On linear differential equations with transformed argument solvable by means of right invertible operators

by D. PRZEWORSKA-ROLEWICZ (Warszawa)

Abstract. Two types of differential equations of order N with variable coefficients and with delayed argument are solved by reduction to an initial value problem for equations with right invertible operators. The first of them is one in which the coincidence conditions are of a special form, namely $x^{(k)}(t_0 + 0) = \varphi^{(k)}(t_0) = 0$ ($k = 0, 1, \ldots$ $\ldots, N - 1$), where φ is an initial function determined on the initial set E_{t_0}. The second one concerns a singular case, when E_{t_0} is one-point set, i.e. the case when it is impossible to apply the method "step by step". A corollary for equations with advanced argument is also given.

In the papers [3] and [5] theorems about existence and uniqueness of a solution of an initial value problem for equations with right invertible operators (obtained in the papers [2] and [4]) have been applied to obtain conditions of solvability of linear differential equations with argument transformed by means of an involution. The purpose of the present paper is to indicate other linear differential equations with transformed argument which can be solved using the solution of an initial value problem for equations with right invertible operators.

The following notions and results given in [4] will be applied in our subsequent considerations.

Let X be a linear space over field of real or complex scalars. Let A be a linear (i.e. additive and homogeneous) operator defined on a linear subset $\mathscr{D}_A$ of X, called the *domain* of A, and mapping $\mathscr{D}_A$ into X. The collection of all such operators will be denoted by $L(X)$. Denote by Z_A the *kernel* of an $A \in L(X)$ i.e. the set $Z_A = \{x \in \mathscr{D}_A : Ax = 0\}$.

DEFINITION 1. An operator $D \in L(X)$ is said to be *right invertible*, if there is an operator $R \in L(X)$ such that

$$\mathscr{D}_R = X, \quad RX \subset \mathscr{D}_D \quad \text{and} \quad DR = I,$$

where I denotes the identity operator.

The operator R is called *right inverse* of D. The collection of all right invertible operators belonging to $L(X)$ will be denoted by $R(X)$.

DEFINITION 2. An operator $F \epsilon L(X)$ is said to be *initial operator* for $D \epsilon R(X)$ corresponding to a right inverse R of D if

$$FX = Z_D, \quad F^2 = F \quad \text{and} \quad FR = 0.$$

The kernel Z_D of the operator $D \epsilon R(X)$ is said to be the *space of constants* for D. Write

$$(1) \qquad Q(D) = \sum_{k=0}^{N} Q_k D^k, \quad \text{where } D \epsilon R(X), Q_0, \ldots, Q_{N-1} \epsilon L(X),$$

$$Q_N = I.$$

An *initial value problem* for the operator $Q(D)$ defined by (1) is to find all solutions of the equation

$$(2) \qquad\qquad Q(D)x = y, \quad y \epsilon X$$

satisfying the initial condition

$$(3) \qquad FD^k x = y_k, \quad \text{where } y_k \epsilon Z_D \ (k = 0, 1, \ldots, N-1),$$

F is initial operator for D.

The initial value problem is said to be *well-posed*, if this problem has a unique solution for every $y \epsilon X, y_0, \ldots, y_{N-1} \epsilon Z_D$. This means that a well-posed homogeneous initial value problem has only zero as a solution.

THEOREM 1 (cf. Corollary 3.1 and Theorem 3.2 of [4]). *Let $D \epsilon R(X)$ and let F be an initial operator for D corresponding to a right inverse R of D. If the operator $I + \hat{Q}$ is invertible, where we write $\hat{Q} = \sum_{k=0}^{N-1} Q_k R^{N-k}$, then the initial value problem (2)-(3) is well-posed and its unique solution is*

$$(4) \qquad x = R^N (I+\hat{Q})^{-1}\left(y - \sum_{j=0}^{N-1} Q_j \sum_{k=j}^{N-1} R^{k-j} y_k \right) + \sum_{k=0}^{N-1} R^k y_k.$$

Now suppose that we are given a projection operator $P_0 \epsilon L(X)$ such that $P_0 D = D P_0$ if both superpositions $P_0 D$ and $D P_0$ are well defined. Write

$$(5) \qquad P_1 = I - P_0, \quad X_0 = P_0 X, \quad X_1 = P_1 X.$$

P_1 is obviously a projection operator. Observe that for an arbitrary $x \epsilon X$ and a positive integer N we have

(6) $P_0 D^k x = x_k$, where $x_k \epsilon X_0$, implies $x_k = D^k x_0$ $(k = 0, 1, \ldots, N-1)$.

Indeed, by definition we have $P_0 x = x_0$ and $x_k = P_0 D^k x = D^k P_0 x = D^k x_0$.

THEOREM 2. *If the operator* $I + P_1 \hat{Q}$ *(which maps the set* $D_{I+P_1\hat{Q}}$ *into* X_1) *is invertible (where* $\hat{Q}$ *is defined in Theorem* 1), *then the following problem*

(7)
$$P_1 Q(D) x = y, \quad y \epsilon X_1,$$

(8)
$$P_0 D^k x = D^k x_0, \quad x_0 \epsilon X_0 \cap \mathscr{D}_{D^{N-1}} \quad (k = 0, 1, \ldots, N-1)$$

with the conditions

(9)
$$F P_0 D^k x = F P_1 D^k x \quad (k = 0, 1, \ldots, N-1)$$

has the unique solution

(10)
$$x = x_0 + R^N (I + P_1 \hat{Q})^{-1} \Big(y - P_1 \sum_{m=0}^{N-1} Q_m D^m x_0 -$$
$$- \sum_{m=0}^{N-1} Q_m \sum_{k=m}^{N-1} R^{k-m} F D^k x_0 \Big) + \sum_{k=0}^{N-1} R^k F D^k x_0.$$

Proof. Observe that from Property (6) it follows that the imposed conditions (8) are of a sufficiently general form. Formulae (8) and (9) together imply that

(11)
$$F D^k P_1 x = F P_1 D^k x = F P_0 D^k x = F D^k x_0 \quad (k = 0, 1, \ldots, N-1).$$

By our assumption $P_1 D = (I - P_0)D = D(I - P_0) = DP_1$. Conditions (8) imply

$$P_1 Q(D) P_1 x$$

$$= P_1 \sum_{m=0}^{N-1} Q_m D^m P_1 x = P_1 D^N P_1 x + P_1 \sum_{m=0}^{N-1} Q_m D^m P_1 x$$

$$= P_1^2 D^N + P_1 \sum_{m=0}^{N-1} Q_m D^m (I - P_0) x$$

$$= P_1 D^N x + P_1 \sum_{m=0}^{N-1} Q_m D^m x - P_1 \sum_{m=0}^{N-1} Q_m D^m P_0 x$$

$$= P_1 D^N x + P_1 \sum_{m=0}^{N-1} Q_m D^m x - P_1 \sum_{m=0}^{N-1} Q_m P_0 D^m x$$

$$= P_1 Q(D) x - P_1 \sum_{m=0}^{N-1} Q_m D^m x_0$$

$$= y - P_1 \sum_{m=0}^{N-1} Q_m D^m x_0.$$

Write: $\tilde{x} = P_1 x$, $\tilde{y} = y - P_1 \sum\limits_{m=0}^{N-1} Q_m D^m x_0$, $\tilde{Q}_m = P_1 Q_m$ $(m = 0, 1, \ldots$

$\ldots, N-1)$, $\tilde{Q}_N = I$, $\tilde{Q}(D) = \sum\limits_{m=0}^{N} \tilde{Q}_m D^m$. Using these notations we obtain the equation

(12) $\tilde{Q}(D)\tilde{x} = \tilde{y}$, where $\tilde{x}, \tilde{y} \in X_1$,

together with the initial conditions

(13) $FD^k \tilde{x} = FD^k x_0$ $(k = 0, 1, \ldots, N-1)$.

These last conditions follow immediately from Formulae (11). Observe that the operator

$$I + \sum_{m=0}^{N-1} \tilde{Q}_m R^{N-m} = I + \sum_{m=0}^{N-1} P_1 Q_m R^{N-m} = I + P_1 \sum_{m=0}^{N-1} Q_m R^{N-m} = I + P_1 \hat{Q}$$

is invertible by our assumption in the space X_1. Thus Theorem 1 implies that the initial value problem (12)–(13) has the unique solution

$$\tilde{x} = R^N (I + P_1 \hat{Q})^{-1} \Big(\tilde{y} - \sum_{m=0}^{N-1} Q_m \sum_{k=m}^{N-1} R^{k-m} FD^k x_0 \Big) + \sum_{k=0}^{N-1} R^k FD^k x_0.$$

Hence problem (7)–(8)–(9) has a unique solution of the form:

$$x = P_0 x + P_1 x = x_0 + \tilde{x}$$

$$= x_0 + R^N (I + P_1 \hat{Q})^{-1} \Big(y - P_1 \sum_{m=0}^{N-1} Q_m D^m x_0 - \sum_{m=0}^{N-1} Q_m R^{k-m} FD^k x_0 \Big) +$$

$$+ \sum_{k=0}^{N-1} R^k FD^k x_0,$$

which was to be proved.

Now consider the following differential equation with *delayed argument*:

(14) $\quad x^{(N)}(t) + \sum\limits_{k=0}^{N-1} \sum\limits_{j=0}^{M} a_{kj}(t) x^{(k)}\big(h_j(t)\big) = y(t)$, $\quad t_0 < t \leqslant T$,

where

(15) $\quad h_j(t) \leqslant h_0(t) \equiv t$ $\quad (j = 1, 2, \ldots, M)$ for $t_0 \leqslant t \leqslant T$,

with the initial conditions

(16) $x^{(k)}(t) = \varphi^{(k)}(t)$ on the initial set E_{t_0} [1] $(k = 0, 1, \ldots, N-1)$, and

(17) $\quad x^{(k)}(t_0 + 0) = \varphi^{(k)}(t_0) = 0$ $\quad (k = 0, 1, \ldots, N = 1)$.

[1] I.e. on the set $E_{t_0} = \{t_0\} \cap \{h_1(t), \ldots, h_M(t) : h_j(t) \leqslant t_0$ for $t_0 \leqslant t \leqslant T$ and $j = 1, 2, \ldots, M\}$.

Write

(18) $$X = \{x \in C(E_{t_0} \cup [t_0, T]): x(t_0) = 0\},$$

$$P_0 x = x|_{E_{t_0}}, \qquad P_1 x = x|_{[t_0, T]}, \qquad X_0 = P_0 X, \qquad X_1 = P_1 X.$$

The projection operators P_0 and P_1 are well defined on the space X. We shall assume that

(19) $$y, a_{kj} \in X_1, \qquad h_j \in C[t_0, T] \qquad (k = 0, 1, \ldots, N-1; \; j = 0, 1, \ldots, M),$$

$$\varphi, \varphi', \ldots, \varphi^{(N-1)} \in X_0.$$

These assumptions immediately imply that

(20) $$y(t_0) = 0, \qquad a_{kj}(t_0) = 0, \qquad \varphi^{(k)}(t_0) = 0 \qquad (k = 0, 1, \ldots, N-1;$$

$$j = 0, 1, \ldots, M).$$

We also write

(21) $$(Q_k x)(t) = \begin{cases} \displaystyle\sum_{j=0}^{M} a_{kj}(t) x(h_j(t)) & \text{for } t_0 \leqslant t \leqslant T \\ 0 & \text{for } t \in E_{t_0} \end{cases} \qquad (k = 0, 1, \ldots, N-1),$$

$$(Q_N x)(t) = \begin{cases} x(t) & \text{for } t_0 \leqslant t \leqslant T, \\ 0 & \text{for } t \in E_{t_0}; \end{cases}$$

(22) $$D = \frac{d}{dt}, \qquad (Rx)(t) = \int_{t_0}^{t} x(s)\,ds, \qquad (Fx)(t) = x(t_0) = 0 \quad \text{for all } x \in X,$$

$$\hat{Q} = \sum_{k=0}^{N-1} Q_k R^{N-k}.$$

By definition, $P_0 Q_k = 0$, hence

(23) $$P_1 Q_k = Q_k \quad (k = 0, 1, \ldots, N-1) \quad \text{and} \quad I + P_1 \hat{Q} = I + \hat{Q}.$$

By usual estimations we prove that the operator $I + \hat{Q}$ is invertible in the space X_1. Thus we obtain the following

THEOREM 3. *If conditions* (19) *are satisfied, then the initial value problem* (14)–(16)–(17) *has the unique solution*

(24) $$x = \varphi + R^N (I + \hat{Q})^{-1} y$$

in the space X_1, *where* R *and* $\hat{Q}$ *are defined by Formulae* (20), (21).

Proof. Observe that the space X, the operators $D, R, F, Q_0, \ldots, Q_N$,
$Q(D) = \sum_{k=0}^{N} Q_k D^k, \hat{Q}, P_0, P_1$ satisfy all conditions of Theorem 2. Put $x_0 = \varphi$.
By definition of the initial function φ and of the operators Q_k (Formulae
(20)) we have $\sum_{m=0}^{N-1} Q_m D^m x_0 = \sum_{m=0}^{N-1} Q_m D^m \varphi = 0$. Since $(FD^k x_0)(t) = (FD^k \varphi)(t)$
$= \varphi^{(k)}(t_0) = 0$ for $k = 0, 1, \ldots, N-1$, Theorem 2 implies that our problem
has a unique solution which is of the required form.

By an obvious change of variables we obtain

THEOREM 4. *The differential equation with advanced argument*

$$x^{(N)}(t) + \sum_{k=0}^{N-1} \sum_{j=0}^{M} a_{kj}(t) x^{(k)}(h_j(t)) = y(t), \quad T \leqslant t < t_0,$$

where $t \equiv h_0(t) \leqslant h_j(t)$ $(j = 1, \ldots, M)$ *for* $T \leqslant t \leqslant t_0$. *with the conditions*

$$x^{(k)}(t) = \varphi^{(k)}(t) \quad on \ the \ set \ \hat{E}_{t_0} \quad (k = 0, 1, \ldots, N-1),$$

defined by $E_{t_0} = \{t_0\} \cup \{h_1(t), \ldots, h_M(t): h_j(t) \geqslant t_0 for \ T \leqslant t \leqslant t_0 \ (j = 1, 2, \ldots, \\ \ldots, M)\}$,

$$x^{(k)}(t_0 - 0) = \varphi^{(k)}(t_0) = 0 \quad (k = 0, 1, \ldots, N-1),$$

has the unique solution

$$x = \varphi + (-1)R^N (I + \tilde{Q})^{-1} y$$

in the space $\hat{X} = \{x \in C([T, t_0] \cup E_{t_0}): x(t_0) = 0, where$

$1°$ $y, a_{kj} \in \hat{X}_1, \quad h_j \in C[T, T_0] \quad (k = 0, 1, \ldots, N-1; \ j = 0, 1, \ldots, M),$
$\varphi, \varphi', \ldots, \varphi^{N-1} \in \hat{X}_0,$

$2°$ $\hat{X}_0 = \hat{P}_0 \hat{X}, \quad \hat{X}_1 = \hat{P}_1 \hat{X}, \quad \hat{P}_0 x = x|_{\hat{E}_{t_0}}, \quad \hat{P}_1 x = x|_{[T, t_0]} \quad for \quad x \in \hat{X},$
D, R, F *are defined as before*,

$3°$ $(\tilde{Q}_k x)(t) = \begin{cases} \sum_{j=0}^{M} a_{kj}(t) x(h_j(t)) & for \ T \leqslant t \leqslant t_0 \\ 0 & for \ t \in \hat{E}_{t_0} \end{cases}$ $(k = 0, 1, \ldots, N-1),$

$\tilde{Q}_N x = X$ *for* $T \leqslant t \leqslant t_0$ *and* 0 *for* $t \in \hat{E}_{t_0}$ *and* $\tilde{Q} = \sum_{k=0}^{N-1} \tilde{Q}_k R^{N-k}.$

Consider now a differential-difference equation with *delayed argument*:

$$(25) \qquad x^{(N)}(t) + \sum_{k=0}^{N-1} \sum_{j=0}^{M_k} a_{kj}(t) x^{(k)}(t - h_{kj}(t)) = y(t) \quad for \ t \geqslant 0,$$

where $h_{kj}(t)$ are given continuous increasing functions defined for $t \geqslant 0$ and such that $h_{kj}(t) \leqslant t$ for $t > 0$, $h_{kj}(0) = 0$, $h_{k0}(t) \equiv 0$ $(j = 0, 1, \ldots$ $\ldots, M_k; \ k = 0, 1, \ldots, N-1)$.

Suppose that the given functions belong to the space X of all piecewise continuous functions defined for $t \geqslant 0$.

Since $t - h_{kj}(t) \geqslant 0$ for all $t \geqslant 0$ $(j = 0, 1, \ldots, M_k; \ k = 0, 1, \ldots, N-1)$, we conclude that in our case the initial set E_0 contains only the point 0. Since no initial function is given, we only can assume that the values of the unknown function and its derivatives up to and including the order $N-1$ are given at 0:

(26) $x^{(k)}(0) = x_k$,   where x_k are arbitrarily fixed constants

$(k = 0, 1, \ldots, N-1)$.

Put

(27) $D = \dfrac{d}{dt}$, $(Rx)(t) = \displaystyle\int\limits_0^t x(s)\,ds$, $(Fx)(t) = x(0)$ for $x \in X$

and

(28) $(A_k x)(t)$

$= \displaystyle\sum_{j=0}^{M_k} a_{kj}(t) x\big(t - h_{kj}(t)\big)$ $(k = 0, 1, \ldots, N-1)$, $Q_N = I$ for $x \in X$,

$$Q(D) = \sum_{k=0}^{N} Q_k D^k, \qquad \hat{Q} = \sum_{k=0}^{N-1} Q_k R^{N-k}.$$

By similar estimations, as those used in Theorem 3, we conclude, that the operator $I + \hat{Q}$ is invertible in the space X. Thus Theorem 1 implies immediately the following

THEOREM 5. *If the given functions satisfy the above conditions, then the initial value problem* (25)–(26) *has the unique solution*

$$x = R^N (I + \hat{Q})^{-1}\Big(y - \sum_{m=0}^{N-1} Q_m \sum_{k=m}^{N-1} R^{k-m} x_k \Big) + \sum_{k=0}^{N-1} R^k x_k.$$

For instance, this theorem can be applied to the equation

$$x^{(N)}(t) + \sum_{k=0}^{N-1} \sum_{j=1}^{M} a_{kj}(t) x^{(k)}\Big(\frac{j}{M} t \Big) = y(t) \quad \text{for } t \geqslant 0.$$

All the results obtained here are true for vector valued functions (if the coefficients are square matrices satisfying the above conditions) and for some partial differential operators (compare with [4]).

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