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Kato–Protter type inequalities, bounded vectors and the exponential function

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Abstract. We show how a densely defined operator in a Hilbert space can generate an exponential function whose values are unbounded operators.

Introduction. The differentiation operator. Let $\hat{f}$ be an entire function of the exponential order ω such that

$$\hat{f}(z) = (2\pi)^{-1/2} \int_{-\omega}^{\omega} f(t)e^{-itz}dy \tag{1}$$

for $f \in \mathscr{L}^2(\boldsymbol{R})$. Denote by $\mathscr{PW}_\omega$ the Paley–Wiener space of restrictions of $\hat{f}$ to the real axis considered as a linear subspace of $\mathscr{L}^2(\boldsymbol{R})$. The Plancherel theorem gives us that each $\mathscr{PW}_\omega$ is a closed subspace of $\mathscr{L}^2(\boldsymbol{R})$ and that the set

$$\mathscr{PW} = \bigcup_{\omega > 0} \mathscr{PW}_\omega$$

is a dense linear subspace of $\mathscr{L}^2(\boldsymbol{R})$.

Consider the differentiation operator

$$A = id/dx,$$

with domain $\mathscr{D}(A) = \mathscr{PW}$. Then A is essentially selfadjoint. Moreover, A generates a group $U(t) = e^{itA}$, $t \in \boldsymbol{R}$, of unitary operators in $\mathscr{L}^2(\boldsymbol{R})$, which is nothing but the group of translation operators in $\mathscr{L}^2(\boldsymbol{R})$. It is an everlasting temptation to *extend* this group, or rather this exponential function, from $\boldsymbol{R}$ to all the complex plane $\boldsymbol{C}$. Though this is not possible within the class of bounded operators, the Paley–Wiener theorem makes it possible within the class of unbounded operators in $\mathscr{L}^2(\boldsymbol{R})$. More precisely, due to formula (1), translations of functions in $\mathscr{PW}$ can be considered in the complex domain. This is the simplest example of the notion of the *exponential function of unbounded operators*, one of the topics of this paper.

The Schwarz inequality yields, for $f \in \mathcal{PW}$,

$$\|Af\|^2 \leqslant \|f\| \|A^2 f\|$$

which is nothing but the simplest version of the classical Hardy–Littlewood inequality ([1], see also [2]) which may be viewed as prototype of the *Kato–Protter inequality.*

Finally, the Plancherel Theorem implies

$$\|A^k \hat{f}\| = \left\| i^k \frac{d^k}{dx^k} \hat{f} \right\| = \|x^k f|_{[-\omega,\omega]}\| \leqslant (\omega)^k \|f|_{[-\omega,\omega]}\| = (\omega)^k \|\hat{f}\|,$$

where $\hat{f} \in \mathcal{PW}_\omega$ for some ω. Since $\mathcal{PW}_\omega$ is invariant for the differentiation operator A, this inequality implies that A, when restricted to $\mathcal{PW}_\omega$, is a bounded operator. This justifies the fact that we can think of elements of $\mathcal{PW}$ as *bounded vectors* of the operator A.

These are the other two of our topics. The aim of this paper is to discuss the interplay between these topics.

Inequalities of Kato–Protter type. Suppose $\mathcal{H}$ is a complex Hilbert space. Let A be a densely defined operator in $\mathcal{H}$ and $\mathcal{D}(A)$ its domain. The inequality in question asserts, in its simplest form, that for $f \in \mathcal{D}(A)^2$

$$\|Af\|^2 \leqslant \|f\| \, \|A^2 f\|. \tag{2}$$

The repeated use of (2) leads us to

PROPOSITION 1. *Suppose A satisfies* (2) *for any $f \in \mathcal{D}(A^2)$. If $f \in \mathcal{D}^\infty(A) = \bigcap_{n=0}^{\infty} \mathcal{D}(A^n)$, then*

$$\|Af\| \leqslant \|f\|^{n/(n+1)} \|A^{n+1} f\|^{1/(n+1)}, \quad n = 1, 2, \ldots \tag{3}$$

Proof. First we prove that

$$\|A^n f\| \leqslant \|f\|^{1/(n+1)} \|A^{n+1} f\|^{n/(n+1)}, \quad n = 1, 2, \ldots \tag{4}$$

If $n = 1$ this is precisely (2). Assume (4) for $n-1$. Then, by (2),

$$\|A^n f\|^2 = \|AA^{n-1} f\|^2 \leqslant \|A^{n-1} f\| \|A^2 A^{n-1} f\| \leqslant \|f\|^{1/n} \|A^n f\|^{(n-1)/n} \|A^{n+1} f\|.$$

Dividing both sides by $\|A^n f\|^{(n-1)/n}$, we get (4) for n.

Now pass to the proof of (3). If $n = 1$ this is again (2). Assume (3) for $n-1$. Then, by (4),

$$\|Af\| \leqslant \|f\|^{(n-1)/n} \|A^n f\|^{1/n} \leqslant \|f\|^{(n-1)/n} \{\|f\|^{1/(n+1)} \|A^{n+1} f\|^{n/(n+1)}\}^{1/n}$$
$$= \|f\|^{n/(n+1)} \|A^{n+1} f\|^{1/(n+1)}. \quad \blacksquare$$

More general form of the inequality in question is

$$\|Af\| \leqslant c_n \|f\|^{n/(n+1)} \|A^{n+1} f\|^{1/(n+1)}, \quad n = 1, 2, \ldots \tag{5}$$

We are interested in the case where

(6) $$\lim_{n\to\infty} c_n = c < +\infty .$$

This happens in the following situation:

PROPOSITION 2 (Protter [5]). *Suppose* $f \in \mathscr{D}(A^{n+1})$ *and*

$$(-1)^j \operatorname{Re}\langle A^j f, f\rangle \geqslant 0, \quad j = 1, 2, \ldots, n.$$

Then (5) *holds true with*

$$c_n^2 = (n+1)n^{-n/(n+1)} \to 1 \quad \text{as } n \to +\infty .$$

The case $n = 1$ has been proved by Kato [4].

We wish to present one more example of a class of operators for which inequality (2) holds. A densely defined operator A in a Hilbert space is said to be *hyponormal* if

$$\mathscr{D}(A) \subset \mathscr{D}(A^*) \quad \text{and} \quad \|A^* f\| \leqslant \|Af\| \quad \text{for} \quad f \in \mathscr{D}(A).$$

Notice that hyponormality is equivalent to the inequality

$$\sum_{i,j} \langle A^i f_j, A^j f_i\rangle \geqslant 0, \quad f_0, f_1 \text{ in } \mathscr{D}(A).$$

(Indeed, if A is hyponormal, then the left-hand side of the inequality above dominates $\|f_0 + A^* f_1\|^2$. Conversely, the quadratic form argument applied to the above inequality leads to

$$|\langle f_0, A^* f_1\rangle|^2 = |\langle A f_0, f_1\rangle|^2 \leqslant \|f_0\|^2 \|A f_1\|^2$$

and this implies hyponormality.)

The class of hyponormal operators contains symmetric, formally normal and subnormal ones.

PROPOSITION 3. *A hyponormal operator* A *satisfies inequality* (2) *for* $f \in \mathscr{D}(A^2)$. *Consequently, for* $f \in \mathscr{D}^\infty(A)$ *it satisfies inequality* (3).

Proof. Since $f \in \mathscr{D}(A^2)$, $Af \in \mathscr{D}(A^*)$ (and, of course, $f \in \mathscr{D}(A)$). Thus we have

$$\|Af\|^2 = \langle A^* Af, f\rangle \leqslant \|A^* Af\| \|f\| \leqslant \|A^2 f\| \|f\| . \quad \blacksquare$$

Notice that if A satisfies (2) for some f, then so does αA for any complex α.

Bounded vectors. Let A be a densely defined operator in $\mathscr{H}$. For $a > 0$ define

$$\mathscr{B}_a(A) = \{f \in \mathscr{D}^\infty(A) \text{ for some } b > 0,\ \|A^n f\| \leqslant b a^n,\ n = 1, 2, \ldots\}.$$

The family of linear subspaces $\{\mathscr{B}_a(A)\}_{a>0}$ is increasing in a. Put

$$\mathscr{B}(A) = \bigcup_{a>0} \mathscr{B}_a(A)$$

and call members of $\mathscr{B}(A)$ *bounded vectors* of A. The linear subspaces $\mathscr{B}_a(A)$, $a > 0$ as well as $\mathscr{B}(A)$ are invariant for A. Also $\mathscr{B}_a(\alpha A) = \mathscr{B}_{a/|\alpha|}(A)$ for complex α.

The following observation is crucial in our considerations.

PROPOSITION 4. *Let $\mathscr{H}$ be a Hilbert space and A a densely defined operator in it. Suppose $f \in \mathscr{B}_a(A)$ and satisfies* (5) *and* (6). *Then*

$$\|Af\| \leqslant ca\|f\|.$$

Proof. Since $f \in \mathscr{D}^{\infty}(A)$, we can use (5). Due to (6), after passing to infinity with n, we get the conclusion. ∎

Proposition 4 says in other words that if the vectors of $\mathscr{B}_a(A)$ satisfy (5) and (6), then the restriction of A to $\mathscr{B}_a(A)$ becomes a bounded operator.

The typical example of an operator having a rich enough collection of bounded vectors is a normal one. Indeed,

if N is an unbounded normal operator in a Hilbert space $\mathscr{H}$ and E its spectral measure, then vectors of the form $E(\sigma)f, f \in \mathscr{D}(N)$ and σ is a bounded Borel subset of $\mathbf{C}$, belong to $\mathscr{B}(N)$ and $\mathscr{B}(N)$ is dense in $\mathscr{H}$. These vectors form a core for N.

The exponential function of bounded type and its generator. Let $\mathscr{H}$ be, as considered so far, a Hilbert space and $\boldsymbol{B}(\mathscr{H})$ the algebra of all bounded linear operators acting on it. Let $\{T(t)\}_{t \geqslant 0}$ be a semigroup of class C_0. Then the following facts concerning the semigroup $\{T(t)\}_{t \geqslant 0}$ are known, cf. for instance [6], Theorem 13.36, to be equivalent:

(a) *the domain of its generator A is equal to $\mathscr{H}$,*

(b) *the semigroup is continuous in the operator norm topology,*

(c) *$A \in \boldsymbol{B}(\mathscr{H})$ and $T(t) = e^{tA}$.*

As a consequence (of (c)), the exponential function

$$z \to e^{zA} = I + \frac{zA}{1!} + \frac{z^2A^2}{2!} + \ldots, \qquad z \in \mathbf{C},$$

which uniquely extends the semigroup $\{T(t)\}_{t \geqslant 0}$ to the whole complex plane $\mathbf{C}$, can be defined and their values belong to $\boldsymbol{B}(\mathscr{H})$. We call this function the *exponential function of bounded operators.*

As an immediate consequence of the definition we have

$$\|e^{zA}\| \leqslant e^{|z|\,\|A\|} \quad \text{and} \quad \|e^{(z+h)A} - e^{zA}\| \leqslant e^{|z|\,\|A\|}(e^{|h|\,\|A\|} - 1), \qquad z, h \in \mathbf{C}. \tag{7}$$

The Paley–Wiener pattern suggests to define the exponential function, whose values are unbounded operators in $\mathscr{H}$, as follows. Suppose we are given linear operators $e(z)$: $\mathscr{D} \to \mathscr{H}$, $z \in \mathbf{C}$, where $\mathscr{H}$ is a Hilbert space and $\mathscr{D}$ is its dense linear subspace. Then $\{e(z)\}_{z\in C}$ is said to be an *exponential function of bounded type on* $\mathscr{D}$ if there is a nested family $\{\mathscr{H}_\omega\}_{\omega>0}$ of closed subspaces of $\mathscr{H}$ such that

(A_1) $\mathscr{D} = \bigcup_{\omega>0} \mathscr{H}_\omega$,

(A_2) *for each* $\omega\in\Omega$ *and each* $z\in \boldsymbol{C}$, $e(z)\mathscr{H}_\omega \subset \mathscr{H}_\omega$,

(A_3) *the family* $\{e(z)|_{\mathscr{H}_\omega}\}_{z\in C}$, *for each* $\omega > 0$, *is the exponential function of bounded operators on* $\mathscr{H}_\omega$.

In other words, an exponential function of bounded type is an operator valued function defined on the complex plane, which can be *filled up* by a nested family of exponential functions of *bounded operators*. This justifies the name we have chosen for it. (The filling-up procedure has been exploited in [7] under somewhat different circumstances of a single subnormal operator.)

We can think of a generator of an exponential function of bounded type. Define it as usual by

$$\mathscr{D}(A) = \{f\in\mathscr{H}:\ \exists\ g\in\mathscr{H},\ \|((e(z)f-f)/z)-g\| \to 0 \text{ as } z\to 0\},$$
$$Af = \lim_{z\to 0}(e(z)f-f)/z.$$

Call A the *generator* of the exponential function $\{e(z)\}_{z\in C}$. Some classical features of such a generator can be obtained from [3]. Since for each $\omega > 0$, $\{e(z)|_{\mathscr{H}_\omega}\}_{z\in C}$, when restricted to $z\in \boldsymbol{R}$, is a C_0 semigroup, we can define the generator A_ω of $\{e(z)|_{\mathscr{H}_\omega}\}_{z\in R}$ in the classical way. Because of analyticity of $\langle e(z)f, f\rangle$ in z and due to (b)$\Leftrightarrow$(c), $A_\omega f = \lim_{z\to 0}(e(z)f-f)/z$ for $f\in\mathscr{H}_\omega$. Since the family $\{e(z)|_{\mathscr{H}_\omega}\}_{z\in C}$, $\omega > 0$, is nested, so is the family $\{A_\omega\}_{\omega>0}$ and, consequently, the operator A defined on $\bigcup_{\omega>0}\mathscr{H}_\omega$ by $Af = A_\omega f$, if $f\in\mathscr{H}_\omega$ is the desired generator of $\{e(z)\}_{z\in C}$ (in particular $\bigcup_{\omega>0}\mathscr{H}_\omega = \mathscr{D}(A)$).

Characterization of the exponential function of bounded type. Suppose now we are given a function $z\to e(z)$, $z\in \boldsymbol{C}$; $e(z)$ being an unbounded closable operator in $\mathscr{H}$ and $\mathscr{D}(e(z))$ its domain. Define

$$\mathscr{D} = \bigcap_{z,u\in C} \mathscr{D}(e(z)e(u))$$

and assume

(M) $\mathscr{D}$ *is dense in* $\mathscr{H}$, $e(0)f = f$ *and* $e(z)e(u)f = e(z+u)f$, $z, u\in \boldsymbol{C}$, $f\in\mathscr{D}$.

The second of inequalities (7) suggests to consider, for $f\in\mathscr{D}$ and $\omega > 0$,

$$M_\omega(f) = \sup\{e^{-\omega|z|}(e^{\omega|h|}-1)^{-1}\|e(z+h)f-e(z)f\|;\ z, h\in \boldsymbol{C},\ h\neq 0\};$$

define

$$\mathscr{D}_\omega = \{f; M_\omega(f) < \infty\}.$$

It is easy to check that $\mathscr{D}_\omega$ is a linear space, $\mathscr{D}_\omega \subset \mathscr{D}_{\omega'}$ if $\omega \leqslant \omega'$ and

$$M_\omega(e(u)f) \leqslant e^{\omega|u|} M_\omega(f), \quad u \in \mathbf{C}.$$

This means that the linear space $\mathscr{D}_\omega$ is invariant for $e(u)$. Moreover, it is plain that

$$\|e(h)f - f\| \leqslant (e^{\omega|h|} - 1) M_\omega(f). \tag{8}$$

Assume

(B_1) *for any* $\omega > 0$ *there is a linear subspace* $\tilde{\mathscr{D}}_\omega$ *of* $\mathscr{D}_\omega$ *such that*

$$M_\omega(f) \leqslant \|f\|, \quad f \in \tilde{\mathscr{D}}_\omega,$$

which is invariant for $e(z)$, $z \in \mathbf{C}$.

Then, by (8), the operator $e(u)|_{\tilde{\mathscr{D}}_\omega}$ is bounded and

$$\|e(u)|_{\tilde{\mathscr{D}}_\omega}\| \leqslant e^{\omega|u|}.$$

Finally, putting all these spaces together we have to know that

(B_2) $\mathscr{D} = \bigcup_{\omega > 0} \tilde{\mathscr{D}}_\omega$.

Besides this consider another set of conditions. Define for $f \in \mathscr{D}$ and $\omega > 0$

$$N_\omega(f) = \sup\{e^{-\omega|z|}\|e(z)f\|;\ z \in \mathbf{C}\}$$

and

$$\mathscr{E}_\omega = \{f;\ N_\omega(f) < \infty\}.$$

It is easy to check that $\mathscr{E}_\omega$ is a linear space, $\mathscr{E}_\omega \subset \mathscr{E}_{\omega'}$ if $\omega \leqslant \omega'$ and

$$N_\omega(e(u)f) \leqslant e^{\omega|u|} N_\omega(f), \quad u \in \mathbf{C}.$$

This means that the linear space $\mathscr{E}_\omega$ is invariant for $e(u)$. Moreover,

$$\|e(z)f\| \leqslant e^{\omega|z|} N_\omega(f). \tag{9}$$

Assume

(C_1) *for any* $\omega > 0$ *there is a linear subspace* $\tilde{\mathscr{E}}_\omega$ *of* $\mathscr{E}_\omega$ *such that*

$$N_\omega(f) \leqslant \|f\|, \quad f \in \tilde{\mathscr{E}}_\omega,$$

which is invariant for $e(z)$, $z \in \mathbf{C}$.

Then, by (9), the operator $e(z)|_{\tilde{\mathscr{E}}_\omega}$ is bounded and

$$\|e(z)|_{\tilde{\mathscr{E}}_\omega}\| \leqslant e^{\omega|z|}.$$

So $e(z)_{\tilde{\mathscr{E}}_\omega}$ extends uniquely to a bounded operator on $\mathscr{K}_\omega = \tilde{\mathscr{E}}_\omega^-$, which, because $e(z)$ is closable, is nothing else than $e(z)^-|_{\mathscr{K}_\omega}$. Thus assuming

(C_2) $\mathscr{D} = \bigcup_{\omega > 0} \tilde{\mathscr{E}}_\omega^-$,

we also suppose that

(C_3) *for any* ω *and any* $f\in\tilde{\mathscr{H}}_\omega$ *there is* $g\in\mathscr{H}$ *such that* $\|((e(z)^- f - f)/z) - g\| \to 0$ *as* $z\to 0$ *and* $\|g\| \leqslant \omega\|f\|$.

The following statement (as well as Example below and Corollary 2) sheds some light on assumption (C_1).

PROPOSITION 5. *Suppose we are given, in a Hilbert space* $\mathscr{H}$, *a family* $e = \{e(z)\}_{z\in C}$ *of linear closable operators satisfying* (M). *If each operator* $e(z)$, $z\in \boldsymbol{C}$, *satisfies* (3) *for* $f\in\mathscr{D}$(1), *that is if*

$$\|e(z)f\| \leqslant \|f\|^{1-1/n}\|e(z)^n f\|^{1/n}, \quad n = 2, 3, \ldots, z\in \boldsymbol{C}, \tag{10}$$

then $N_\omega(f) = \|f\|$ *for* $f\in\mathscr{E}_\omega$ *and for any* $\omega > 0$.

Proof. The only thing which requires some proof is the last part of the conclusion. Suppose for $f\in\mathscr{D}$ (10) holds true. Since $e(z)^n f = e(nz)f$, fixing f in some $\mathscr{E}_\omega$, and using inequality (9), we get, for $n = 2, 3, \ldots$, $z\in\boldsymbol{C}$,

$$e^{-\omega|z|}\|e(z)f\| \leqslant e^{-\omega|z|}\|f\|^{1-1/n}\|e(nz)f\|^{1/n} = \|f\|^{1-1/n}N_\omega(f)^{1/n}.$$

The limit passage, because z is arbitrary, gives us $N_\omega(f) \leqslant \|f\|$. Thus $\tilde{\mathscr{E}}_\omega = \mathscr{E}_\omega$.

Since always $\|f\| \leqslant N_\omega(f)$, the equality $N_\omega(f) = \|f\|$ follows immediately for $f\in\mathscr{E}_\omega$.

The characterization (a)–(c) leads us directly, after little technical work, to the following

PROPOSITION 6. *Suppose we are given, in a Hilbert space* $\mathscr{H}$, *a family* $e = \{e(z)\}_{z\in C}$ *of linear closable operators satisfying* (M). *Then e satisfies* (B_1)–(B_2) *if and only if it does* (C_1)–(C_3). *If the above happens, then one can choose* $\tilde{\mathscr{D}}_\omega = \tilde{\mathscr{E}}_\omega$, $\omega > 0$.

The equivalence (b)$\Leftrightarrow$(c) relates the aforesaid to exponential functions of bounded type as follows.

PROPOSITION 7. *Suppose we are given, in a Hilbert space* $\mathscr{H}$, *a family* $e = \{e(z)\}_{z\in C}$ *of linear closable operators satisfying* (M). *If e satisfies either* (B_1)–(B_2) *or, equivalently,* (C_1)–(C_3) *with some* $\tilde{\mathscr{D}}_\omega = \tilde{\mathscr{E}}_\omega$, $\omega > 0$ *and both families* $\{\tilde{\mathscr{D}}_\omega\}_\omega$, $\{\tilde{\mathscr{E}}_\omega\}_\omega$ *being nested, then* $e^- = \{e(z)^-|_{\mathscr{D}_-}\}_{z\in C}$ *is an exponential function of bounded type on* $\mathscr{D}_- = \bigcup_{\omega>0}\tilde{\mathscr{D}}_\omega^- = \bigcup_{\omega>0}\tilde{\mathscr{E}}_\omega^-$.

Conversely, suppose $\{e(z)\}_{z\in C}$ *is an exponential function of bounded type. Then it satisfies* (B_1)–(B_2) *or, equivalently,* (C_1)–(C_3) *with a suitably chosen* $\tilde{\mathscr{D}}_\omega = \tilde{\mathscr{E}}_\omega$, $\omega > 0$ *and both families* $\{\tilde{\mathscr{D}}_\omega\}_\omega$, $\{\tilde{\mathscr{E}}_\omega\}_\omega$ *being nested.*

Proof. Suppose $\{e(z)\}_{z\in C}$ satisfies (B_1)–(B_2) and the operators $e(z)$, $z\in\boldsymbol{C}$, are closable. Take $\mathscr{H}_\omega = \mathscr{D}_\omega^-$. We show that $\mathscr{H}_\omega \subset \mathscr{D}_-$. Suppose $f_n \to f$, $f_n\in\mathscr{D}_\omega$. Since $e(h)$ is a bounded operator on $\mathscr{D}_\omega$ for any $h\in\boldsymbol{C}$,

(1) Cf. Corollary 2 at the end of the paper.

each of the sequences $\{e(u)f_n\}$ and $\{e(z+u)f_n\}$ is a Cauchy one. Moreover, since $e(z)e(u)f_n = e(z+u)f_n$ and the operators $e(h)$, $h \in C$, are closable, we get that f belongs to $\mathscr{D}_- \subset \bigcap_{z,u\in C} \mathscr{D}(e(z)^- e(u)^-)$ and $e(z)^- e(u)^- f = e(z+u)^- f$. Moreover, $e(z)^- \mathscr{H}_\omega \subset \mathscr{H}_\omega$, $\forall z \in C^-$.

On the other hand, the equivalence (b)$\Leftrightarrow$(c) leads us immediately to the conclusion that the nested family $\{\mathscr{H}_\omega\}_{\omega>0}$ as well as $\{e(z)^-|_{\mathscr{H}_\omega}\}_{z\in C}$ is what is required for e to be an exponential function of bounded type.

Now suppose $\{e(z)\}_{z\in C}$ is an exponential function of bounded type on $\mathscr{D}$ being dense in $\mathscr{H}$. Take $f \in \mathscr{D}$. Since $f \in \mathscr{H}_\omega$ for some $\omega > 0$ again invoking the equivalence (b)$\Leftrightarrow$(c), we get, via inequalities (7), $M_{\|A_\omega\|}(f) \leqslant \|f\|$ as well as $N_{\|A_\omega\|}(f) \leqslant \|f\|$, where A_ω is the (only) operator such that $e(z)|_{\mathscr{H}_\omega} = e^{zA_\omega}$.

Now we want to define both $\tilde{\mathscr{D}}_\alpha$ and $\tilde{\mathscr{E}}_\alpha$. First suppose $\sup\{\|A_\omega\|: \omega > 0\} < +\infty$. Then set $\tilde{\mathscr{D}}_\alpha = \tilde{\mathscr{E}}_\alpha = \{0\}$ for $0 < \alpha \leqslant \sup\{\|A_\omega\|: \omega > 0\}$ and $\tilde{\mathscr{D}}_\alpha = \tilde{\mathscr{E}}_\alpha = \mathscr{D}$ for $\alpha > \sup\{\|A_\omega\|: \omega > 0\}$. Now suppose $\sup\{\|A_\omega\|: \omega > 0\} = +\infty$. Then we can find a sequence $\{\omega_n\}_n$ increasing to $+\infty$ such that $\|A_{\omega_n}\| < \|A_{\omega_{n+1}}\|$ $\forall n$. Set $\tilde{\mathscr{D}}_\alpha = \tilde{\mathscr{E}}_\alpha = \{0\}$ for $0 < \alpha < \|A_{\omega_0}\|$ and $\tilde{\mathscr{D}}_\alpha = \tilde{\mathscr{E}}_\alpha = \mathscr{H}_{\omega_0}$ for $\|A_{\omega_n}\| \leqslant \alpha < \|A_{\omega_{n+1}}\|$.

Then we conclude that $\mathscr{D} = \bigcup_{\omega>0} \tilde{\mathscr{D}}_\omega = \bigcup_{\omega>0} \tilde{\mathscr{E}}_\omega$. This means that $\{e(z)\}_{z\in C}$ satisfies (B_1)-(B_2) or, equivalently, (C_1)-(C_3). ∎

Example. Let A be an operator such that $\mathscr{D} = \mathscr{D}(A) = \mathscr{D}(A^2)$, $A^2 f = 0$ for $f \in \mathscr{D}$ and there is a net of closed subspaces $\mathscr{H}_\omega$, $\omega > 0$, of $\mathscr{H}$ and bounded operators A_ω in $\mathscr{H}_\omega$, $\omega > 0$, such that $Af = A_\omega f$ if $f \in \mathscr{H}_\omega$ (such an operator can be realized, for instance as the direct sum of bounded nilpotents of norms tending to infinity). Then e^{zA} is an exponential function of bounded type, $\mathscr{D}_\omega = \mathscr{E}_\omega = \mathscr{D}$ for any ω, however, there is no $\tilde{\mathscr{D}}_\omega$ such that $\tilde{\mathscr{D}}_\omega = \mathscr{D}_\omega$ whatever ω is.

An exponential function of bounded type, when restricted to nonnegative z's becomes a semigroup of unbounded operators as considered in [3]. So the theory developed in [3] is applicable here. On the other hand, one may want to know when such a semigroup, and – in particular – a classical C_0 semigroup, is extendible to an exponential function of bounded type. The following statement, which is a semigroup version of the preceding one, gives the answer.

Proposition 8. *Suppose $\{T(t)\}_{t>0}$ is a family of unbounded closable operators in $\mathscr{H}$. Define*

$$\mathscr{D} = \bigcap_{s,t>0} \mathscr{D}(T(s)T(t)).$$

Suppose $\mathscr{D}$ is dense in $\mathscr{H}$, $T(0)f = f$ and $T(s)T(t)f = T(s+t)f$, $s, t > 0, f \in \mathscr{D}$.

Then $\{T(t)\}_{t>0}$ *extends to an exponential function of bounded type on* $\mathscr{D}_- = \bigcup_{\omega>0} \tilde{\mathscr{D}}_\omega^- = \bigcup_{\omega>0} \tilde{\mathscr{E}}_\omega^-$ *if either* (B$_1$)–(B$_2$) *hold true with*

$$M_\omega(f) = \sup\{e^{-\omega t}(e^{\omega s}-1)^{-1}\| T(t+s)f - T(t)f\|;\ s, t > 0\},$$

or (C$_1$)–(C$_3$) *hold true with*

$$N_\omega(f) = \sup\{e^{-\omega t}\|T(t)f\|;\ t > 0\},$$

for some $\tilde{\mathscr{D}}_\omega = \tilde{\mathscr{E}}_\omega$, $\omega > 0$ *and both families* $\{\tilde{\mathscr{D}}_\omega\}_\omega$, $\{\tilde{\mathscr{E}}_\omega\}_\omega$ *being nested.*

Generating exponential functions of bounded type. Here we wish to focus our interest on the question of when an unbounded operator A generates an exponential function of bounded type.

THEOREM. *Let A be a closed densely defined operator in a Hilbert space $\mathscr{H}$. Suppose that for any $f \in \mathscr{D}^\infty(A)$ conditions* (5) *and* (6) *hold true. Suppose, moreover, that $\mathscr{B}(A)$ is dense in $\mathscr{H}$. Then for any $f \in \mathscr{B}(A)$*

$$e^{zA}f = f + \frac{z}{1!}Af + \frac{z^2}{2!}A^2 f + \ldots, \quad z \in \boldsymbol{C}, \tag{11}$$

defines an exponential function of bounded type on $\mathscr{D} = \bigcup_{\omega>0} \mathscr{B}_\omega(A)$.

Proof. Take $f \in \mathscr{B}(A)$. Since $f \in \mathscr{B}_a(A)$ for some $a > 0$, Proposition 4 gives us $\|Af\| \leqslant ca\|f\|$. This means, since $\mathscr{B}_a(A)$ is invariant for A, that the operator A restricted to $\mathscr{B}_a(A)$ is bounded (with the bound ca). To fit it in with our previous notation set $\mathscr{H}_\alpha = \mathscr{B}_a(A)^-$ where $\alpha = ca$. Since A is closed and the restriction of A to $\mathscr{B}_a(A)$ is bounded, $\mathscr{H}_\alpha \subset \mathscr{D}(A)$ and the closure of the restriction of A to $\mathscr{B}_a(A)$ is equal to the restriction of A to $\mathscr{H}_\alpha$. Thus, the series in (11) converges for $f \in \mathscr{H}_\alpha$ to, say, $e_a(z)f \in \mathscr{H}_\alpha$ and the operator $e(z)$: $f \to e(z)f$, $f \in \mathscr{H}_\alpha$ is bounded (with norm $\leqslant e^{ca|z|}$) in $\mathscr{H}_\alpha$. Letting a vary over $(0, +\infty)$ we get the required exponential function of bounded type on $\mathscr{B}(A)$. ∎

Theorem and Propositions 2 and 3 give us at once

COROLLARY 1. *Suppose A is an unbounded closed operator with $\mathscr{B}(A)$ being dense. If A is either hyponormal or satisfies*

$$(-1)^j \operatorname{Re}\langle A^j f, f\rangle \geqslant 0, \quad j = 1, 2, \ldots, f \in \mathscr{B}(A),$$

then $\{e^{zA}\}_{z\in C}$ *defined by* (11) *for* $f \in \mathscr{B}(A)$ *is an exponential function of bounded type on* $\mathscr{D} = \bigcup_{\omega>0} \mathscr{B}_\omega(A)^-$.

Because any power of a subnormal operator is again subnormal, so is the exponential function it generates. Thus we get, via Proposition 3, immediately the following (cf. Proposition 5)

Corollary 2. *Suppose A is an unbounded closed subnormal operator with $\mathscr{B}(A)$ being dense. Then each operator e^{zA}, $z \in C$, satisfies* (10) *for f in* $\mathscr{D} = \bigcup_{\omega > 0} \mathscr{B}_{\omega}(A)$.

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