normality condition.

In this way the family $A_* = \{A_{*a}\}$ is the required system of T.

Example 1. System A_* for $n+1 = 8$.

$$T_1 = \{1, 2, 3, 4), \qquad T_2 = \{5, 6, 7, 8\},$$

$$A_* = \{A_{*1}, A_{*2}, A_{*3}, A_{*4}, A_{*5}, A_{*6}, A_{*7}\},$$

$$C = \{A_{*2}\}, \qquad E = \{A_{*3}\}, \qquad D = \{A_{*4}, A_{*6}\}, \qquad F = \{A_{*5}, A_{*7}\},$$

$$A_{*1} = \{(1, 2), (3, 4), (5, 6), (7, 8)\},$$

$$A_{*2} = \{(1, 3), (2, 4), (5, 7), (6, 8)\},$$

$$A_{*3} = \{(1, 4), (2, 3), (5, 8), (6, 7)\},$$

$$A_{*4} = \{(1, 5), (2, 6), (3, 7), (4, 8)\},$$

$$A_{*5} = \{(1, 6), (2, 5), (3, 8), (4, 7)\},$$

$$A_{*6} = \{(1, 7), (2, 8), (3, 5), (4, 6)\},$$

$$A_{*7} = \{(1, 8), (2, 7), (3, 6), (4, 5)\}.$$

LEMMA 2'. *For each $n \equiv 3 \pmod 8$ there exists a system A_* of pairs of T.*

Proof. In virtue of 8°, if $(i, j) \in A_{*a} \in C \cup E$ and $i \in T_2$, then $j \in T_2$. Each column A_{*a} of the class $D \cup F$ consists of n pairs (i, j). By virtue of 8° and the equality $|T_1| = |T_2| - 4$, if $i \in T_2$, then for $n-2$ pairs (i, j) we have $j \in T_1$, and for the remaining two $j \in T_2$.

Let $K = \{(n-1)/2, \quad (n+3)/2, \ldots, n\}$. Since $|K| = (n+5)/4 \equiv 0 \pmod 2$, we can construct a system A of pairs of K. Let the construction be that of Reiss [2]; denote it by $L = \{L_a\}$, $a = 1, \ldots, (n+1)/4$. The system L contains two columns L_{a_0} and L_{β_0} such that

$$L_{a_0} \cup L_{\beta_0} = \{(a_1, a_2), (a_2, a_3), (a_3, a_4), \ldots, (a_{(n+5)/4}, a_1)\},$$

where $a_1, \ldots, a_{(n+5)/4}$ is a permutation of K. We may assume that $a_0 = (n-3)/4$ and $\beta_0 = (n+1)/4$. Since $|T_1| \equiv 0 \pmod 4$, there exists a normal system $M = \{M_a\}$, $a = 1, \ldots, (n-5)/2$, of pairs of T_1.

Now, define a system $N = \{N_a\}$, $a = 1, \ldots, (n+3)/2$, of pairs of T_2 as follows:

$$N_1 = \left\{\left(\frac{n-1}{2}, \frac{n+1}{2}\right), \left(\frac{n+3}{2}, \frac{n+5}{2}\right), \ldots, (n, n+1)\right\},$$

and if $(i, j) \in L_a$, then $(i, j) \in N_{2a}$, $(i+1, j+1) \in N_{2a}$ and $(i, j+1) \in N_{2a+1}$, $(j, i+1) \in N_{2a+1}$. The system N is normal. Columns A_{*a} belonging to $A_{*1} \cup C \cup E$ we define by $A_{*a} = M_a \cup N_a$. Since $|N| - |M| = 4$, there remained columns $N_{(n-3)/2}, N_{(n-1)/2}, N_{(n+1)/2}, N_{(n+3)/2}$. In view of the construction of L and of N, it follows that

$$\bigcup_{i=(n-3)/2}^{i=(n+3)/2} N_i = \left\{\left(\frac{n-1}{2}, \frac{n+3}{2}\right), \left(\frac{n+3}{2}, \frac{n+7}{2}\right), \left(\frac{n+7}{2}, \frac{n+11}{2}\right), \ldots,\right.$$
$$\left(n, \frac{n-1}{2}\right), \left(\frac{n+1}{2}, \frac{n+5}{2}\right), \left(\frac{n+5}{2}, \frac{n+9}{2}\right), \left(\frac{n+9}{2}, \frac{n+13}{2}\right), \ldots,$$
$$\left(n+1, \frac{n+1}{2}\right), \left(\frac{n-1}{2}, \frac{n+5}{2}\right), \left(\frac{n+3}{2}, \frac{n+9}{2}\right), \left(\frac{n+7}{2}, \frac{n+13}{2}\right), \ldots,$$
$$\left(n, \frac{n+1}{2}\right), \left(\frac{n+1}{2}, \frac{n+3}{2}\right), \left(\frac{n+5}{2}, \frac{n+7}{2}\right), \left(\frac{n+9}{2}, \frac{n+11}{2}\right), \ldots,$$
$$\left.\left(n+1, \frac{n-1}{2}\right)\right\}.$$

Columns A_{*a} belonging to D we define as follows:

$$A_{*(n-3)/2} = \left\{\left(1, \frac{n-1}{2}\right), \left(2, \frac{n+1}{2}\right), \left(3, \frac{n+3}{2}\right), \left(4, \frac{n+5}{2}\right), \ldots,\right.$$
$$\left.\left(\frac{n-3}{2}, n-3\right); (n-2, n), (n-1, n+1)\right\},$$

$$A_{*(n+1)/2} = \left\{\left(1, \frac{n+3}{2}\right), \left(2, \frac{n+5}{2}\right), \left(3, \frac{n+7}{2}\right), \left(4, \frac{n+9}{2}\right), \ldots,\right.$$
$$\left.\left(\frac{n-3}{2}, n-1\right), \left(n, \frac{n-1}{2}\right), \left(n+1, \frac{n+1}{2}\right)\right\},$$

$$A_{*(n+5)/2} = \left\{\left(1, \frac{n+7}{2}\right), \left(2, \frac{n+9}{2}\right), \left(3, \frac{n+11}{2}\right), \left(4, \frac{n+13}{2}\right), \ldots, \right.$$
$$\left.\left(\frac{n-3}{2}, n+1\right), \left(\frac{n-1}{2}, \frac{n+3}{2}\right), \left(\frac{n+1}{2}, \frac{n+5}{2}\right)\right\},$$

. . . .

$$A_{*n-1} = \left\{(1, n), (2, n+1), \left(3, \frac{n-1}{2}\right), \left(4, \frac{n+1}{2}\right), \ldots, \right.$$
$$\left.\left(\frac{n-3}{2}, n-5\right), (n-4, n-2), (n-3, n-1)\right\}.$$

Columns A_{*a} belonging to F are obtained from those of E by the normality condition.

Definitions were chosen in a way to assert that all pairs from the set $\bigcup_{i=(n+3)/2}^{(n+1)/2} N_i$ be now in columns A_{*a} belonging to $D \cup F$. It is easy to check that $A_* = \{A_{*a}\}$ is a system of pairs of T. The proof is complete.

Example 2. We shall give an example of the just described construction in the case of $n = 11$. Here $T_1 = \{1, 2, 3, 4\}$, $T_2 = \{5, 6, 7, 8, 9, 10, 11, 12\}$, and the system A_* consists of the following 11 columns:

$$A_{*1} = \{(1, 2), (3, 4), (5, 6), (7, 8), (9, 10), (11, 12)\},$$
$$A_{*2} = \{(1, 3), (2, 4), (5, 9), (6, 10), (7, 11), (8, 12)\},$$
$$A_{*3} = \{(1, 4), (2, 3), (5, 10), (6, 9), (7, 12), (8, 11)\},$$
$$A_{*4} = \{(1, 5), (2, 6), (3, 7), (4, 8), (9, 11), (10, 12)\},$$
$$A_{*5} = \{(1, 6), (2, 5), (3, 8), (4, 7), (9, 12), (10, 11)\},$$
$$A_{*6} = \{(1, 7), (2, 8), (3, 9), (4, 10), (5, 11), (6, 12)\},$$
$$A_{*7} = \{(1, 8), (2, 7), (3, 10), (4, 9), (5, 12), (6, 11)\},$$
$$A_{*8} = \{(1, 9), (2, 10), (3, 11), (4, 12), (5, 7), (6, 8)\},$$
$$A_{*9} = \{(1, 10), (2, 9), (3, 12), (4, 11), (5, 8), (6, 7)\},$$
$$A_{*10} = \{(1, 11), (2, 12), (3, 5), (4, 6), (7, 9), (8, 10)\},$$
$$A_{*11} = \{(1, 12), (2, 11), (3, 6), (4, 5), (7, 10), (8, 9)\}.$$

These columns are divided into 5 classes A_1, C, D, E and F as follows:

$$A_1 = \{A_{*1}\}, \quad C = \{A_{*2}\}, \quad E = \{A_{*3}\}, \quad D = \{A_{*4}, A_{*6}, A_{*8}, A_{*10}\},$$
$$F = \{A_{*5}, A_{*7}, A_{*9}, A_{*11}\}.$$

A system A of pairs of T will be called *quasi-normal* with the parameters $(\beta_0, \ldots, \beta_{[(n+1)/8]})$ if it satisfies 5° and the two following conditions:

$6^{\circ\prime}$ There are precisely β_k columns $A_{2a} \in C$ which contain $2k$ odd pairs (i, j) with $(i, j) \in T_1$ and all remaining pairs of which are even.

$7^{\circ\prime}$ There are precisely β_k columns $A_{2a+1} \in E$ which contain $2k$ even pairs (i, j) with $i, j \in T_1$ and all remaining pairs of which are odd.

A quasi-normal system A with the parameters $(\beta_0, \ldots, \beta_{[(n+1)/8]})$ will be denoted by $A(\beta_0, \ldots, \beta_{[(n+1)/8]})$.

Similarily, as in the case of a normal system, we shall divide columns of a quasi-normal system into 5 classes A_1, C, D, E and F.

A quasi-normal system will be denoted by A_{**} if it satisfies also the following condition:

$8^{\circ\prime}$ If $(i, j) \in A_{**a} \in C \cup E$ and $i \in T_1$, then $j \in T_1$, and if $(i, j) \in A_{**a} \in D \cup F$ and $i \in T_1$, then $j \in T_2$.

It should be clear that each normal system is quasi-normal with the parameters $\big((n-3)/4, 0, \ldots, 0\big)$.

LEMMA 3. *For each sequence of natural numbers* $(\beta_0, \ldots, \beta_{[(n+1)/8]})$ *such that*

$$\sum_{j=0}^{(n+1)/8} \beta_j = \frac{n-3}{4} \quad \text{if } n \equiv 7 \pmod 8$$

or

$$\sum_{j=0}^{(n+1)/8} \beta_j = \frac{n-7}{4} \quad \text{if } n \equiv 3 \pmod 8,$$

there exists (at least one) a quasi-normal system $A_{**}(\beta_0, \ldots, \beta_{[(n+1)/8]})$.

Proof. Take a sequence $(\beta_0, \ldots, \beta_{[(n+1)/8]})$ satisfying the hypothesis. From Lemmas 1 and 2 or 2′ we infer that there exists a normal system of pairs in which columns A_{2a} consist of even pairs only, and columns A_{2a+1} of odd pairs only. The proof will be completed if we shall show that one can exchange some $2k$ even pairs from β_k columns A_{2a} for some $2k$ odd pairs from β_k columns A_{2a+1}. By the hypothesis that the sum $\sum \beta_j$ equals the number of columns in the class C, we can restrict ourselves to columns $A_{2a} \in C$. Let then be given a column $A_{2a} \in C$. The column contains pairs $(2i-1, 2j-1)$ and $(2i, 2j)$. By definitions of a normal system and of the class E, it follows that the column A_{2a+1} must contain pairs $(2i-1, 2j)$ and $(2j-1, 2i)$. Take pairs $(2i, 2j)$ and $(2i-1, 2j-1)$ from the column A_{2a} to the column A_{2a+1} and pairs $(2i-1, 2j)$ and $(2j-1, 2i)$ from the column A_{2a+1} to the column A_{2a}. We can do it because both pairs consist of the same elements. Now, to complete the proof, it suffices to repeat the procedure k times on various sets of pairs.

3. Let h be an arbitrary mapping from T into itself. We shall say that h is *even at element* i if i and $h(i)$ are either both even or both odd. Otherwise, h is said to be *odd at element* i.

LEMMA 4. *Let* $(\beta_0, \ldots, \beta_{[(n+1)/8]})$ *and* $(\beta_0', \ldots, \beta_{[(n+1)/8]}')$ *be two sequences of natural numbers satisfying hypothesis of Lemma* 3. *If, for some* i, $\beta_i \neq \beta_i'$, *then the systems* $A_{**}(\beta_0, \ldots, \beta_{[(n+1)/8]})$ *and* $A_{**}(\beta_0', \ldots, \beta_{[(n+1)/8]}')$ *are not isomorphic to each other.*

Proof. Assume $n \equiv 7 \pmod 8$. As follows from the construction of a quasi-normal system, if $(i,j) \in A_{**2a} \in D$, then (i,j) is even, and if $(i,j) \in A_{**2a+1} \in F$, then (i,j) is odd. Put

$$B = \{B_a\} = A_{**}(\beta_0, \ldots, \beta_{[(n+1)/8]}), \quad B' = \{B_a'\} = A_{**}(\beta_0', \ldots, \beta_{[(n+1)/8]}').$$

If B and B' were isomorphic, then there would exist a mapping $h\colon T \to T$ yielding isomorphism between B and B'. Since B differs from B' for the numbers of odd and even pairs $(i,j) \in B_a \in C \cup E$ such that $i,j \in T_1$, there must exist an element $i \in T_1$ such that h is odd at i.

Consider three cases:

I. For each i, if $h(i) \neq i$, then $i \in T_1$ and $h(i) \in T_1$.

II. For each i, if $h(i) \neq i$ and $i \in T_1$, then $h(i) \in T_1$, and if $i \in T_2$, then $h(i) \in T_2$.

III. There exist $i \in T_1$ and $j \in T_2$ such that $h(i) = j$ or $h(j) = i$.

Let the system B be divided into classes A_1, C, D, E, F, and the system B' into analogous classes A_1', C', D', E', F' (see p. 151).

Case I. In virtue of the construction of these classes, the isomorphism h can transfer columns from D into columns from D' and F' only, and columns from F into columns from F' and D' only. It follows that if h is odd at element i for some $i \in T_1$, then it is such for each $i \in T_1$. Consequently, the isomorphism h preserves numbers of odd and even pairs for each column from C. Since h moves only elements of T_1, columns from C go into columns from C', but the system B differs from the system B' for the numbers of odd and even pairs in some columns from C and C'. A contradiction.

Case II. In virtue of the construction of classes A_1, C, D, E, F and classes A_1', C', D', E', F', the isomorphism h can map columns from $C \cup A_1$ into columns from C' and E', and columns from E into columns from C' and E'. It follows that if h is odd for some $i \in T_2$, then either h is odd for each element from T_2 or it is such for precisely half of elements from T_2. In the first case, h must be for each element of T_1 either even or odd. Consequently, for each column from C the isomorphism keeps the number of odd pairs as well as of that of even pairs unchanged. Hence, columns from C must go onto columns from C', but this, as we have seen, is impossible.

In the second case, columns $B_a \in A_1 \cup C \cup E$ can go onto columns $B'_a \in A'_1 \cup C' \cup E'$. If $(i,j) \in B_a \in C \cup E \cup A_1$, where $i, j \in T_2$, and $h(i)$, $h(j)$ are odd at i and j, then for each pair (i,j) $(i,j \in T_2)$ either $h(i)$ and $h(j)$ are both odd at i and j or $h(i)$ and $h(j)$ are both even at i and j. If in $B_a \in C \cup E \cup A_1$ there exists a pair such that $i, j \in T_2$, $h(i)$ is odd at i, and $h(j)$ is even at j, then for each pair $(i,j) \in B_a \in C \cup E \cup A_1$, where $i, j \in T_2$, $h(i)$ is odd at i and $h(j)$ is even at j.

And now, if columns $B_a \in D \cup F$ can go onto columns $B'_a \in D' \cup F'$, then either $h(i)$ is odd at each $i \in T_1$ or $h(i)$ is even at each $i \in T_1$, or $h(i)$ is odd at $(n+1)/4$ i's belonging to T_1 and $h(i)$ is even at $(n+1)/4 i$'s belonging to T_1. The subcases of $h(i)$ being odd for all $i \in T_1$ or even for all $i \in T_1$ have been already considered in the first case.

Now, if $h(i)$ is odd at $(n+1)/4$ i's of T_1, and if in column $B_a \in D \cup F$ there exists a pair (i,j) such that $i \in T_1, j \in T_2$, and $h(i)$ and $h(j)$ are odd at i and j, then for $(n+1)/4$ pairs $(i,j) \in B_a$ $h(i)$ and $h(j)$ are odd at i and j, and for $(n+1)/4$ pairs $(i,j) \in B_a$ $h(i)$ and $h(j)$ are even at i and j. If in $B_a \in D \cup F$ there exists a pair (i,j) such that $h(i)$ is odd at $i \in T_1$ and $h(j)$ is even at $j \in T_2$, then for $(n+1)/4$ pairs $(i,j) \in B_a$ $h(i)$ is odd at $i \in T_1$ and $h(j)$ is even at $j \in T_2$, and for $(n+1)/4$ pairs $(i,j) \in B_a$ $h(i)$ is even at $i \in T_1$ and $h(j)$ is odd at $j \in T_2$. Hence, in virtue of the construction (see p. 152), it follows that if $h(i)$ is odd at $i \in T_1$, then either $h(k+i)$ is odd at $k+i$, and for $h(i)$ being even at $i \in T_1$ $h(k+i)$ is even at $k+i$ or, for $h(i)$ being odd at $i \in T_1$, $h(k+i)$ is even at $k+i$, and for $h(i)$ being even at $i \in T_1$ $h(k+i)$ is odd at $k+i$.

If systems of pairs $\bar{A}$ and A are isomorphic, and mapping $h\colon T_1 \to T_2$ yielding an isomorphism between $\bar{A}$ and $\bar{\bar{A}}$ is such that $h(i) = k+i$, then columns $B_a \in C \cup E \cup A_1$ could go onto columns $B'_a \in C' \cup E' \cup A'_1$, but the number of odd and even pairs in each column will not be changed. However, this is a contradiction because the numbers of odd and even pairs in these columns differ. The proof of case II is complete.

Case III. Consider two subcases:

(a) $C \cup E \cup A_1$ does not go onto $C' \cup E' \cup A'_1$,

(b) $C \cup E \cup A_1$ does go onto $C' \cup E' \cup A'_1$.

In the subcase (a), there exists a column $B_{a_1} \in C \cup E \cup A_1$ mapped onto a column $B'_{a_2} \in D' \cup F'$. It follows that both numbers, that of $i \in T_1$ such that $h(i) \in T_2$ and that of $i \in T_2$ such that $h(i) \in T_1$, are equal to $(n+1)/4$, and h at $i \in T_2$ such that $h(i) \in T_1$ is for every i odd or for every i even. Since the cardinality of the set $D \cup F$ is greater than that of the set $C \cup E \cup A_1$, there must exist a_3 such that a column $B_{a_3} \in D \cup F$ goes onto a column $B'_{a_4} \in D' \cup F'$. Hence, the number of $i \in T_1$ such that $h(i) \in T_2$ and h is at i odd is equal to the number of $i \in T_1$ such that $h(i) \in T_1$ and h is at this i is odd, and the number of $i \in T_1$ such that $h(i) \in T_1$ and h at

i is odd is equal to the number of $i \in T_2$ such that $h(i) \in T_2$ and h at this i is odd. Now, since the numbers of column in $C \cup E \cup A_1$ and $C' \cup E' \cup A_1'$ are equal to each other, so are the numbers of columns in $D \cup F$ and $D' \cup F'$, and since the column $B_{\alpha_1} \in C \cup E$ goes onto the column $B'_{\alpha_2} \in D' \cup F'$, there must exist a column $B_{\alpha_5} \in D \cup F$ which goes onto a column $B'_{\alpha_0} \in C' \cup E' \cup A_1'$. It follows that the number $i \in T_1$ such that $h(i) \in T_2$ is equal to the number $i \in T_2$ such that $h(i) \in T_2$ and $h(i)$ is odd at this i.

Now, we are able to show that h cannot map B onto B'. Let in C be β_{k_0} columns and in C' be β'_{k_0} columns, each of which contains $2k_0$ odd pairs. Suppose $\beta'_{k_0} > \beta_{k_0}$. Assume that by the mapping of B onto B' we have obtained β_{k_0} columns in C': $B'_{\alpha_1}, \ldots, B'_{\alpha_{\beta_{k_0}}}$. None was obtained from a column in $D \cup F$. In fact, the number of $i \in T_1$ such that $h(i) \in T_2$ equals $(n+1)/4$ and h is at each such i either odd or even, and the number of $i \in T_1$ such that $h(i) \in T_1$ and h is odd — equals the number of those $i \in T_2$ for which $h(i) \in T_2$ and h at this i is odd. But for a column from $D \cup F$ could go onto a column from C', it is necessary that the number of $i \in T_2$ such that $h(i) \in T_2$ and h at this i is odd, be either 0 or $(n+1)/4$. Hence, the number of $i \in T_1$ such that $h(i) \in T_1$ and h at this i is odd equals either 0 or $(n+1)/4$ and, therefore, a column from $D \cup F$ cannot go onto any of the columns $B'_{\alpha_1}, \ldots, B'_{\alpha_{\beta_{k_0}}}$ from C' unless each pair in it is odd. Thus columns $B'_{\alpha_1}, \ldots, B'_{\alpha_{\beta_0}}$ could be obtained only from columns in $C \cup E \cup A_1$.

But columns $B'_{\alpha_1}, \ldots, B'_{\alpha_{\beta_0}}$ could not be obtained from columns in $E \cup A_1$, for otherwise the number of $i \in T_2$ such that $h(i) \in T_2$ and h is odd at i would be equal to $(n+1)/8$, and we have shown that it equals either 0 or $(n+1)/4$.

Hence, columns $B'_{\alpha_1}, \ldots, B'_{\alpha_{\beta_0}}$ could be obtained only from columns in C. Since h is for each $i \in T_1$ such that $h(i) \in T_2$ all the time even or all the time odd, each pair $(i, j) \in B_{\alpha_0} \in C$ such that $i \in T_1$, $j \in T_1$, $h(i) \in T_2$, $h(j) \in T_2$, where B_{α_0} is a column mapped onto a column in B'_{α_0}, must be even. Hence, the number of odd and even pairs will not be changed and B'_{α_0} could not be obtained from columns in C. The proof of subcase (a) is complete.

Consider now subcase (b). If $h(i) \in T_2$ for some $i \in T_1$, then it must be for each $i \in T_1$. The same for $i \in T_2$ and $h(i) \in T_1$. Suppose that C goes onto C' and D onto D'. Since B is not normal, there exist $\alpha_0, \alpha_1, \alpha_2$ and i_0, i_1, i_2 such that $2i_0-1, 2i_1-1, 2i_2-1, 2i_0, 2i_1, 2i_2 \in T_1$ and $(2i_0-1, 2i_1)$, $(2i_1-1, 2i_0) \in B_{\alpha_0}$, $(2i_0-1, 2i_2-1)$, $(2i_0-1, 2i_2) \in B_{\alpha_1}$, $(2i_1-1, 2i_2-1)$, $(2i_1, 2i_2) \in B_{\alpha_2}$, and that all these pairs are mapped onto even pairs. In particular, pairs $(2i_0-1, 2i_1)$ and $(2i_1-1, 2i_0)$ are mapped so, and thus h must be even at two elements and odd at two elements of the four, for instance, at $2i_0-1$ and $2i_0$ odd, and at $2i_1-1$ and $2i_1$ even, but then

h at $2i_2$ and $2i_2-1$ must be simultaneously odd and even. Hence, there is no $i \in T_1$ such that $h(i) \in T_2$. A contradiction with the assumption of case III.

Thus the proof of the lemma is complete in the case $n \equiv 7 \pmod 8$. The proof for $n \equiv 3 \pmod 8$ is analogous.

LEMMA 5. *There are*

$$\frac{\left(\left[\frac{n+1}{8}\right]+\frac{n-7}{4}-1\right)!}{\left[\frac{n+1}{8}-1\right]!\,\frac{n-7}{4}!} \tag{4}$$

non-isomorphic systems of pairs in T.

Proof follows from Lemma 4 and from the well-known combinatorial formula $(r+p-1)!/p!\,(r-1)!$ giving the number of p-element subsets with repetition of r-element set.

4. Now we come to a construction of systems of Steiner triples, based on systems of pairs from 2 and 3.

Let $U = \{n+2, \ldots, 2n-1\}$. Construct on U a system of Steiner triples $V = C(2, 3, n)$ which has the property that V does not contain $C\big(2, 3\,(n-1)/2\big)$. Such a system V does exist for $n \equiv 1 \pmod 6$ (cf. [4]) or for $n \equiv 3 \pmod 6$ (cf. [3]). Let $A = \{A_a\}$, $a = 1, \ldots, n$ be a system of pairs for the set $T = \{1, \ldots, n+1\}$. To each pair $(i, j) \in A_a$ adjoin now the element $n+a+1$. Let Z denote the set of triples constructed in this way.

LEMMA 6. *$Z \cup V$ is a system $C(2, 3, 2n+1)$ of Steiner triples for the set $S = T \cup U$.*

For a proof, see [2].

Now, let be given two non-isomorphic systems B_1 and B_2 of pairs and let Z_1 and Z_2 be the sets of triples constructed for each of them just as Z above.

LEMMA 7. *$Z_1 \cup V$ and $Z_2 \cup V$ are not isomorphic.*

Proof. Assume to the contrary that $Z_1 \cup V$ and $Z_2 \cup V$ are isomorphic. Hence, there is a mapping $S \to S$ transforming $Z_1 \cup V$ onto $Z_2 \cup V$. Since the mapping cannot transform T onto T and U onto U, there are elements in T which go to elements of U. Thus V cannot be mapped onto itself. Now, since $Z_2 \cup V$ contains subsystem V, $Z_1 \cup V$ must contain a subsystem V^* isomorphic to V and such that $Z_1 \cap V^* \neq \emptyset$. But Z does not contain any subsystem, because no triple of elements of U belongs to it. Hence $V \cap V^* \neq \emptyset$. However, in such a case if $V \neq V^*$, then V must contain a subsystem on a set of cardinality $(n-1)/2$. Contradiction.

THEOREM. *For each $m \equiv 7$ or 15 (mod 24), inequality* (2) *holds.*

Proof. We have $m = 2n+1$. By virtue of Lemma 5, on an n-element set there are (4) non-isomorphic systems of pairs and, by Lemma 7, if two systems of pairs are non-isomorphic, then the related systems of triples are non-isomorphic. The theorem follows.

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