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A UNIPOTENT GROUP ASSOCIATED WITH CERTAIN LINEAR GROUPS

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1. Introduction. We consider* linear groups G with the property that the eigenvalues of each element are of modulus one. For each $g \in G$ we form the Jordan decomposition $g = s(g) \cdot u(g)$, where $u(g)$ is unipotent, $s(g)$ acts semisimply, and they commute. We show that the group $U \subseteq GL(V)$ generated by the set $\{u(g): g \in G\}$ of unipotent parts is unipotent. In particular, it follows that the unipotent elements of G, $G_u = \{g \in G: g = u(g)\}$, form a (normal) subgroup in G since $G_u = G \cap U$ and U is a group.

Groups of this kind turn up in the study of distal actions of automorphisms, as explained in [4] and [1], and also in analysis on connected Lie groups having polynomial growth (see [3] and [2]). In the latter situation, a connected Lie group G has polynomial growth if and only if its adjoint representation satisfies the eigenvalue condition above.

The proofs in the sequel require only that the trace function $\mathrm{Tr}(g)$ be bounded on G. This condition on the trace is used directly in [1]. However, it is not very hard to show that *the trace is bounded on G if and only if all eigenvalues are of modulus one.*

Proof. Only the necessity requires comment. If $g \in G$ has eigenvalues $a_1, \ldots, a_n$, let $r = \max\{|a_j|, |a_j|^{-1}\}$. By switching to g^{-1}, if necessary, we may assume that r occurs among the $|a_1|, \ldots, |a_n|$. If $r = 1$, then $|a_i| = 1$ for all i and we are done. Otherwise, consider $|\mathrm{Tr}(g^k)| = \left|\sum a_j^k\right|$ as $k \to +\infty$. The terms involving the eigenvalues $\{|a_j|: j \in J\}$ with $|a_j| = r$ dominate all others eventually; thus

$$\left|\sum\{a_j^k: j \in J\}\right| = r^k \left|\sum\{\exp[ik\theta_j]: j \in J\}\right|$$

is bounded. This cannot happen unless the trigonometric polynomial

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$\sum \exp[ik\theta_j]$ vanishes as $k \to +\infty$. This is impossible unless it vanishes for all $k \in \mathbf{Z}$, which is a contradiction. This completes the proof.

2. Main theorem. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We shall prove the following

THEOREM. *Let $G \subseteq GL(n, \mathbf{K})$ be a group such that all eigenvalues are of modulus one for each $g \in G$. Let $g = s(g)\cdot u(g)$ be the Jordan decomposition and let U be a group generated by unipotent parts $\{u(g): g \in G\}$. Then U is a unipotent group.*

The idea is to show that $\mathrm{Tr}(h) = n$ for all $h \in U$. We start with a few lemmas.

LEMMA 1. *Let G be a subgroup of $GL(n, \mathbf{K})$ such that* $\mathrm{Tr}(g) = n$ *for all $g \in G$. Then G is unipotent.*

Proof. It suffices to show that $g - I$ is nilpotent for each $g \in G$. Recall that if A is an $(n \times n)$-matrix over $\mathbf{K}$, then A is nilpotent whenever $\mathrm{Tr}(A^i) = 0$ for $1 \leqslant i \leqslant n$. Now,

$$(g-I)^i = \sum_{j=0}^{i} \binom{i}{j} g^j(-I)^{i-j},$$

$$\mathrm{Tr}(g-I)^i = \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j}\mathrm{Tr}(g^j) = 0\cdot n = 0,$$

as required.

LEMMA 2. *If $u \in GL(n, \mathbf{K})$ is unipotent, then there exist matrices A_l $(0 \leqslant l \leqslant n)$ such that*

$$u^k = \sum_{l=0}^{n} A_l k^l \quad \textit{for all } k \in \mathbf{Z}.$$

Proof. Since u lies on a 1-parameter subgroup, $u = \mathrm{Exp}(X)$ for a nilpotent X. Then $u^k = \mathrm{Exp}(kX)$ is a finite exponential series, so

$$u^k = \mathrm{Exp}(kX) = \sum_{l=0}^{n} A_l k^l.$$

LEMMA 3. *For integers $n \geqslant 0$ and $r \geqslant 1$ consider the set of multi-indices of length r:*

$$S(n, r) = S = \{\mathbf{i} \in \mathbf{Z}_+^r:\ 0 \leqslant i_j \leqslant n,\ \textit{all } 1 \leqslant j \leqslant r\}.$$

Consider any function of the form

$$f(\mathbf{k}) = \sum_{\mathbf{i}\in S} p(\mathbf{i}, \mathbf{k})\mathbf{k}^{\mathbf{i}},$$

where each coefficient $p(\mathbf{i}, *)$ is an (almost) periodic function of $\mathbf{k} \in \mathbf{Z}^r$. If $f = 0$ on $\mathbf{Z}^r$, then so is each coefficient: $p(\mathbf{i}, \mathbf{k}) = 0$ for all $\mathbf{i} \in S$, $\mathbf{k} \in \mathbf{Z}^r$.

Proof. We work by induction on the degree r. When $r = 1$ (any n), the argument is simple. Each coefficient in $f = \sum p(i, k)k^i$ is bounded in $k \in \mathbf{Z}$; unless $p(n, *) = 0$ for all k, the term involving k^n is dominant on a recurrent set in $\mathbf{Z}$ and we could not have $f = 0$ for all k. Similarly, we get $p(n, k) = \ldots = p(1, k) = 0$ for all $k \in \mathbf{Z}$, and then it is clear that the constant term $p(0, k)$ must also be identically zero.

Assuming the result true for all degrees less than r (and for all $n \geqslant 0$), consider

$$f(\boldsymbol{k}) = \sum_{\boldsymbol{i} \in S} p(\boldsymbol{i}, \boldsymbol{k}) k_1^{i_1} \ldots k_r^{i_r}. \tag{1}$$

Let us write $\boldsymbol{k} = (\boldsymbol{k}', k_r) \in \mathbf{Z}^{r-1} \times \mathbf{Z}$, $\boldsymbol{i} = (\boldsymbol{i}', i_r) \in \mathbf{Z}_+^{r-1} \times \mathbf{Z}_+$ and fix a $\boldsymbol{k}' \in \mathbf{Z}_+^{r-1}$, letting $k_r \in \mathbf{Z}$ vary. The lead terms in (1) are those involving k_r^n:

$$\Big[\sum_{\boldsymbol{i}' \in S'} p\big((\boldsymbol{i}', n), (\boldsymbol{k}', k_r)\big) k_1^{i_1} \ldots k_{r-1}^{i_{r-1}}\Big] k_r^n.$$

The expression $[\ldots]$ is almost periodic in k_r, so if it is not identically zero, there will be a sequence of terms $|k_r| \to \infty$ for which $[\ldots]k_r^n$ dominates all terms involving lower powers of k_r. We are thus led to a contradiction unless

$$\sum_{\boldsymbol{i}' \in S'} p\big((\boldsymbol{i}', n), (\boldsymbol{k}', k_r)\big) k_1^{i_1} \ldots k_{r-1}^{i_{r-1}} = 0 \qquad \text{for all } k_r \in \mathbf{Z}.$$

Then, by successively considering terms involving lower powers of k_r, we get the system of identities

$$\sum_{\boldsymbol{i}' \in S'} p\big((\boldsymbol{i}', j), (\boldsymbol{k}', k_r)\big) k_1^{i_1} \ldots k_{r-1}^{i_{r-1}} = 0 \qquad \text{for all } k_r \in \mathbf{Z},\ 0 \leqslant j \leqslant n. \tag{2}$$

These remain true for any choice of $\boldsymbol{k}'$.

Now consider (2), holding $k_r \in \mathbf{Z}$ fixed and letting $\boldsymbol{k}' \in \mathbf{Z}^r$ vary. We apply induction to each equation in the system to conclude that $p\big((\boldsymbol{i}', j), (\boldsymbol{k}', k_r)\big) = 0$ for all $\boldsymbol{i}' \in S'$, $0 \leqslant j \leqslant n$, $\boldsymbol{k}' \in \mathbf{Z}^{r-1}$, and for any $k_r \in \mathbf{Z}$. This proves the lemma.

Proof of the Theorem. To show that $\mathrm{Tr}(h) = n$ for all $h \in U$, we note that any $h \in U$ is of the form

$$h = u_1^{k_1} \ldots u_r^{k_r}, \qquad r < \infty,\ k_i \in \mathbf{Z}_+$$

(taking non-negative exponents since $u(g^{-1}) = u(g)^{-1}$). Take r elements $g_1, \ldots, g_r$ in G, decompose them ($g_i = s_i u_i$), and find similarity transforms putting each s_i in the diagonal form:

$$s_i = a_i \cdot \mathrm{diag}(b_{i1}, \ldots, b_{in}) \cdot a_i^{-1}$$

(a_i invertible on C^n). Let $B \in C^{nr}$ be the array $\{b_{ij}\colon 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant n\}$

whose rows are the eigenvalues of $s_1, \ldots, s_r$; we propose to let B vary throughout all arrays Z with $|z_{ij}| = 1$. When we do this, we write

$$s_i(Z) = a_i \cdot \operatorname{diag}(z_{i1}, \ldots, z_{ir}) \cdot a_i^{-1}.$$

If $j \in Z^r$ is any multi-index, put

$$Z^j = \begin{bmatrix} z_{11}^{j_1} & \ldots & z_{1n}^{j_1} \\ z_{r1}^{j_r} & \ldots & z_{rn}^{j_r} \end{bmatrix}.$$

This notation insures that $s_i(Z^j) = s_i(Z)^{j_i}$ for $1 \leqslant i \leqslant r$. For each Z and $j, k \in Z^r$ form the product in $GL(n, C)$:

$$G(Z, j, k) = s_1(Z)^{j_1} u_1^{k_1} \ldots s_r(Z)^{j_r} u_r^{k_r} = s_1(Z^j) u_1^{k_1} \ldots s_r(Z^j) u_r^{k_r}.$$

Now, writing $u_i^k = A_0^i + A_1^i k + \ldots + A_n^i k^n$, as in Lemma 2, we compute

$$(3) \quad F(Z, j, k) = \operatorname{Tr}\big(G(Z, j, k)\big) = \operatorname{Tr}\Big(s_1(Z^j)\Big(\sum_{i_1=0}^{n} A_{i_1}^1 k_1^{i_1}\Big)\ldots\Big)$$

$$= \sum_{i \in S} \operatorname{Tr}\big(s_1(Z^j) A_{i_1}^1 \ldots s_r(Z^j) A_{i_r}^r\big) k^i = \sum_{i \in S} p(Z, i, j) k^i.$$

Here $p(Z, i, j) = p(Z^j, i)$ is a polynomial in the entries of Z^j, hence for fixed Z and

$$i \in S = \{i \in Z_+^r \colon 0 \leqslant i_j \leqslant n, \text{ all } j = 1, 2, \ldots, r\}$$

the map $j \to p(Z^j, i)$ is a finite sum of characters on Z^r, hence is periodic. There is also a uniform bound $|p(Z, i, j)| \leqslant M$ for all $i \in S$, $j \in Z^r$, and Z.

Now fix $Z = B$ and take $k = j$ in (3) to get

$$H(k) = F(B, k, k) = \sum_{i \in S} p(B, i, k) k^i.$$

H is bounded on Z^r, since

$$H(k) = \operatorname{Tr}(s_1^{k_1} u_1^{k_1} \ldots s_r^{k_r} u_r^{k_r}) = \operatorname{Tr}(g_1^{k_1} \ldots g_r^{k_r})$$

is the trace of an element in G, hence $|H| \leqslant n$ due to the eigenvalue requirements on G.

Consider the behavior of H if we fix $k' \in Z^{r-1}$ and vary $k_r \in Z$, and examine the terms involving k_r^n:

$$H = (\ldots) + \sum p\big(B, (i_1, \ldots, i_{r-1}, n), k\big) k_1^{i_1} \ldots k_r^n.$$

We know that H is bounded; after dividing by k_r^n, both H/k_r^n and $(\ldots)/k_r^n$ approach zero as $k_r \to +\infty$ due to boundedness of the coefficients $p(B, i, k)$, so

$$(4) \quad \sum p\big(B, (i_1, \ldots, i_{r-1}, n), k\big) k_1^{i_1} \ldots k_{r-1}^{i_{r-1}} \to 0$$

as $k_r \to +\infty$. Since $\boldsymbol{k}' = (k_1, \ldots, k_{r-1})$ is fixed, (4) is an almost periodic function of $k_r \in \boldsymbol{Z}$, and so must identically be zero for all k_r. Since this is so for any $\boldsymbol{k}'$, (4) is zero for all $\boldsymbol{k} \in \boldsymbol{Z}^r$. Next, work on terms involving k_r^{n-1} and so on to get the system of identities

$$\sum p\big(B, (i_1, \ldots, i_{r-1}, n), \boldsymbol{k}\big) k_1^{i_1} \ldots k_{r-1}^{i_{r-1}} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\sum p\big(B, (i_1, \ldots, i_{r-1}, 1), \boldsymbol{k}\big) k_1^{i_1} \ldots k_{r-1}^{i_{r-1}} = 0$$

for all $\boldsymbol{k} \in \boldsymbol{Z}^r$. (The argument does not apply to the zero power of k_r.) By Lemma 3 and fixing $k_r \in \boldsymbol{Z}$, all coefficient functions are zero:

$$p(B, \boldsymbol{i}, \boldsymbol{k}) = 0 \quad \text{for all } \boldsymbol{k} \in \boldsymbol{Z}^r, \boldsymbol{i} \in S \text{ with } i_r > 0.$$

However, the argument above can be applied anew fixing attention on any component k_j $(1 \leqslant j \leqslant r)$ instead of k_r. This forces all coefficient functions to be zero $(p(B, \boldsymbol{i}, \boldsymbol{k}) = 0)$ except for the "constant term" corresponding to $\boldsymbol{i} = (0, \ldots, 0)$.

Now, since the coefficient functions are zero, we conclude that

$$p(B, \boldsymbol{i}, \boldsymbol{j}) = p(B^{\boldsymbol{j}}, \boldsymbol{i}) = 0 \quad \text{for all } \boldsymbol{i} \neq (0, \ldots, 0) \text{ in } S, \boldsymbol{j} \in \boldsymbol{Z}^r,$$

so that

$$F(B, \boldsymbol{j}, \boldsymbol{k}) = \sum_{\boldsymbol{i} \in S} p(B^{\boldsymbol{j}}, \boldsymbol{i}) \boldsymbol{k}^{\boldsymbol{i}} = p(B^{\boldsymbol{j}}, \boldsymbol{0}) \quad \text{for all } \boldsymbol{j}, \boldsymbol{k} \in \boldsymbol{Z}^r.$$

Now set $\boldsymbol{j} = \boldsymbol{0} = (0, \ldots, 0)$ and recall (3); for all $\boldsymbol{k} \in \boldsymbol{Z}^r$ we get

$$\mathrm{Tr}(Iu_1^{k_1} Iu_2^{k_2} \ldots Iu_r^{k_r}) = \mathrm{Tr}(u_1^{k_1} \ldots u_r^{k_r}) = p(B^{\boldsymbol{0}}, \boldsymbol{0}) = c,$$

c being some constant. The constant is evaluated by taking $k_1 = \ldots = k_r = 0$: $c = \mathrm{Tr}(I) = n$. Thus

$$\mathrm{Tr}(u_1^{k_1} \ldots u_r^{k_r}) = n \quad \text{for all } \boldsymbol{k} \in \boldsymbol{Z}^r, \ r < \infty,$$

which proves the theorem.

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