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ON THE SEQUENCE OF FACILITY INSTALLATION

1. Consider a system of n elements, one of which is working and all other ones being in reserve. The reserve elements are switched over to work one by one, in a fixed sequence, at the moment of breakdown of the actual working element. The change-over from reserve into working order takes place immediately, and the elements which have failed to work are never repaired. The whole system is said to be in breakdown when all its elements have already broken down.

An element is called in cold reserve if it does not break down while in reserve, and it is called in hot reserve if the breakdown intensity is the same both in working order and in reserve. It may happen in practice that reserve elements have a different breakdown intensity than working elements. If the reserve breakdown intensity is smaller than the working order breakdown intensity then such a reserve is called a warm one.

The sequence of installing elements into work is inessential both in cold and in hot reserves. It appears, however, to be essential in a warm reserve, provided the elements are not identical.

This note discusses two criteria of sequencing elements for work; first, the maximization of the system reliability in the initial working period, and second, the maximization of the mean working time of the system. Numerical examples show that even under such strong assumptions as exponential distributions of working times those criteria are not equivalent. In the third part of the paper a method of forming subsystems of pairs of elements which would assure the maximum mean working time of all elements is presented.

2. Let us numerate the elements of the system in the sequence of their installation for work. For mathematical convenience we shall assume hat if an element underlies a breakdown while in reserve then its workingt time will be zero, regardless of whether it was really installed or not.

Let t_k, $k = 1, 2, \ldots, n$, denote the moment of the installation of the k-th element, and let $F_k(x)$ be the cumulative distribution function of this random variable. Thus, $F_n(x)$ represents the sought distribution of the working time of a system composed of n elements. We shall assume in the sequel that 1° the breakdown probability of element no. k, while being in reserve, equals $1-\exp(-\lambda_k x)$ in a time interval of length x, 2° the length of time an element is in reserve does not influence its working time, 3° the breakdown probability of element no. k, while working, equals $1-\exp(-\Lambda_k x)$ in a time interval of length x, 4° the elements are statistically independent.

The distribution functions $F_k(x)$ satisfy the recurrent equation system (see [2], p. 317)

$$
\begin{aligned}
F_1(x) &= 1-\exp(-\Lambda_1 x),\\
F_{k+1}(x) &= \int_0^x \left[1-\exp\left(-\lambda_{k+1}u-\Lambda_{k+1}(x-u)\right)\right]dF_k(x),
\end{aligned}
\tag{1}
$$

$$k = 1, 2, \ldots, n-1.$$

Introducing Laplace-Stieltjes transforms

$$f_k^*(s) = \int_0^\infty e^{-sx}dF_k(x)$$

we obtain from (1)

$$
\begin{aligned}
f_1^*(s) &= \frac{\Lambda_1}{s+\Lambda_1},\\
f_{k+1}^*(s) &= f_k^*(s)-\frac{s}{s+\Lambda_{k+1}}f_k^*(s+\lambda_{k+1}), \qquad k = 1, 2, \ldots, n-1.
\end{aligned}
\tag{2}
$$

Since

$$T_k = Et_k = \lim_{s\to 0}[1-f_k^*(s)]/s,$$

hence we have

$$
\begin{aligned}
T_1 &= \frac{1}{\Lambda_1},\\
T_{k+1} &= T_k+\frac{1}{\Lambda_{k+1}}f_k^*(\lambda_{k+1}), \qquad k = 1, 2, \ldots, n-1.
\end{aligned}
\tag{3}
$$

To determine the optimum sequence of element installation the following criterion is used in practice (see [2], p. 319).

Criterion 1. The sequence of element installation is called optimal if it maximizes the system reliability in the initial working period of the system.

Speaking more precisely, the following formula is proved in [2]

$$(4)\qquad F_n(x) = \Lambda_1(\Lambda_2+\lambda_2)(\Lambda_3+2\lambda_3)\dots\big(\Lambda_n+(n-1)\lambda_n\big)x^n/n!+o(x^n);$$

then it is demanded that the first term of the right hand side of (4) be as small as possible. An easy proof shows that a system of elements satisfies criterion 1 if and only if the following inequalities are satisfied

$$(5)\qquad \frac{\lambda_1}{\Lambda_1}\geqslant\frac{\lambda_2}{\Lambda_2}\geqslant\dots\geqslant\frac{\lambda_n}{\Lambda_n}.$$

Consider now

CRITERION 2. The sequence of element installation is called optimal if it maximizes the mean working time of the system.

THEOREM 1. *The necessary condition for a system to be optimum in the sense of criterion 2 is that it satisfies the following inequality*

$$(6)\qquad \frac{\lambda_{n-1}}{\Lambda_{n-1}}\geqslant\frac{\lambda_n}{\Lambda_n}\,\frac{\lambda_{n-1}+\Lambda_n}{\Lambda_{n-1}+\lambda_n}.$$

Proof. From (3) follows

$$T_n = T_{n-1}+\frac{1}{\Lambda_n}f^*_{n-1}(\lambda_n) = T_{n-2}+\frac{1}{\Lambda_{n-1}}f^*_{n-2}(\lambda_{n-1})+$$

$$+\frac{1}{\Lambda_n}\left[f^*_{n-2}(\lambda_n)-\frac{\lambda_n}{\lambda_n+\Lambda_{n-1}}f^*_{n-2}(\lambda_{n-1}+\lambda_n)\right].$$

An exchange of the order of the two last elements in the system results in getting a new system the mean working time of which equals

$$T'_n = T_{n-2}+\frac{1}{\Lambda_n}f^*_{n-2}(\lambda_n)+\frac{1}{\Lambda_{n-1}}\left[f^*_{n-2}(\lambda_{n-1})-\frac{\lambda_{n-1}}{\lambda_{n-1}+\Lambda_n}f^*_{n-2}(\lambda_{n-1}+\lambda_n)\right].$$

The condition $T_n\geqslant T'_n$ implies the necessity of (6), and this ends the proof of the theorem.

For $n = 2$ condition (6) is also a sufficient condition for a system to be optimum in the sense of criterion 2.

We shall show now by example that criteria 1 and 2 are not equivalent. Take two element A and B with the following reserve and working order breakdown intensities: $\lambda_A = 1$, $\Lambda_A = 3$, $\lambda_B = 10$, $\Lambda_B = 60$. From

(4) we have (the upper indices in parentheses indicate the sequence of installation):

$$F_2^{(AB)}(x) \cong \Lambda_A(\Lambda_B + \lambda_B)x^2/2 = 105x^2,$$

$$F_2^{(BA)}(x) \cong \Lambda_B(\Lambda_A + \lambda_A)x^2/2 = 120x^2.$$

From (3) we obtain

$$T_2^{(AB)} = \frac{1}{\Lambda_A} + \frac{\Lambda_A}{\Lambda_B(\lambda_B + \Lambda_A)} = 0{,}3371, \tag{7}$$

$$T_2^{(BA)} = \frac{1}{\Lambda_B} + \frac{\Lambda_B}{\Lambda_A(\lambda_A + \Lambda_B)} = 0{,}3446.$$

The above shows that an installation of element A before element B is better in the sense of criterion 1, and the opposite order of installation is better in the sense of criterion 2.

In practice criterion 1 is used while designing systems with great reliability. There are situations where the system does not have to work till its breakdown but only for a short time period and where a breakdowu of the system means the failure of a whole project. Criterion 2 is better in situations where the system works till it breaks down and where the damaged systems are exchanged for new ones. From the point of view of renewal theory [1] the maximum mean working time of the system is then better.

3. Consider a set of $4n$ elements consisting of two types of elements in the number of $2n$ each. These elements are grouped into subsystems of pairs consisting of a main and a reserve element each, the reserve being a warm one. Assume that a subsystem works till its breakdown, at which moment it is exchanged for a new one. The reserve subsystems are in a cold reserve.

We shall now answer the question how to match elements in order that the mean working time of all subsystems formed from the elements of the given set be maximum. Without loss of generality we may restrict ourselves to a set of 4 elements consisting of 2 elements of each type I and II. Two strategies of matching are possible: 1° pairs consisting of identical elements (identical pairs), 2° mixed pairs, optimally ordered in the sense of criterion 2.

THEOREM 2. *If the reserve and working order breakdown intensities satisfy the inequalities $\lambda_{\mathrm{I}} < \Lambda_{\mathrm{I}}$ and $\lambda_{\mathrm{II}} < \Lambda_{\mathrm{II}}$ then mixed pairs are not less effective then identical ones.*

Proof. Assume that $\lambda_{\mathrm{I}} = a$, $\Lambda_{\mathrm{I}} = A$, $\lambda_{\mathrm{I}_\lambda} = b$, $\Lambda_{\mathrm{II}} = B$, and also, that in a mixed pair the element with parameters (a, A) precedes the element with parameters (b, B). Thus, the following inequality is satisfied

$$\frac{a}{A} \geqslant \frac{b}{B} \cdot \frac{a+B}{A+b}. \tag{8}$$

The mean working time of an optimum mixed pair is given by the right-hand side of (7). To prove the theorem it is sufficient to show that

$$2\left[\frac{1}{A} + \frac{A}{B(b+A)}\right] \geqslant \frac{1}{A} + \frac{1}{a+A} + \frac{1}{B} + \frac{1}{b+B}. \tag{9}$$

Assuming (without loss of generality) that $A = 1$ and transfering all terms of (9) to the left side we obtain

$$W(a, b, B) = 1 + \frac{2}{B(b+1)} - \frac{1}{1+a} - \frac{1}{B} - \frac{1}{b+B}.$$

From (8) we have

$$a \geqslant a^* = \frac{Bb}{Bb+B-b},$$

hence $W(a, b, B) \geqslant W(a^*, b, B)$. Putting now $B = b+x$ with $b > 0$, $x > 0$ we get after some simple transformations

$$\frac{1}{b} W(a^*, b, B) = \frac{(b+1)x^3 + (4b^2-2)x^2 + (b-1)(5b^2-1)x + 2b^2(b-1)^2}{(b+1)(b+x)(2b+x)(2b^2+2bx+x)}.$$

The nominator of this fraction is positive for $b > 0$ and $x > 0$, and the numerator has a local minimum equal to zero in the point $x = 1-b$. Thus, for $x \geqslant 0$ the minimum value of the numerator is zero for $0 < b \leqslant 1$ and $2b^2(1-b)^2 > 0$ for $b > 1$. This ends the proof of the theorem.

References

[1] D. R. Cox, *Renewal theory*, J. Wiley, New York 1963.

[2] Б. В. Гнеденко, Ю. К. Беляев, А. Д. Соловьев, *Математические методы в теории надежности*, Наука, Москва 1965.

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O KOLEJNOŚCI INSTALOWANIA ELEMENTÓW DO PRACY

STRESZCZENIE

W pracy rozpatruje się zespoły elementów złożone z elementu zasadniczego, zainstalowanego do pracy, i pewnej liczby elementów rezerwowych. Elementy rezerwowe znajdują się w rezerwie letniej, tzn. mogą w rezerwie ulegać awarii, lecz niekoniecznie z tą samą intensywnością, co w stanie pracy. Elementy rezerwowe są włączane do pracy natychmiast w chwilach awarii elementu pracującego. Zespół ulega awarii w chwili awarii ostatniego elementu.

Niezawodność zespołu jest zależna od kolejności instalowania elementów do pracy, jeżeli tylko elementy te nie są jednakowe. W niniejszej pracy przeanalizowano dwa kryteria optymalności zespołów. W pierwszym kryterium żąda się, żeby niezawodność zespołu w pierwszym okresie jego pracy była maksymalna; w drugim żąda się, żeby średni czas pracy zespołu był maksymalny.

Drugie kryterium optymalności zespołów zastosowano do konstruowania podzespołów z danego zbioru elementów. Pokazano, że w sensie maksymizacji średniego czasu pracy wszystkich podzespołów utworzonych z danego zbioru elementów, zawierającego elementy dwojakiego rodzaju, efektywniejsze od konstruowania par jednakowych elementów jest konstruowanie par mieszanych, uporządkowanych optymalnie w sensie drugiego kryterium.

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О ПОРЯДКЕ ВКЛЮЧЕНИЯ ЭЛЕМЕНТОВ В РАБОТУ

РЕЗЮМЕ

Рассмотренные в этой работе системы сложены из основного элемента включённого в работу и некоторых резервных элементов. Резервные элементы находятся в облегчённом режиме до момента их включения вместо основного элемента. Они включаются в работу немедленно в моменте аварии основного элемента. Эта система отказывается тогда, когда отказывается последний элемент.

Надежность системы зависит от порядка включения элементов в работу, если эти элементы неодинаковые. В этой работе рассмотрены два критерия оптимальности систем. В первом критерии требуем, чтобы надёжность систем на первом этапе работы была максимальная. Во втором критерии требуем, чтобы среднее время работы систем было максимальное.

Второй критерий использован в конструкции подсистем из данного множества элементов. В работе показано, что в смысле максимальности среднего времени работы всех подсистем сложенных из данного множества элементов, содержащего элементы двойного сорта-имеется—лучше соединения одинаковых элементов—соединение смешанных элементов, приведенных в порядок оптимально в смысле второго критерия.