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# Solutions of a functional equation in a special class of functions

by M. Kuczma and J. Matkowski (Katowice)

In the present paper we consider the problem of the existence and uniqueness of solutions of the functional equation

$$\varphi(x) = h(x, \varphi[f(x)]), \tag{1}$$

where $\varphi$ is an unknown real-valued function of a real variable, belonging to a certain function class $\mathcal{B}$, which is defined below.

Our Theorem 1 generalizes the result on the uniqueness of solutions of (1) fulfilling a Lipschitz condition (cf. [7]) as well as those on the uniqueness of differentiable solutions of (1) (cf. [1], [6]). The results of the present paper are also related to the investigations of B. Choczewski (cf. [2] and the references quoted therein) and Kuczma [5] regarding the so-called regular solutions of equation (1) or of its particular cases, and more generally, solutions characterized by a specific asymptotic behaviour at the fixed point $\xi$ of the function $f$.

In general the solution of equation (1) depends on an arbitrary function (cf. [3]), therefore conditions ensuring the uniqueness of a solution are of a particular importance in the theory of functional equations in a single variable (cf. [3], p. 44–45, and [4]).

**1.** Let $I$ be an interval. We assume that

(i) $f$ is defined in $I$ and there exists a $\xi \in I$ such that

$$0 < \frac{f(x)-\xi}{x-\xi} < 1 \quad \text{for } x \in I,\ x \neq \xi;$$

moreover, $f(\xi) = \xi$.

Remark 1. This condition implies that for every interval $I_1 \subset I$ such that $\xi \in I_1$ we have $f(I_1) \subset I_1$. If, moreover, $f$ is continuous in $I$, then for every $x \in I$ we have $\lim_{n\to\infty} f^n(x) = \xi$, where $f^n$ denotes the $n$-th iterate of the function $f$ (cf. [3], p. 20–21).

Setting $x = \xi$ in equation (1) we obtain for the value $\eta = \varphi(\xi)$ the condition

$$h(\xi, \eta) = \eta. \tag{2}$$

Thus we assume:

(ii) $h$ is defined in a domain $\Omega$ containing $(\xi, \eta)$ such that condition (2) is fulfilled.

(iii) For every $x \in I$ the set $\Omega_x = \{y: (x, y) \in \Omega\}$ is a non-empty open interval and $h(f(x), \Omega_{f(x)}) \subset \Omega_x$.

DEFINITION. Denote by $\mathscr{B}$ the class of functions $\varphi$ such that every $\varphi \in \mathscr{B}$ is defined in a neighbourhood of $\xi$ (relative to $I$) and there exist positive numbers $\mu$ and $c$ (depending on $\varphi$) such that

$$(3) \qquad |\varphi(x) - \eta| \leqslant \mu |x - \xi| \quad \text{for } x \in I \cap \langle \xi - c, \xi + c \rangle .$$

Condition (3) may be alternatively expressed in terms of the asymptotic relation

$$\varphi(x) = \eta + O(|x - \xi|), \quad x \to \xi, \ x \in I.$$

We shall prove the following

THEOREM 1. *Assume that hypotheses* (i) *and* (ii) *are fulfilled and there exist positive numbers $\alpha, \beta, s, l$ such that* [1]

$$(4) \qquad |f(x) - \xi| \leqslant s|x - \xi| \quad \textit{for } x \in I \cap \langle \xi - \alpha, \xi + \alpha \rangle ,$$

$$(5) \qquad |h(x, y) - h(x, \bar{y})| \leqslant l |y - \bar{y}|$$
$$\textit{for } (x, y), (x, \bar{y}) \in \Omega \cap \langle \xi - \alpha, \xi + \alpha \rangle \times \langle \eta - \beta, \eta + \beta \rangle ,$$

$$(6) \qquad ls < 1 .$$

*Then:*

(a) *for every two solutions $\varphi_1, \varphi_2 \in \mathscr{B}$ of equation* (1) *there exists a neighbourhood $U$ of the point $x = \xi$ such that $\varphi_1(x) = \varphi_2(x)$ for $x \in I \cap U$;*

(b) *if $s < 1$, then every two solutions $\varphi_1, \varphi_2 \in \mathscr{B}$ of equation* (1) *in $I$ are equal in the interval $I \cap \langle \xi - \alpha, \xi + \alpha \rangle$;*

(c) *if $f$ is continuous in $I$, then there exists at most one solution $\varphi \in \mathscr{B}$ of equation* (1) *in $I$.*

Proof. (a) Suppose that $\varphi_1$ and $\varphi_2$ are two solutions of equation (1) in $I$ belonging to the class $\mathscr{B}$. It follows by the definition of the class $\mathscr{B}$ that there exists a $\mu > 0$ and a $c > 0$ such that

$$(7) \qquad |\varphi_i(x) - \eta| \leqslant \mu |x - \xi| \quad \text{for } x \in I \cap \langle \xi - c, \xi + c \rangle, \ i = 1, 2.$$

Hence $\varphi_i$ are continuous at the point $x = \xi$ and $\varphi_i(\xi) = \eta$, $i = 1, 2$. Therefore we may assume that $c$ in (7) has been chosen in such a manner that

$$(8) \qquad |\varphi_i(x) - \eta| \leqslant \beta \quad \text{for } x \in I \cap \langle \xi - c, \xi + c \rangle, \ i = 1, 2.$$

[1] Condition (4) means that $f$ belongs to a class analogous to $\mathscr{B}$, but here the specification of the constants $s$ and $\alpha$ is necessary.

Without loss of generality we may assume that $c \leqslant a$. Since $f(I \cap \langle \xi - c, \xi + c\rangle) \subset I \cap \langle \xi - c, \xi + c\rangle$ (see Remark 1), it follows by (8), (1) and (5) that for $x \in I \cap \langle \xi - c, \xi + c\rangle$

$$|\varphi_1(x) - \varphi_2(x)| = \big|h\big(x, \varphi_1[f(x)]\big) - h\big(x, \varphi_2[f(x)]\big)\big| \leqslant l\,|\varphi_1[f(x)] - \varphi_2[f(x)]|.$$

By induction we obtain

$$(9)\qquad |\varphi_1(x) - \varphi_2(x)| \leqslant l^n |\varphi_1[f^n(x)] - \varphi_2[f^n(x)]| \quad \text{for } x \in I \cap \langle \xi - c, \xi + c\rangle,\ n = 1, 2, \ldots$$

Now, (9), (7) and (4) yield

$$\begin{aligned} |\varphi_1(x) - \varphi_2(x)| &\leqslant l^n \big|\big(\varphi_1[f^n(x)] - \eta\big) + \big(\eta - \varphi_2[f^n(x)]\big)\big| \\ &\leqslant l^n \big(|\varphi_1[f^n(x)] - \eta| + |\varphi_2[f^n(x)] - \eta|\big) \\ &\leqslant 2\mu l^n |f^n(x) - \xi| \leqslant 2\mu (ls)^n |x - \xi|. \end{aligned}$$

Thus we get

$$|\varphi_1(x) - \varphi_2(x)| \leqslant 2\mu c (ls)^n \quad \text{for } x \in I \cap \langle \xi - c, \xi + c\rangle,\ n = 1, 2, \ldots$$

Letting $n \to \infty$ in the last inequality we obtain by (6) $\varphi_1(x) = \varphi_2(x)$ for $x \in I \cap \langle \xi - c, \xi + c\rangle$. This completes the proof of part (a) of the assertion of the theorem.

(b) Since $s < 1$, we obtain by (4) that $\lim_{n\to\infty} f^n(x) = \xi$ for $x \in I \cap \langle \xi - a, \xi + a\rangle$. Let $\varphi_1, \varphi_2 \in \mathscr{B}$ be two solutions of equation (1) in $I$. By (a) there exists a neighbourhood $U$ of $\xi$ such that $\varphi_1(x) = \varphi_2(x)$ for $x \in I \cap U$. For every $x \in I \cap \langle \xi - a, \xi + a\rangle$ there exists an index $n$ such that $f^n(x) \in I \cap U$. Thus $\varphi_1[f^n(x)] = \varphi_2[f^n(x)]$, whence we obtain by a repeated use of (1) $\varphi_1(x) = \varphi_2(x)$.

(c) This follows by the same argument as above, since according to Remark 1 we have $\lim_{n\to\infty} f^n(x) = \xi$ for every $x \in I$.

Note that in case (a) there need not exist a common neighbourhood $U$ of $\xi$ in which all solutions $\varphi \in \mathscr{B}$ of equation (1) in $I$ coincide. This may be seen from the following

Example. Let $I = \langle 0, 1\rangle$, $\xi = 0$, and put $x_n = 2^{-n}$, $n = 0, 1, 2, \ldots$ Define $f$ on $\langle 0, 1\rangle$ by

$$f(x) = x_n + \tfrac{1}{2}(x - x_n) \quad \text{for } x_n < x \leqslant x_{n-1},\ n = 1, 2, \ldots; \quad f(0) = 0,$$

and consider the equation

$$(10)\qquad \varphi(x) = \tfrac{1}{2}\varphi[f(x)].$$

Then hypotheses (i) and (ii) are fulfilled (with $\Omega$ being the whole plane and $\eta = 0$) as well as conditions (4)–(6) ($s = 1$, $l = \frac{1}{2}$). The functions

$$\varphi_N(x) = \frac{1}{x - x_n} \quad \text{for } x_n < x \leqslant x_{n-1},\ n = 1, \ldots, N, \tag{11}$$

$$\varphi_N(x) = 0 \quad \text{for } 0 \leqslant x \leqslant x_N,$$

all satisfy equation (10) in $\langle 0, 1\rangle$ and are in the class $\mathscr{B}$, but there is no common neighbourhood of 0 (in $\langle 0, 1\rangle$) in which all functions (11) would vanish.

**2.** Now we shall prove two theorems on the existence of solutions of equation (1) in the class $\mathscr{B}$.

THEOREM 2. *Suppose that hypotheses* (i) *and* (ii) *are fulfilled and that there exist positive numbers* $a, \beta, s, l$ *such that inequalities* (4)–(6) *hold. Further suppose that there exists a* $k > 0$ *such that*

$$|h(x, \eta) - h(\xi, \eta)| \leqslant k|x - \xi| \quad \textit{for } x \in I \cap \langle \xi - a, \xi + a\rangle. \tag{12}$$

*Then:*

(a) *There exists a neighbourhood* $U$ *of* $x = \xi$ *and a function* $\varphi \in \mathscr{B}$ *such that* $\varphi$ *is a solution of equation* (1) *in* $U \cap I$.

(b) *Suppose that also* (iii) *is fulfilled. If* $f$ *is continuous in* $I$ *or if* $s < 1$ *and* $I \subset \langle \xi - a, \xi + a\rangle$, *then there exists exactly one function* $\varphi \in \mathscr{B}$ *satisfying equation* (1) *in* $I$.

Proof. (a) Without loss of generality we may assume that $\xi$ is the left end-point of $I$.

Put

$$\mu = \frac{k}{1 - sl} \tag{13}$$

and choose a number $c$,

$$0 < c < \min(a, \beta/\mu). \tag{14}$$

Evidently, the number $c$ can be chosen in such a manner that the following inclusion holds

$$D \stackrel{\text{df}}{=} \{(x, y):\ \xi \leqslant x \leqslant \xi + c,\ |y - \eta| \leqslant \mu(x - \xi)\} \subset \Omega.$$

It follows by (14) that

$$D \subset \langle \xi, \xi + a\rangle \times \langle \eta - \beta, \eta + \beta\rangle. \tag{15}$$

Let $\mathscr{A}$ denote the set of functions $\varphi$ defined in $\langle \xi, \xi + c\rangle$ fulfilling the condition

$$|\varphi(x) - \eta| \leqslant \mu|x - \xi| \quad \text{for } x \in \langle \xi, \xi + c\rangle. \tag{16}$$

We shall prove that the transformation $\psi = T(\varphi)$ defined by the formula

$$\psi(x) = h\big(x, \varphi[f(x)]\big) \tag{17}$$

maps $\mathscr{A}$ into itself. Let $\varphi \in \mathscr{A}$. Applying in turn (17), (2), (5), (12), (16), (4) and (13) we obtain for $x \in \langle \xi, \xi + c\rangle$

$$\begin{aligned} |\psi(x) - \eta| &= \big|h\big(x, \varphi[f(x)]\big) - h(\xi, \eta)\big| \\ &\leqslant \big|h\big(x, \varphi[f(x)]\big) - h(x, \eta)\big| + |h(x, \eta) - h(\xi, \eta)| \\ &\leqslant l|\varphi[f(x)] - \eta| + k|x - \xi| \leqslant l\mu|f(x) - \xi| + k|x - \xi| \\ &\leqslant (l\mu s + k)|x - \xi| = \mu|x - \xi|, \end{aligned}$$

so $\psi \in \mathscr{A}$.

Now we define the sequence of functions $\{\varphi_n\}$ by the recurrence relation

$$\varphi_{n+1}(x) = h\big(x, \varphi_n[f(x)]\big), \quad \varphi_0(x) = \eta, \quad x \in \langle \xi, \xi + c\rangle. \tag{18}$$

Since $\varphi_0 \in \mathscr{A}$ and $T(\mathscr{A}) \subset \mathscr{A}$, we have $\varphi_n \in \mathscr{A}$ for $n = 1, 2, \ldots$ By (5) we have

$$\begin{aligned} |\varphi_{n+1}(x) - \varphi_n(x)| &= \big|h\big(x, \varphi_n[f(x)]\big) - h\big(x, \varphi_{n-1}[f(x)]\big)\big| \\ &\leqslant l|\varphi_n[f(x)] - \varphi_{n-1}[f(x)]|. \end{aligned}$$

Repeating this procedure $n$ times we obtain

$$|\varphi_{n+1}(x) - \varphi_n(x)| \leqslant l^n|\varphi_1[f^n(x)] - \eta|, \quad x \in \langle \xi, \xi + c\rangle,\ n = 0, 1, 2, \ldots$$

Now relations $\varphi_1 \in \mathscr{A}$ and (4) imply

$$|\varphi_{n+1}(x) - \varphi_n(x)| \leqslant \mu(ls)^n|x - \xi| \leqslant \mu c(ls)^n, \quad x \in \langle \xi, \xi + c\rangle, \quad n = 0, 1, 2, \ldots \tag{19}$$

Since

$$\varphi_k(x) = \varphi_0(x) + \sum_{n=0}^{k-1}\big(\varphi_{n+1}(x) - \varphi_n(x)\big),$$

inequality (19) implies the convergence of the sequence $\varphi_n(x)$ in $\langle \xi, \xi + c\rangle$. It follows from the continuity of $h$ in $D$ with respect to the second variable (cf. (5) and (15)) and from (18) that $\varphi(x) = \lim_{n\to\infty}\varphi_n(x)$, $x \in \langle \xi, \xi + c\rangle$, is a solution of equation (1) in $\langle \xi, \xi + c\rangle$. In order to show that the transformation $T$ maps $\mathscr{A}$ into itself we have proved that

$$\big|h\big(x, \varphi[f(x)]\big) - h(\xi, \eta)\big| \leqslant \mu|x - \xi| \quad \text{for } x \in \langle \xi, \xi + c\rangle$$

i. e., in view of (1) and (2),

$$|\varphi(x) - \eta| \leqslant \mu|x - \xi| \quad \text{for } x \in \langle \xi, \xi + c\rangle.$$

This means that $\varphi \in \mathscr{A}$ and hence $\varphi \in \mathscr{B}$. This completes the proof of (a).

For the proof of (b) notice that the conditions assumed imply that $\lim_{n\to\infty} f^n(x) = \xi$ for $x \in I$. By a use of an argument similar to that in the proof of Theorem 3.2 in [3], p. 70, we can extend the local solution obtained in (a) onto the whole $I$ (²). Now Theorem 1 completes the proof of (b).

THEOREM 3. *Let hypotheses* (i) *and* (ii) *be fulfilled and suppose that there exist positive constants* $a, \beta, s, k, l$ *such that inequalities* (4) *and* (6) *hold and*

$$|h(x, y) - h(\xi, \eta)| \leqslant k|x-\xi| + l|y-\eta| \quad \text{for } (x, y) \in \Omega_0, \tag{20}$$

$$h(x, y_1) \leqslant h(x, y_2) \quad \text{for } (x, y_1), (x, y_2) \in \Omega_0,\ y_1 \leqslant y_2, \tag{21}$$

$$h \text{ is continuous with respect to } y \text{ in } \Omega_0, \tag{22}$$

*where* $\Omega_0 = \Omega \cap \langle \xi - a, \xi + a\rangle \times \langle \eta - \beta, \eta + \beta\rangle$, *and finally*

$$h(\xi, \eta) \leqslant h(x, \eta) \quad \text{for } x \in I \cap \langle \xi - a, \xi + a\rangle. \tag{23}$$

*Then*:

(a) *There exists at least one function* $\varphi \in \mathscr{B}$ *fulfilling equation* (1) *in a neighbourhood of* $x = \xi$.

(b) *If, moreover,* (iii) *is fulfilled, and if* $f$ *is continuous in* $I$ *or* $s < 1$ *and* $I \subset \langle \xi - a, \xi + a\rangle$, *then there exists at least one function* $\varphi \in \mathscr{B}$ *satisfying equation* (1) *in* $I$.

Proof. Again we may assume that $\xi$ is the left end-point of $I$. In the same manner as in the proof of Theorem 2 we define the numbers $\mu$ and $c$, the sets $D$ and $\mathscr{A}$, the transformation $T$ and the sequence $\{\varphi_n\}$. By the same argument as previously (one must use this time relation (20)) it follows that $T(\mathscr{A}) \subset \mathscr{A}$. Relations (23) and (21) imply that

$$\varphi_n(x) \geqslant \varphi_{n-1}(x) \quad \text{for } x \in \langle \xi, \xi + c\rangle. \tag{24}$$

On the other hand, since $T(\mathscr{A}) \subset \mathscr{A}$, all $\varphi_n$ are in $\mathscr{A}$, which shows that the sequence $\varphi_n(x)$ is bounded in $\langle \xi, \xi + c\rangle$ and consequently, it converges in $\langle \xi, \xi + c\rangle$. In view of (22) the function $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$ is a solution of equation (1) in $\langle \xi, \xi + c\rangle$. The remaining part of the proof is the same as in Theorem 2.

Remark 2. Theorem 3 remains valid if conditions (21) and (23) are replaced by

$$h(x, y_1) \geqslant h(x, y_2) \quad \text{for } (x, y_1), (x, y_2) \in \Omega_0,\ y_1 \leqslant y_2,$$

and

$$h(\xi, \eta) \geqslant h(x, \eta) \quad \text{for } x \in I \cap \langle \xi - a, \xi + a\rangle,$$

respectively. Then, in the proof, only the direction of the inequality in (24) is changed, which does not affect the further argument.

(²) In Theorem 3.2 in [3] the continuity of the functions $f$ and $h$ was needed only in order to obtain a continuous extension.

Remark 3. If $I = \langle \xi - a, \xi + a\rangle$, $a > 0$, then we may assume less than (i), namely that

$$|f(x) - \xi| < |x - \xi| \quad \text{for } x \in I,\ x \neq \xi;\ f(\xi) = \xi.$$

This condition implies that for every interval $I_1 = \langle \xi - b, \xi + b\rangle$, $0 < b \leqslant a$, we have $f(I_1) \subset I_1$. Furthermore, the continuity of $f$ yields $\lim_{n\to\infty} f^n(x) = \xi$ for $x \in I$.

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