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HOMOCLINIC AND PERIODIC SOLUTIONS OF SCALAR DIFFERENTIAL DELAY EQUATIONS

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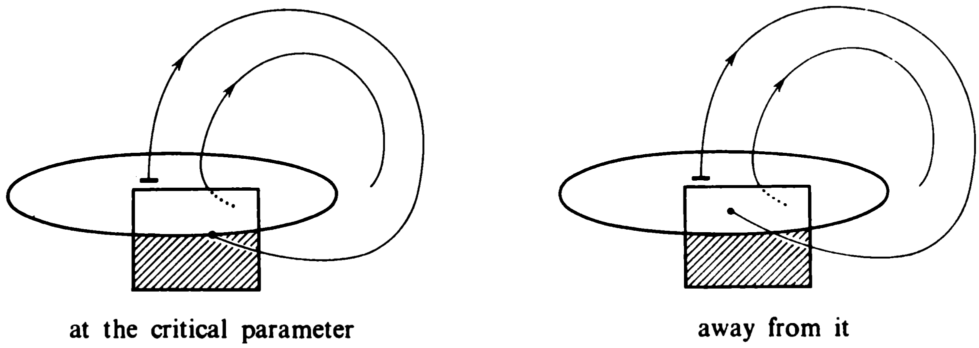
1. Introduction

Consider a saddle point with one unstable direction in a one-parameter family of vectorfields, with contraction in the local stable manifolds stronger than expansion in the local unstable manifolds. Suppose that at some critical parameter a_0 one branch of the unstable manifold, say the upper one, is homoclinic. If by a change of the parameter this connection is broken in such a way that upper branches now return to a neighborhood of the saddle point above the local stable manifolds, then periodic orbits bifurcate.

In case of two-dimensional spaces this has been known for a long time (compare e.g. ch. XII [1]). For arbitrary finite dimension and analytic vectorfields (C^3 or C^4 is sufficient) the result is due to L. P. Šil'nikov [13]. He obtained the bifurcating periodic solutions from fixed points of modified return maps on a transversal to the homoclinic orbit which – in the situation described – are defined by translation along the flow above the local stable manifolds, and assign to each point on or below the local stable manifold the intersection of the returning branch with the transversal:



at the critical parameter   away from it

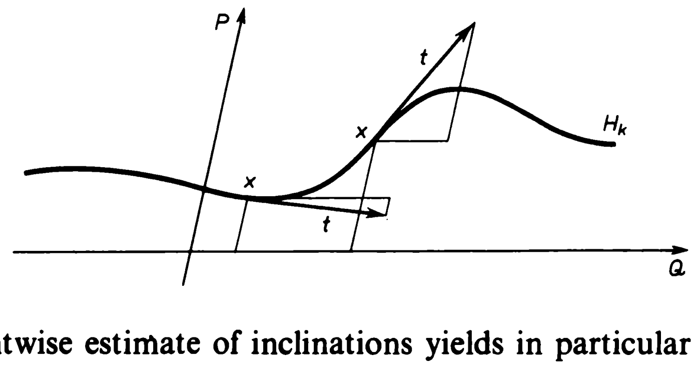
In [15] and [17] we proved analogous results in infinite dimension, for semiflows defined by functional differential equations (F.D.E.s). The onset in [15] is different from Šil'nikov's; for parameters $a \neq a_0$ close to a_0 one obtains return maps $\mathscr{P}_a$: $D_a \to D_a$ so that Schauder's fixed point theorem yields initial values for periodic solutions. This has the advantage that only minimal smoothness is required (of class C^1) – at the cost of assertions on uniqueness and stability for the bifurcating solutions.

The approach in [17] uses a map of Šil'nikov type and guarantees uniqueness and stability properties. Initial values of the bifurcating periodic solutions lie on a differentiable curve. – A major difficulty is to show that the modified return map in smooth enough for the application of an implicit function theorem. For this we need certain continuous second order derivatives of the semiflows. The main tool is a sharpened inclination lemma (λ-lemma) for C^2-maps in arbitrary Banach spaces which are not necessarily reversible, Lemma 2.1 in [16].

We make a digression and describe a special case of the relevant estimate. Consider a C^2-map h: $U \to E$ in a Banach space E with hyperbolic fixed point $0 \in U$ (i.e. the spectrum σ of $Dh(0)$ is disjoint with the unit circle – $0 \in \sigma$ is not excluded here). Assume for simplicity that the unstable space P of $Dh(0)$ has dimension 1, that $Dh(0)$ is a strict contraction on the stable space Q, and that P and Q are invariant also under the nonlinear map h. Let p and q denote the projections given by the decomposition $E = P \oplus Q$.

Then there is a constant $c > 0$ such that for every suitable transversal H to P (we do not make this precise here, but H is not required to be a submanifold) in a sufficiently small neighborhood U' of 0, for all points x in preimages $H_k = (h|U')^{-k}(H)$, $k \in N_0$, and for all nonzero tangent vectors $t \in T_x H_k$ (which can be defined) we have

$$(t, x) \qquad \frac{|pt|}{|qt|} \leqslant c\,|px|.$$



This pointwise estimate of inclinations yields in particular a better result than uniform convergence, as for example in Palis' first λ-lemma [12], since we have geometrical convergence of $\sup\{|px|: x \in H_k\}$ to 0 for $k \to \infty$.

Similar estimates for diffeomorphisms appeared already in the course of

[7]. [16] contains a necessarily different, directly accessible proof for the more general case.

For a generalization of Palis' λ-lemma without pointwise estimates, to arbitrary smooth maps in Banach spaces, see also [6].

Coming back to semiflows, we would like to mention two other new results on bifurcation from homoclinic or heteroclinic solutions in infinite dimension, [2] and [3].

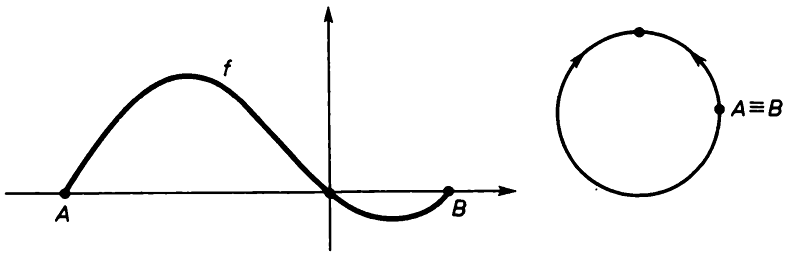
An open problem remains to derive analogoues of Šil'nikov's result in [13] for homoclinic solutions in higher-dimensional unstable manifolds.

Another drawback is still the lack of examples. This is partly due to the difficulty of finding homoclinic orbits, of course.

In [15] we established existence of heteroclinic solutions for F.D.E.s

$$(af) \qquad \dot{x}(t) = af(x(t-1))$$

with parameter $a > 0$ and periodic nonlinearity f: $R \to R$ which describe the simplest case of a state variable on a circle with one attractive rest point, negative feedback, (one other rest point) and a delayed reaction to deviations:



The parameter multiplying the nonlinearity can be interpreted as the delay since *eq.* (af) is equivalent to

$$\dot{y}(t) = f(y(t-a)).$$

Special cases to which the result from [15] applies model prototypes of phase-locked loops [4, 14]. There are also relations to models from mathematical biology.

Let C denote the Banach space of continuous real functions on $[-1, 0]$, equipped with the supremum-norm. *Eq.* (af) defines a semiflow $F(\cdot, \cdot, a)$: $R_0^+ \times C \to C$, by

$$F(t, \varphi, a) = x_t,$$

$$x_t(s) = x(t+s) \quad \text{for all } t \geqslant 0 \text{ and } s \in [-1, 0],$$

with the unique solution x: $[-1, \infty) \to R$ of the initial value problem

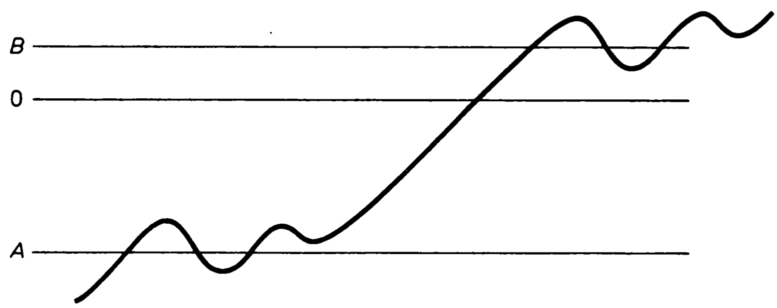
$$\dot{x}(t) = af(x(t-1)) \quad \text{for } t > 0,$$

$$x|[-1, 0] = \varphi, \quad \varphi(0) = x(0+).$$

The heteroclinic solutions which exist for a critical parameter a_0 leave the equilibria given by $A+j(B-A)$, $j \in Z$, monotonically increasing and settle down at $A+(j+1)(B-A)$ in a damped oscillation.

We proved (Theorem 2 [15]) that for $a > a_0$, periodic solutions of the second kind bifurcate off; i.e. solutions with "periods" $p > 0$ such that

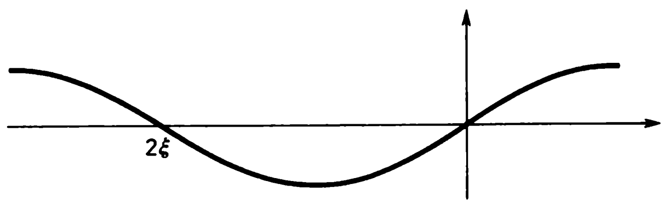
$$x(t+p) = x(t)+(B-A) \quad \text{for all } t \in R.$$



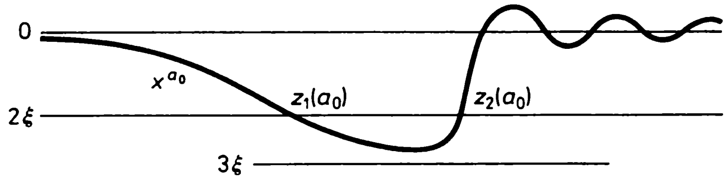
Such solutions correspond to periodic rotations with period p on the circle.

The same example was taken up in [17], and also in [3]. To our knowledge, there has been no other example for bifurcation of periodic orbits from saddle connections in infinite dimension.

In the present paper we state a theorem on bifurcation from *homoclinic* orbits for semiflows defined by F.D.E.s, analogous to Theorem 13.2 [17], construct an example and verify the hypotheses. This construction is more complicated than in [15]. The example is again of type (*af*), with parameter $a > 0$ and with a nonlinearity f: $R \to R$ which now has zeros at $2\xi < 0$ and at 0, with $0 < f$ on $(3\xi, 2\xi) \cup (0, -\xi)$ and $f < 0$ on $(2\xi, 0)$.



We find a critical parameter a_0 with a homoclinic solution x^{a_0} which decreases from 0 to a value in $(3\xi, 2\xi)$, increases then beyond 2ξ and 0, and ends in a damped oscillation around 0.



The two zeros of $x^{a_0} - 2\xi$ are spaced away from each other by more than the delay interval (of length 1).

The bifurcating periodic solutions y^a: $R \to R$ exist for $a < a_0$. Their orbits are unique in a neighborhood of the closure of the orbit $\{x_t^{a_0}: t \in R\}$ in C, stable and attractive with asymptotic phase. Periods tend to ∞ as $a \uparrow a_0$.

The graphs of the solutions y^a with orbit $\{y_t^a: t \in R\}$ close to the homoclinic connection show that y^a oscillates around the constant solution $t \to 2\xi$ with

$$(SO) \qquad |z - z'| > 1 \quad \text{for each pair of zeros } z \neq z'$$

of $y^a - 2\xi$. This indicates a relation to "slowly oscillating periodic solutions"; i.e. to periodic solutions with property (SO). The latter play a prominent role in the dynamics of equation

$$\dot{x}(t) = \tilde{f}(x(t-1))$$

with nonlinearities $\tilde{f}$: $R \to R$ which satisfy the negative feedback condition

$$(NF) \qquad x\tilde{f}(x) < 0$$

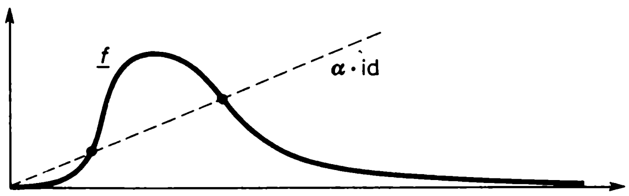
globally, for all $x \in R \setminus \{0\}$. The dynamics in this simplest case of delayed negative feedback is very rich – we do not intend to give here an account of what has been accomplished. In our case, (NF) holds true for $af(\cdot + 2\xi)$ on the interval $(\xi, -2\xi)$ but breaks down at the zero -2ξ. Therefore our result shows how a family of slowly oscillating periodic solutions can vanish in a homoclinic solution when amplitudes increase beyond the domain of negative feedback. This process goes along with small oscillations around the other equilibrium, where one has positive feedback, locally, – the fact behind is that all trajectories in the stable space of the linearization of $F(\cdot, \cdot, a_0)$ at the stationary point $\varphi = 0$ are spiralling in.

Small wiggles in larger amplitude periodic solutions are also observed in more complex scalar F.D.E.s, for example in relatives of models

$$(\underline{f}, \alpha) \qquad \dot{x}(t) = \underline{f}(x(t-1)) - \alpha x(t)$$

for the density of the red blood cell population, due to Lasota and Ważewska–Czyżewska [8], Mackey and Glass [9]. Here, $\alpha > 0$, $\underline{f}(0) = 0$, $\underline{f}(R^+) \subset R^+$, $\underline{f}'(0) = 0$, $\underline{f}' > 0$ on some interval $(0, \xi)$ and $\underline{f}' < 0$ on (ξ, ∞). For a detailed study, see [10].

Linearization at the first positive zero of $\underline{f} - \alpha\mathrm{id}$ and a look to the right hand side of eq. $(\underline{f}, \alpha)$ for $x(t-1)$ and $x(t)$ large reveal



that one may expect parameters with homoclinic solutions and bifurcation of periodic solutions from them also in this class of equations.

Finally, note that our result includes *periodic* nonlinearities as shown above.

2. Scalar autonomous F.D.E.s and the bifurcation theorem

Let $f\colon R \to R$ be a C^2-function with $f(0) = 0$ and $f'(0) = 1$. By a solution of equation

$$(af) \qquad \dot{x}(t) = af\big(x(t-1)\big)$$

with parameter $a \in R$, we mean either a continuous function $x\colon [s-1, \infty) \to R$, $s \in R$, which is differentiable for $t > s$ with (af) for these t, or a differentiable function $x\colon R \to R$ which satisfies *eq.* (af) for all $t \in R$. Recall the notation x_t for the function $[-1, 0] \ni s \to x(t+s) \in R$ whenever $[t-1, t]$ is in the domain of the solution x.

For each $(\varphi, a) \in C \times R$ (see introduction) there is a unique solution $x = x(\varphi, a)\colon [-1, \infty) \to R$ of *eq.* (af) with $x_0 = \varphi$. This follows easily with the aid of the formulas

$$x(t) = x(n) + \int_n^t af\big(x(s-1)\big)\,ds$$

for all $n \in N_0$, $t \in [n, n+1]$. The parameterized semiflow $F\colon R_0^+ \times C \times R \to C$ given by the relation $F(t, \varphi, a) = x_t$, $x = x(\varphi, a)$, is continuous, of class C^1 on $(1, \infty) \times C \times R$ and of class C^2 on $(2, \infty) \times C \times R$. Furthermore, $D_2 F$ exists and is continuous on $R_0^+ \times C \times R$.

Trajectories $X\colon R \ni t \to X_t \in C$, $X\colon [s, +\infty) \ni t \to X_t \in C$ $(s \in R)$ of $F(\cdot, \cdot, a)$, $a \in R$, are defined by the implication

$$s \leqslant t \Rightarrow X_t = F(t-s, X_s, a).$$

Solutions $x\colon R \to R$ $(x\colon [s-1, \infty) \to R)$ and trajectories $X\colon R \to C$ $(X\colon [s, \infty) \to C)$ are in a one-to-one correspondance, with $X_t = x_t$ for all $t \in R$ $(t \geqslant s)$; in particular, $x(t) = X_t(0)$ for all t in the domain of X.

$f(0) = 0$ implies $F(t, 0, a) = 0$ on $R_0^+ \times R$, and we have

$$D_2 F(t, 0, a)\varphi = y_t$$

where $y\colon [-1, \infty) \to R$ is the solution of the linear equation

$$(a \cdot \mathrm{id}) \qquad \dot{y}(t) = ay(t-1)$$

with $y_0 = \varphi$. The semigroups $T(\cdot, \cdot, a)\colon$ $(t, \varphi) \to T(t, \varphi, a) = y_t = D_2 F(t, 0, a)\varphi$, $a \in R$, are strongly continuous. The spectra $\sigma(a)$ of their generators are given by the characteristic equation, i.e. by the zeros of the analytic function

$$z \to z - ae^{-z}$$

(the characteristic equation follows from the Ansatz $y(t) = e^{zt}$ for a complex-valued solution of *eq.* $(a \cdot \mathrm{id})$). For each $a > 0$ there is a unique positive zero $u(a)$. Every such $u(a)$ is a simple zero. The map $0 < a \to u(a) \in R^+$ is analytic and strictly increasing, with $u(a) \downarrow 0$ as $a \downarrow 0$. We have

(2.1) $$\mathrm{Re}\, z < \log a \quad \text{for all } a > 0 \text{ and all } z \in \sigma(a) \setminus \{u(a)\},$$

see Proposition 2 [15]. One can show that for $0 < a < 3\pi/2$, every $z \in \sigma(a) \setminus \{u(a)\}$ satisfies $\mathrm{Re}\, z < 0$ while $\pm 3\pi i/2 \in \sigma(3\pi/2)$. Note also $0 \in \sigma(0)$. It follows that for all a in the interval $(0, 3\pi/2)$ there is a decomposition

$$C = P_a \oplus Q_a$$

into $T(\cdot, \cdot, a)$-invariant closed subspaces $P_a = R\Phi_a$ and Q_a, where

$$\Phi_a(t) = e^{u(a)t} \quad \text{for } t \in [-1, 0].$$

Nonconstant trajectories in the "unstable space" P_a grow exponentially, trajectories in the "stable space" Q_a decay exponentially as $t \to +\infty$. (For $a = 0$ and $a = 3\pi/2$ there are center spaces, containing nontrivial bounded trajectories $R \to C$.)

Let p_a and q_a denote the projections onto P_a and Q_a, defined by the decomposition above. Note that for $\varphi \in P_a$,

(2.2) $$\varphi = \varphi(0)\Phi_a, \quad |\varphi| = |\varphi(0)|$$

so that p_R: $\varphi \to \varphi(0)$ projects P_a onto R, with $p_R \Phi_a = 1$.

In the sequel, the symbols $B_0, B_1, B_2, \ldots, \underline{B}_1, \underline{B}_2, \ldots$ always denote open balls in C with center 0. For the statement of the bifurcation theorem and for the next sections we need the following version of the saddle point property for the parameterized semiflow F:

Let an open set $A \Subset (0, 3\pi/2)$ and some B_0 be given. There exist B_1, B_2 with $B_2 \subset B_1$, constants $c > 0$, $\gamma > 0$ and maps u_a: $P_a \cap B_1 \to Q_a$, s_a: $Q_a \cap B_1 \to P_a$; $a \in A$; with the following properties.

I. For every $a \in A$ we have that

(i) $u_a(0) = 0$, $s_a(0) = 0$. u_a and s_a satisfy a Lipschitz condition with constant 1/2. The graphs

$$U_a := \{\varphi + u_a(\varphi) \colon \varphi \in P_a \cap B_1\} \quad \text{and}$$

$$S_a := \{s_a(\varphi) + \varphi \colon \varphi \in Q_a \cap B_1\}$$

are tangent to P_a and Q_a, respectively, at $\varphi = 0$.

(ii) For every $\varphi \in P_a \cap B_1$ there is a unique trajectory $X^* = X^*(\varphi + u_a(\varphi))$: $R \to C$ of $F(\cdot, \cdot, a)$ with $X_0^* = \varphi + u_a(\varphi)$ and $|X_t^*| \leqslant ce^{\gamma t}|\varphi|$ for all $t \leqslant 0$. If X: $R \to C$ is a trajectory of $F(\cdot, \cdot, a)$ with $X_t \in B_2$ for all $t \leqslant 0$ then $X = X^*(\varphi + u_a(\varphi))$ for some $\varphi = P_a \cap B_1$.

(iii) $\varphi \in Q_a \cap B_1$ implies $|F(t, s_a(\varphi)+\varphi, a)| \leqslant ce^{-\gamma t}|\varphi|$ for all $t \geqslant 0$. If $\varphi \in C$ and $F(t, \varphi, a) \in B_2$ for all $t \geqslant 0$ then $\varphi = s_a(\psi)+\psi$ for some $\psi \in Q_a \cap B_1$.

II. For every $a \in A$, $p_a B_2 \subset B_1$ and $q_a B_2 \subset B_1$. The maps

$$B_2 \times A \ni (\varphi, a) \to u_a(p_a \varphi) \in C, \quad B_2 \times A \ni (\varphi, a) \to s_a(q_a \varphi) \in C$$

are of class C^2.

Here is the

BIFURCATION THEOREM. *Let $a_0 \in (0, 3\pi/2)$ be given with*

$$\operatorname{Re} z < -u(a_0) \quad \textit{for all } z \in \sigma(a_0) \setminus \{u(a_0)\}.$$

Let $A \Subset (0, 3\pi/2)$ be an open neighborhood of a_0. Let balls $B_2 \subset B_1$, constants $c > 0$ and $\gamma > 0$ and two families of maps u_a: $P_a \cap B_1 \to Q_a$, s_a: $Q_a \cap B_1 \to P_a$; $a \in A$; with properties I *and* II *be given.*

Suppose there exist $\eta \in R$ with $\eta \Phi_a \in B_1$ for all $a \in A$, a ball $B_3 \subset B_2$, $t_+ \in R$, $t_- < t_+$, $\varepsilon > 0$ such that the trajectory $X^0 = X^*(\eta \Phi_{a_0} + u_{a_0}(\eta \Phi_{a_0}))$ of $F(\cdot, \cdot, a_0)$ is homoclinic with

$$p_R \, p_{a_0} X_t^0 < p_R \, s_{a_0}(q_{a_0} X_t^0) \quad \textit{for all } t \leqslant t_-$$

and

$$X_t^0 \in B_3 \quad \textit{for all } t \geqslant t_+ \quad (\textit{and } \lim_{t \to +\infty} X_t^0 = 0),$$

while for every $a \in A$ with $a_0 - \varepsilon < a < a_0$ the trajectories $X^a = X^*(\eta \Phi_a + u_a(\eta \Phi_a))$ of $F(\cdot, \cdot, a)$ satisfy

$$p_R \, p_a X_{t_+}^a < p_R \, s_a(q_a X_{t_+}^a).$$

Then there exist a neighborhood V of $\{X_t^0: t \in R\} \cup \{0\}$, an open neighborhood $A' \subset A$ of a_0 and a differentiable curve

$$A' \ni a \to \varphi_a \in C$$

such that

(i) $\varphi_{a_0} \in \{X_t^0: t \in R\}$,

(ii) *for every $a \in A'$ with $a < a_0$ there is a periodic trajectory Y^a: $R \to C$ of $F(\cdot, \cdot, a)$ with $Y_0^a = \varphi_a$ and $\{Y_t^a: t \in R\} \subset V$; Y^a is orbitally asymptotically stable with asymptotic phase,*

(iii) *there is no periodic trajectory Y: $R \to C$ of $F(\cdot, \cdot, a_0)$ with orbit $\{Y_t: t \in R\}$ in V,*

(iv) *for every $a \in A'$ with $a < a_0$ and for every periodic trajectory Y: $R \to C$ of $F(\cdot, \cdot, a)$ with $\{Y_t: t \in R\} \subset V$ there exists $t \in R$ with $Y_s = Y_{t+s}^a$ for all $s \in R$.*

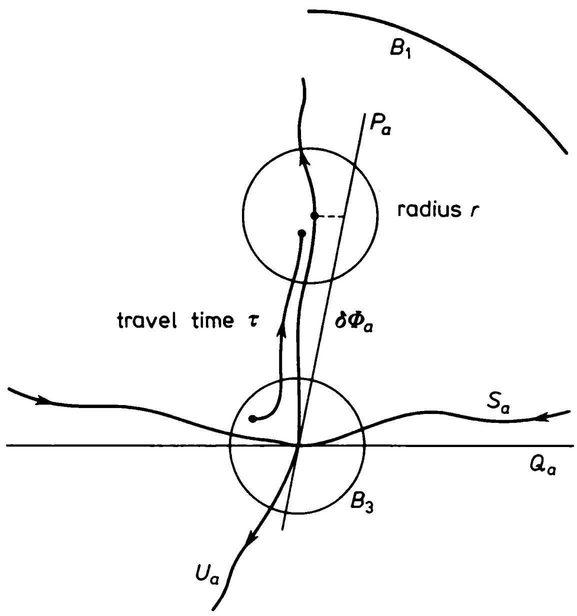
This can be proved exactly as Theorem 13.2 [17]. – Exponential estimates (not contained in the saddle point property above) imply the following lemma which will be useful for the verification of the hypotheses in the bifurcation theorem.

LEMMA 2.1. *Let an open interval $A \Subset (0, 3\pi/2)$ be given. Consider $B_2 \subset B_1$, constants $c > 0$, $\gamma > 0$ and maps u_a: $P_a \cap B_1 \to Q_a$, s_a: $Q_a \cap B_1 \to P_a$; $a \in A$; with properties* I *and* II.

Let $a_0 \in A$ be given. There exist $\delta_0 > 0$ and an open neighborhood $\tilde{A} \subset A$ of a_0 such that for every $\delta \in (0, \delta_0)$ and $r > 0$ there is an open ball $B_3 \subset B_2$ with the following properties.

For every $(\varphi, a) \in B_3 \times \tilde{A}$, $\delta\Phi_a \in B_2$ and $\delta\Phi_a + u_a(\delta\Phi_a) \in B_1$, $q_a\varphi \in B_1$, and $p_R p_a \varphi > p_R s_a(q_a \varphi)$ implies that there is $\tau = \tau(\varphi, a) > 0$ with $|F(\tau, \varphi, a) - [\delta\Phi_a + u_a(\delta\Phi_a)]| < r$ and with $F(t, \varphi, a) \in B_1$ for all $t \in [0, \tau]$.

A proof would be lengthy, without unexpected arguments, so it is omitted here. Readers interested in the technical details are referred to the first sections of [17].



3. Nonlinearities with two zeros

We begin with a parameter $a_+ \in (0, 1)$ such that

(3.1) for all $a \in (0, a_+]$ and $z \in \sigma(a) \setminus \{u(a)\}$, $\operatorname{Re} z < \log a < -u(a)$.

This is possible because of (2.1). Let g: $R \to R$ be a C^2-function as before, with $g(0) = 0$ and $g'(0) = 1$. Assume furthermore that g has another zero at 2ξ, $\xi < 0$, with

(3.2) $0 < g$ on $(3\xi, 2\xi)$, $g < 0$ on $(2\xi, 0)$, $0 < g$ on $(0, -\xi]$,

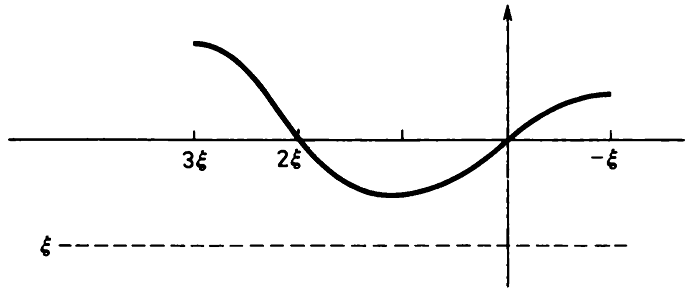
that g is bounded with

(3.3) $$\xi < a_+ g,$$

(3.4) $$a_+ g'(2\xi) < -1,$$

and that there is some $\varkappa \in (0, 1)$ with

(3.5) $$|a_+ g(\eta)| \leqslant \varkappa |\eta| \quad \text{for all } \eta \in [\xi, -\xi].$$



Condition (3.4) will only be used for the proof of part (ii) of the next proposition. It can be weakened to

$$a_+ g'(2\xi) < -1/e,$$

at the cost of a lengthy argument which seems not worth while here. The role of condition (3.4) will soon become clearer when we modify g on the interval $(-\infty, 2\xi)$ in order to start the search for the homoclinic solution.

PROPOSITION 3.1. *Consider a solution x of eq. (ag) with a in $(0, a_+]$. Let $t \in R$.*

(i) *If $2\xi < x_t < 0$ then either $2\xi < x < 0$ and $\dot{x} < 0$ on $(t, +\infty)$, and $\lim_{s \to +\infty} x(s) = 2\xi$, or*

(*) *there exists $z > t$ with $x(z) = 2\xi$, $\dot{x} < 0$ on $(t, z+1)$, $3\xi < x$ on $(t, z+1]$.*

(ii) *$a = a_+$ and $2\xi < x_t < 0$ imply (*).*

(iii) *$x_t(0) \geqslant -\xi$ implies $x > 0$ on $[t, +\infty)$.*

(iv) *If $0 < x_t < -\xi$ then there exists $s > t$ with $0 < \dot{x}$ on $(t, s]$, $x(s) = -\xi$, $0 < x$ on $[t-1, +\infty)$.*

Proof. (i) Note that for all $s > t$ with $2\xi < x < 0$ on $[t, s]$, $\dot{x} < 0$ on $(t, s+1]$, by (3.2), and for $v \in [s, s+1]$, $x(v) = x(s) + \int_s^v ag(x(s'-1))ds' > 2\xi + \xi$, by (3.3). In case $2\xi < x < 0$ on $[t-1, +\infty)$, (3.2) yields $\dot{x} < 0$ on $(t, +\infty)$. Set $b := \lim_{s \to +\infty} x(s)$. Then $2\xi \leqslant b < 0$. By *eq.* (*ag*), $\lim_{s \to +\infty} \dot{x}(s) = ag(b)$. $2\xi < b$ would imply $\lim_{s \to +\infty} \dot{x}(s) \neq 0$ which is impossible because of $2\xi < x < 0$ on $[t-1, +\infty)$. – Therefore $b = 2\xi$.

(ii) Follows from (3.4), by means of the elementary Lemma 2.3 in [11].

(iii) Suppose $x(s) \leqslant 0$ for some $s > t$. Then there are v, w with $t \leqslant v < w \leqslant s$, $x(v) = -\xi$, $x(w) = 0$, $0 < x < -\xi$ on (v, w). By (3.3), $\dot{x} > \xi$ on (v, w). Therefore $w > v+1$, and $0 < x(v+1)$. By (3.2), $0 < \dot{x}$ on $(v+1, w)$; we arrive at a contradiction to $x(w) = 0$.

(iv) For each $v > t$ with $0 < x \leqslant -\xi$ on $[t-1, v]$, $0 < \dot{x}$ on $(t, v+1]$, by (3.2). Moreover, $\dot{x} > \min ag|[\min x_t, -\xi] > 0$ on $(t, v+1]$. This yields existence of $s > t$ with $0 < \dot{x}$ on $(t, s+1]$ and $x(s) = -\xi$, $0 < x$ on $[t-1, s]$. Use (iii).

Now let us examine the lower branch of the local unstable manifold at $0 \in C$, for $a = a_+$. We employ the saddle point property as stated in the preceding section, for the semiflow $G(\cdot, \cdot, a_+)$: $R_0^+ \times C \to C$ defined by solutions of eq. $(a_+ g)$: It follows that there exist balls $\underline{B}_2 \subset \underline{B}_1 \subset \{\varphi \in C: |\varphi| < -\xi\}$, constants $\underline{c} > 0$ and $\underline{\gamma} > 0$, and maps $\underline{u}$: $P_{a_+} \cap \underline{B}_2 \to Q_{a_+}$, $\underline{s}$: $Q_{a_+} \cap \underline{B}_2 \to P_{a_+}$ such that the analogues of I and II hold.

Choose $\underline{\eta} < 0$ with $\underline{\eta}\Phi_{a_+} \in \underline{B}_2$ so small that the trajectory $X^* = X^*(\underline{\eta}\Phi_{a_+} + \underline{u}(\underline{\eta}\Phi_{a_+}))$: $R \to C$ of the semiflow $G(\cdot, \cdot, a_+)$ satisfies $X_t^* \in \underline{B}_2$ for all $t \leqslant 0$. This is possible by the estimate of X^* in I (ii). Define $x(t) := X_t^*(0)$ for all $t \in R$. x is a solution of eq. $(a_+ g)$ with $x_t = X_t^*$ for all $t \in R$ and $\lim_{t \to -\infty} x(t) = 0$.

PROPOSITION 3.2. *There exists* $z > 0$ *with* $2\xi < x < 0$ *on* $(-\infty, z)$, $x(z) = 2\xi$, $\dot{x} < 0$ *on* $(-\infty, z+1)$.

Proof. (We omit the index a_+.)

(a) For each $t \leqslant 0$, $qX_t^* = \underline{u}(pX_t^*)$. Proof: Let $t \leqslant 0$. X: $R \ni s \to X_{s+t}^* \in C$ is a trajectory of $G(\cdot, \cdot, a_+)$ with $X_s \in \underline{B}_2$ for all $s \leqslant 0$. By the analogue of I(ii), $X = X^*(\varphi + \underline{u}(\varphi))$ for some $\varphi \in P \cap \underline{B}_1$. Hence $X_t^* = X_0 = \varphi + \underline{u}(\varphi)$.

(b) $X_t^* \neq 0$ for all $t \leqslant 0$, since otherwise $0 = pX_0^* = \underline{\eta}\Phi$, a contradiction to $\underline{\eta} < 0$.

(c) $pX_t^* \neq 0$ for all $t \leqslant 0$, since otherwise for some $t \leqslant 0$: $0 = \underline{u}(pX_t^*)$, $0 = pX_t^* + \underline{u}(pX_t^*) = X_t^*$ (by (a)), a contradiction to (b).

(d) Recall $pX_t^* = (pX_t^*)(0)\Phi$, for all $t \in R$, and $\Phi > 0$. By continuity, (c) now gives that either $pX_t^* < 0$ for all $t \leqslant 0$, or $pX_t^* > 0$ for all $t \leqslant 0$. $pX_0^* = \underline{\eta}\Phi < 0$ excludes the second possibility. – $\lim_{P \ni \varphi \to 0, \varphi \neq 0} |\underline{u}(\varphi)|/|\varphi| = 0$ allows to find $\delta > 0$ with $2|\underline{u}(\varphi)| \leqslant |\varphi| \cdot e^{-u(a_+)}$ for all $\varphi \in P$ with $|\varphi| < \delta$. By $|\varphi| \cdot e^{-u(a_+)} = |\varphi(0)| e^{-u(a_+)} = \min_{t \in [-1, 0]} |\varphi(t)|$ for $\varphi \in P$, we obtain $\varphi + \underline{u}(\varphi) < 0$ whenever $\varphi \in P$ and $-\delta < \varphi < 0$. With $\lim_{t \to -\infty} X_t^* = 0$ and $pX_t^* < 0$ for $t \leqslant 0$ and (a), we get $x_t = X_t^* < 0$ for all t in some interval $(-\infty, s]$, where $s \leqslant 0$. $x_t = X_t^* \in \underline{B}_2$

for all $t \leqslant 0$ gives $\xi < x(t)$ for all $t \leqslant 0$. Apply parts (i) and (ii) of Proposition 3.1.

Observe that the restriction of x to the interval $(-\infty, z+1]$ does not depend on the values of g on $(-\infty, 2\xi)$ while for t in the interval $[z+1, z+2]$,

$$x(t) = x(z+1) + \int_{z+1}^{t} a_+ g(x(s-1))ds$$

depends on t, the restriction of x to $(z, z+1]$ – where $x < 2\xi$, and on the values of g on $(-\infty, 2\xi)$ only.

We replace g, if necessary, by a bounded C^2-function f with

$$f = g \quad \text{on } [2\xi, +\infty), \tag{3.6}$$

$$\xi < a_+ f, \tag{3.7}$$

$$0 < f \quad \text{on } (3\xi, 2\xi), \tag{3.8}$$

so that

$$-\xi < x(z+1) + \int_{z}^{z+1} a_+ f \circ x. \tag{3.9}$$

Let $\tilde{x}$: $[-1, +\infty) \to R$ be the solution of *eq.* $(a_+ f)$ which coincides with x_{z+1} on $[-1, 0]$. Set $\underline{x}(t) := x(t)$ for all $t \leqslant z+1$, $\underline{x}(t) := \tilde{x}(t-(z+1))$ for all $t > z+1$. $\underline{x}$: $R \to R$ is a solution of *eq.* $(a_+ f)$ with $\lim_{t \to -\infty} \underline{x}(t) = 0$, $2\xi < \underline{x} < 0$ on $(-\infty, z)$, $\underline{x}(z) = 2\xi$, $\dot{\underline{x}} < 0$ on $(-\infty, z+1)$ and with

$$-\xi < \underline{x}(z+2).$$

As in the proof of part (iii) of Proposition 3.1 we infer

$$0 < \underline{x} \quad \text{on } [z+2, +\infty).$$

Let us also note

$$3\xi < \underline{x}(z+1)$$

(since $\xi < a_+ g(x(t-1)) = a_+ f(\underline{x}(t-1)) = \dot{\underline{x}}(t)$ for $z \leqslant t \leqslant z+1$) and

$$0 < \dot{\underline{x}} \quad \text{on } (z+1, z+2],$$

by (3.8) and $3\xi < \underline{x} < 2\xi$ on $(z, z+1]$.

Remark. We changed g to f in order to obtain $-\xi < \underline{x}(z+2)$. This change of the nonlinearity, which depends on $x|(-\infty, z+1]$, can be avoided. Alternatively, one can write down somewhat technical a priori conditions on g which ensure $-\xi < \underline{x}(z+2)$ – compare the hypotheses in section 2 of [15].

4. Critical parameter and homoclinic solution

We fix a_- in $(0, a_+)$ so small that

(4.1) $$a_- f < -\xi,$$

choose an open neighborhood $A \Subset (0, 3\pi/2)$ of $[a_-, a_+]$ and set $B_0: = \{\varphi \in C: |\varphi| < -\xi\}$. The first aim is a suitable continuous family of lower branches of local unstable manifolds at $0 \in C$.

There are open balls $B_2 \subset B_1 \subset B_0$, constants $c > 0$ and $\gamma > 0$, and two families of maps u_a: $P_a \cap B_1 \to Q_a$, s_a: $Q_a \cap B_1 \to P_a$; $a \in A$; with properties I and II from section 2, now for the parameterized semiflow F defined by solutions of the equations (af), $a \in R$.

Pick $\eta < 0$ so small that for every $a \in A$, $\eta\Phi_a \in B_2$, and that the trajectory $X^* = X^*(\eta\Phi_a + u_a(\eta\Phi_a))$: $R \to C$ of the semiflow $F(\cdot, \cdot, a)$ satisfies $X_t^* \in B_2$ for all $t \leqslant 0$. This is possible by the estimate of X^* in I (ii). Define $x^a(t) := X_t^*(0)$ for all $t \in R$. Exactly as in Section 3 we infer that for each $a \in A$, x^a is a solution of eq. (af) with $\lim_{t \to -\infty} x^a(t) = 0$, $2\xi < x^a < 0$ on $(-\infty, 0]$, $\dot{x}^a < 0$ on $(-\infty, 1]$.

Corollary 4.1. *For every $a \in A$ there exists $t_- = t_-(a) \in R$ with $p_a x_t^a < s_a(q_a x_t^a)$ for all $t \leqslant t_-$.*

Proof. Let $a \in A$. The exponential estimate in I (ii) permits to choose an open ball B_4 so small that for (t, φ) in $R_0^+ \times (S_a \cap B_4)$, $|F(t, \varphi, a)| < |x_0^a|$. As in the proof of Proposition 3.2 we see that there exists $t_1 < 0$ such that for all $t \leqslant t_1$, $q_a x_t^a = u_a(p_a x_t^a)$ and $p_a x_t^a < 0$. Choose $t_2 < t_1$ with $x_t^a \in B_4$ for all $t \leqslant t_2$. Consider $t \leqslant t_2$. $x_t^a \in S_a \cap B_4$ would imply $|x_0^a| = |F(-t, x_t^a, a)| < |x_0^a|$, a contradiction. Then either $p_a x_t^a < s_a(q_a x_t^a)$, or $s_a(q_a x_t^a) < p_a x_t^a < 0$. The latter would imply $0 < -p_a x_t^a < -s_a(q_a x_t^a) \leqslant |s_a(u_a(p_a x_t^a))| \leqslant |p_a x_t^a|/4$, a contradiction.

Property II implies that the map

(4.2) $$A \ni a \to x_0^a = \eta\Phi_a + u_a(\eta\Phi_a) \in C \text{ is continuous.}$$

Let us write P_+, Q_+, p_+, q_+, Φ_+, u_+, s_+ instead of $P_{a_+}, \ldots$ from now on. $x^+ := x^{a_+}$ is merely a translate of x, of course. We show

Proposition 4.1. (i) *There are reals $s < 0$, $t < 0$ with $x_t^+ = \underline{x}_s$.*

(ii) *There exists $z^+ > 0$ with $2\xi < x^+ < 0$ on $(-\infty, z^+)$, $x^+(z^+) = 2\xi$, $\dot{x}^+ < 0$ on $(-\infty, z^+ + 1)$, $3\xi < x^+$ on $(-\infty, z^+ + 1]$, $0 < \dot{x}^+$ on $(z^+ + 1, z^+ + 2]$, $-\xi < x^+(z^+ + 2)$, $0 < x^+$ on $[z^+ + 2, +\infty)$.*

Proof. (i) $\lim_{t \to -\infty} \underline{x}(t) = 0$ and I (ii) imply that there exists $t_1 < 0$ with $q_+ x_t = u_+(p_+ x_t)$ for all $t \leqslant t_1$, compare part (a) from the proof of Proposition 3.2. Now $x < 0$ on $(-\infty, 0]$ gives $p_+ x_t \neq 0$ for all $t \leqslant t_1$; i.e. $p_+ x_t < 0$ or

$0 < p_+ x_t$ for each of these t. As in the proof of Proposition 3.2, we have some $\delta = \delta(a_+) > 0$ such that $\varphi \in P_+$ and $0 < \varphi < \delta$ imply $0 < \varphi + u_+(\varphi)$, while $\varphi \in P_+$ and $-\delta < \varphi < 0$ imply $\varphi + u_+(\varphi) < 0$. With $\lim_{t \to -\infty} p_+ x_t = 0$ and with $p_+ x_t + u_+(p_+ x_t) = x_t < 0$ and $p_+ x_t \neq 0$ for all $t \leqslant t_1$, we infer that there exists $t_2 \leqslant t_1$ with $p_+ x_t < 0$ for all $t \leqslant t_2$, and by continuity,

$$\{p_+ x_t:\ t \leqslant t_2\} \supset (\tilde{\eta}, 0) \cdot \Phi_+ \qquad \text{for some } \tilde{\eta} < 0.$$

In the same way we find $s_2 < 0$ and $\eta^* \in (\tilde{\eta}, 0)$ with

$$\{p_+ x_s^+:\ s \leqslant s_2\} \supset (\eta^*, 0) \cdot \Phi_+, \qquad \text{and } q_+ x_s^+ = u_+(p_+ x_s^+) \qquad \text{for all } s \leqslant s_2.$$

Hence there are $t \leqslant t_2 < 0$ and $s \leqslant s_2 < 0$ with $p_+ x_t = \eta^* \Phi_+ = p_+ x_s^+$, and consequently $x_t = x_s^+$.

(ii) By (i),

$$x^+(v+t) = x(v+s) \qquad \text{for all } v \geqslant -1.$$

Now the properties of x stated at the end of section 3 and the properties of x^+ stated above imply the assertions.

For $a \in [a_-, a_+]$, set

$$z_1(a) := \inf\{t \in R:\ x^a(t) \leqslant 2\xi\} \leqslant +\infty,$$

$$z_2(a) := \inf\{t > z_1(a):\ 2\xi \leqslant x^a(t)\} \leqslant +\infty.$$

Proposition 4.2. (i) *If* $z_1(a) = +\infty$ *then* $2\xi < x^a < 0$ *and* $\dot{x}^a < 0$ *on* R, *and* $\lim_{t \to +\infty} x^a(t) = 2\xi$.

(ii) *If* $z_1(a) < +\infty$ *then* $0 < z_1(a)$, $2\xi < x^a < 0$ *on* $(-\infty, z_1(a))$,

$$x^a(z_1(a)) = 2\xi,\ \dot{x}^a < 0 \qquad on\ (-\infty, z_1(a)+1),$$

$$3\xi < x^a(z_1(a)+1) \leqslant x^a < 2\xi \qquad on\ (z_1(a), z_1(a)+1].$$

(iii) *If* $z_1(a) < +\infty$ *and* $z_2(a) = +\infty$ *then* $x^a < 2\xi$ *on* $(z_1(a), +\infty)$,

$$0 < \dot{x}^a \qquad on\ (z_1(a)+1, +\infty), \qquad \lim_{t \to +\infty} x^a(t) = 2\xi.$$

(iv) *If* $z_1(a) < +\infty$ *and* $z_2(a) < +\infty$ *then* $z_2(a) > z_1(a)+1$,

$$3\xi < x^a < 2\xi \qquad on\ (z_1(a), z_2(a)), \qquad x^a(z_2(a)) = 2\xi,$$

$$0 < \dot{x}^a \qquad on\ [z_1(a)+1, z_2(a)+1).$$

(v) $z_1(a_+) = z^+ < +\infty$, $z_2^+ := z_2(a_+) < +\infty$, $x^+(z_2^+ + 1) > -\xi$,

$$x^+ > 0 \qquad on\ [z_2^+ + 1, +\infty).$$

(vi) $x^- := x^{a_-} < 0$ *on* R.

Proof. (i) $z_1(a) = +\infty$ means that $2\xi < x^a$. Recall $2\xi < x^a < 0$ on $(-\infty, 0]$, and $f = g$ on $[2\xi, +\infty)$. Apply Proposition 3.1 (i).

(ii) follows similarly. For the last estimate, see (3.3).

(iii). By (ii), $3\xi < x^a$ on $[z_1(a), z_1(a)+1]$. Recall $0 < af$ on $(3\xi, 2\xi)$. Proceed as in the proof of part (i) of Proposition 3.1.

(iv). $z_1(a) < +\infty$ and (ii) yield $z_1(a)+1 < z_2(a)$ and $3\xi < x^a < 2\xi$ on $(z_1(a), z_1(a)+1]$. Now $0 < af$ on $(3\xi, 2\xi)$ gives $0 < \dot{x}^a$ on $(z_1(a)+1, z_2(a)+1)$.

(v). Proposition 4.1 (ii) shows $z^+ = z_1(a_+)$. $\dot{x}^+ < 0$ on $[z^+, z^+ + 1)$ and $-\xi < x^+(z^+ + 2)$ imply $z_2(a_+) < z^+ + 2 < +\infty$. $z^+ + 1 < z_2(a_+)$ gives $z^+ + 2 < z_2(a_+)+1$. By $0 < \dot{x}^+$ in $(z^+ + 1, z_2(a_+)+1)$ and by $-\xi < x^+(z^+ + 2)$, $-\xi < x^+(z_2(a_+)+1)$. Use Proposition 3.1 (iii) and $f = g$ on $[2\xi, +\infty)$.

(vi). In case $z_1(a_-) = +\infty$, apply (i). In case $z_1(a_-) < +\infty$, $x^{a_-} < 0$ on $(-\infty, z_1(a_-))$ and $x^{a_-}(z_1(a_-)) = 2\xi$. We show $x^{a_-} < \xi$ on $[z_1(a_-), +\infty)$: Otherwise, there exist $s \geqslant z_1(a_-)$, $t > s$ with $x_{a_-}(t) = \xi$, $x^{a_-}(s) = 2\xi$, $2\xi < x^{a_-} < \xi$ on (s, t). Hence $0 \leqslant \dot{x}^{a_-}(t)$. This excludes $s < t-1$ since $a_- f < 0$ on $(2\xi, \xi)$. By the choice of a_- (see (4.1)), $-\xi = x^{a_-}(t) - x^{a_-}(s) = \int_s^t a_- f(x^{a_-}(v-1))\,dv < (t-s)\cdot(-\xi)$, a contradiction.

PROPOSITION 4.3. *There exists $a_* \in [a_-, a_+)$ such that for all $a \in [a_*, a_+]$, $z_1(a) < +\infty$ and $z_2(a) < +\infty$, and $x^{a_*}(z_2(a_*)+1) < 0$. The maps $[a_*, a_+] \ni a \to z_1(a) \in R$ and $[a_*, a_+] \ni a \to z_2(a) \in R$ are continuous.*

Proof. (a) The set $A_1 := \{a \in [a_-, a_+]:\ z_1(a) < +\infty\}$ is open in $[a_-, a_+]$ (use $\dot{x}^a(z_1(a)) < 0$), as well as the set

$$A_2 := \{a \in A_1:\ z_2(a) < +\infty\}.$$

Continuity of the parameterized semiflow, (4.2) and

$$x^a(z_1(a)) = 2\xi, \qquad \dot{x}^a(z_1(a)) < 0 \qquad \text{for } a \in A_1$$

imply that the map $A_1 \ni a \to z_1(a) \in R$ is continuous. By a similar argument, $A_2 \ni a \to z_2(a) \in R$ is continuous, too. We have $[a_-, a_+] \supset A_1 \supset A_2 \ni a_+$.

(b) In case $A_2 = [a_-, a_+]$, set $a_* := a_-$. By Proposition 4.2 (vi), $x^{a_*}(z_2(a_*)+1) < 0$.

(c) The case $A_2 \subsetneqq [a_-, a_+]$. Openness and $a_+ \in A_2$ imply that there exists $a' \in [a_-, a_+)$ with $(a', a^+] \subset A_2$ and $a' \notin A_2$. Choose $\varepsilon_1 > 0$ with $\max a_+ f|[2\xi - \varepsilon_1, 2\xi] < -2\xi$, and $\varepsilon_2 > 0$ with $-\varepsilon_1 < \min a_+ f|[2\xi, 2\xi + \varepsilon_2]$.

(c.1) Suppose $a' \notin A_1$. Then $z_1(a') = +\infty$, $2\xi < x^{a'} < 0$ on R, $\lim\limits_{t \to +\infty} x^{a'}(t)$

$= 2\xi$. Choose $s > 1$ with $2\xi < x_s^{a'} < 2\xi + \varepsilon_2$. By continuity, there exists $a_* \in (a', a_+)$ with $2\xi < x^{a_*} < 0$ on $(-\infty, s]$, $2\xi < x_s^{a_*} < 2\xi + \varepsilon_2$. We have $z_1(a_*) < +\infty$, and $z_2(a_*) < +\infty$, by the choice of a'. It follows that $s < z_1(a_*)$. With $\dot{x}^{a_*} \leq 0$ on $(s, z_1(a_*)]$, $2\xi < x^{a_*} < 2\xi + \varepsilon_2$ on $[z_1(a_*) - 1, z_1(a_*))$. By eq. $(a_* f)$, $-\varepsilon_1 < \dot{x}^{a_*}$ on $(z_1(a_*), z_1(a_*)+1]$. Hence $2\xi - \varepsilon_1 < x^{a_*} < 2\xi$ on $(z_1(a_*), z_1(a_*)+1]$. With $0 < \dot{x}^{a_*}$ on $(z_1(a_*)+1, z_2(a_*)]$, $2\xi - \varepsilon_1 < x^{a_*} < 2\xi$ on $(z_2(a_*)-1, z_2(a_*))$. Equation $(a_* f)$ yields $\dot{x}^{a_*} < -2\xi$ on $(z_2(a_*), z_2(a_*)+1]$. Hence

$$x^{a_*}(z_2(a_*)+1) < 2\xi - 2\xi = 0.$$

(c.2) Suppose $a' \in A_1$. Then $z_1(a') < +\infty$ but $z_2(a') = +\infty$, $x^{a'} < 2\xi$ on $(z_1(a'), +\infty)$, $\lim_{t \to +\infty} x^{a'}(t) = 2\xi$. Choose $s > z_1(a')+1$ with $2\xi - \varepsilon_1 < x_s^{a'} < 2\xi$. By continuity, there exists $a_* \in (a', a_+)$ with $s > z_1(a_*)+1$, $x^{a_*} < 2\xi$ on $(z_1(a_*), s]$, $2\xi - \varepsilon_1 < x_s^{a_*} < 2\xi$. We have $z_2(a_*) < +\infty$, by the choice of a'.

It follows that $s < z_2(a_*)$. $0 < \dot{x}^{a_*}$ on $(z_1(a_*)+1, z_2(a_*)]$ gives $2\xi - \varepsilon_1 < x^{a_*} < 2\xi$ on $[z_2(a_*)-1, z_2(a_*))$. Therefore $\dot{x}^{a_*} < -2\xi$ on $[z_2(a_*), z_2(a_*)+1)$, and $x^{a_*}(z_2(a_*)+1) < 0$.

Finally, consider the set

$$A_0 := \{a \in [a_*, a_+]\text{: there exists } t > z_2(a)+1 \text{ with } x_t^a < 0\}.$$

A_0 is open in $[a_*, a_+]$. We have $a_+ \notin A_0$, by Proposition 4.2 (v), and $a_* \in A_0$, since $2\xi < x^a < 0$ on $(z_2(a), z_2(a)+1]$ implies $\dot{x}^a < 0$ on $(z_2(a)+1, z_2(a)+2]$.

It follows that there exists $a_0 \in (a_*, a_+]$ such that $[a_*, a_0) \subset A_0$ while $a_0 \notin A_0$.

$x^0 := x^{a_0}$ will be the desired homoclinic solution – see Proposition 4.4 below.

For $a \in [a_*, a_0]$ define

$$b_1(a) := \inf\{t \in (z_2(a), z_2(a)+1]\text{: } 0 \leqslant x^a(t)\} \leqslant +\infty$$

(i.e., $b_1(a) = +\infty$ does *not* imply $x^a < 0$ on $(z_2(a), +\infty)$.) We have

(4.3) $\qquad b_1(a) = +\infty$ if and only if $x^a < 0$ on $(z_2(a), z_2(a)+1]$,

by Proposition 4.2 (iv). In particular,

(4.4) $$b_1(a_*) = +\infty,$$

see Proposition 4.2 (iv) and Proposition 4.3. – Again by Proposition 4.2 (iv),

$$(4.5) \qquad b_1(a)\begin{Bmatrix} < \\ = \end{Bmatrix} z_2(a)+1 \text{ implies } 0 \begin{Bmatrix} < \\ = \end{Bmatrix} \dot{x}^a(t) \text{ for } \begin{Bmatrix} \text{all } t \in [b_1(a), z_2(a)+1) \\ t = b_1(a) \end{Bmatrix}.$$

PROPOSITION 4.4. (i) $b_1(a_0) < z_2(a_0)+1$.

(ii) *For all* $t \geqslant b_1(a_0)$, $x^{a_0}(t) \in (\xi, -\xi)$.

(iii) *Let* $x = x^{a_0}$, $z_2 = z_2(a_0)$. *The zeros of x form a sequence of points* $b_n = b_n(a_0)$, $n \in N$, *such that*

$$b_n < b_{n+1} < b_n + 1 < b_{n+2} \text{ for all } n \in N$$

and

$$0 \begin{Bmatrix} < \\ > \end{Bmatrix} \dot{x} \text{ in } \begin{Bmatrix} [b_1, z_2+1) \\ (z_2+1, b_1+1) \end{Bmatrix} \text{ and in } (b_n+1, b_{n+1}+1) \quad \text{for all } \begin{Bmatrix} odd \\ even \end{Bmatrix} n \in N.$$

(iv) $\lim_{t \to +\infty} x^{a_0}(t) = 0$.

Proof. (We omit the index or argument a_0 when convenient.)

(a) *Proof of* (i). Suppose $b_1 = +\infty$. Then $2\xi < x < 0$ on $(z_2, z_2+1]$. Therefore $\dot{x} < 0$ on $(z_2+1, z_2+2]$. Hence $x_{z_2+2} < 0$, a contradiction to $a_0 \notin A_0$. – Suppose $b_1 = z_2+1$. As above, $\dot{x} < 0$ on (z_2+1, z_2+2), $x < 0$ on $(z_2+1, z_2+2]$, and for some $t > z_2+1$ close to z_2+2, $x_t < 0$ which contradicts $a_0 \notin A_0$.

(b) On $[b_1, +\infty)$, $\xi < x$.

Proof: Otherwise there exists $t > z_2+1$ $(> b_1)$ with $x(t) = \xi < 0$ and with $\xi < x$ on $[b_1, t)$. We have $x < -\xi$ on $[b_1, t]$ (since $-\xi \leqslant x(s)$ for some $s \leqslant t$ would imply $0 < x(t)$, compare the proof of Proposition 3.1 (iii)). Observe that for all $s \in [t, t+1]$, $z_2 < s-1 \leqslant t$. Therefore $2\xi < x(s-1) < -\xi$ for such s. With (3.2) and (3.5), we infer that either $2\xi < x(s-1) \leqslant 0$ and $\dot{x}(s) \leqslant 0$, or $0 < x(s-1) < -\xi$ and $0 < \dot{x}(s) = a_0 f(x(s-1)) \leqslant x(s-1) < -\xi$. Hence $x_{t+1} < 0$, a contradiction to $a_0 \notin A_0$.

(c) For every $t \geqslant b_1$ there exists $s \in (t, t+1)$ with $0 < x(s)$. *Proof*: Otherwise $x_{t+1} \leqslant 0$ for some $t \geqslant b_1$. $x(b_1) = 0$, $b_1 < z_2+1$ and $0 < \dot{x}$ on $[b_1, z_2+1)$ imply $t \geqslant z_2+1$. We have $t^* := \sup\{s < t\colon 0 < x(s)\} > -\infty$, $b_1 < z_2+1 < t^* \leqslant t$, $x(t^*) = 0$, and there is a point $t' \in [t^*-1, t^*)$ with $0 < x(t') <$

$-\xi$, $0 < f(x(t'))$; see (3.2). Obviously, $x \leqslant 0$ on $[t^*, t+1]$. We have

$$(*) \qquad x(t'+1) < 0.$$

Proof of this: $t'+1 \in [t^*, t^*+1) \subset [t^*, t+1)$ gives

$$x(t'+1) \leqslant 0. \ x(t'+1) = 0 \quad \text{and} \quad 0 < a_0 f(x(t')) = \dot{x}(t'+1)$$

would result in a contradiction to $x \leqslant 0$ on $[t^*, t+1]$.

With (b), $\xi < x \leqslant 0$ on $[t^*, t^*+1]$. Therefore $\dot{x} \leqslant 0$ on $[t^*+1, t^*+2]$. Using $(*)$ and $t'+1 \in [t^*, t^*+1]$ we infer $x(t^*+2) < x(t^*+1) \leqslant 0$, and $x \leqslant 0$ on $[t^*+1, t^*+2)$. Again by (b), $\xi < x \leqslant 0$ on $[t^*+1, t^*+2]$. Hence $\dot{x} \leqslant 0$ on $[t^*+2, t^*+3]$. With $x(t^*+2) < 0$, $x_{t^*+3} < 0$, a contradiction to $a_0 \notin A_0$.

(d) On $[b_1, +\infty)$, $x < -\xi$. *Proof*: Otherwise, $x(t) \geqslant -\xi$ for some $t \geqslant b_1$, and $x > 0$ on $[t, +\infty)$, compare the proof of Proposition 3.1 (iii). – Suppose $0 < x \leqslant -\xi$ on $[t, +\infty)$. Then $0 < \dot{x}$ on $[t+1, +\infty)$, $\lim_{s \to +\infty} x(s) \in (0, -\xi]$, $\lim_{s \to +\infty} \dot{x}(s) = a_0 f(\lim_{s \to +\infty} x(s)) > 0$, and we obtain a contradiction. – It follows that for some $s \geqslant t$, $x(s) > -\xi$. (4.2) and continuity of the parameterized semiflow F imply that there is a sequence of points a_n in $[a_*, a_0) \subset A_0$, $n \in N$, with $\lim_{n \to +\infty} a_n = a_0$ and $x^{a_n}(s) > -\xi$ for all $n \in N$. Arguing as in the proof of Proposition 3.1 (iii) we get

$$(**) \qquad 0 < x^{a_n} \quad \text{on } [s, +\infty) \quad \text{for all } n \in N.$$

$a_n \in A_0$ implies $x_{t_n}^{a_n} < 0$ for some $t_n > z_2(a_n)+1$. $(**)$ gives $t_n < s$ for all $n \in N$. We have $\lim_{n \to +\infty} z_2(a_n) = z_2(a_0)$, and there is a subsequence $(t_{n(k)})_{k \in N}$ which converges to a limit t' in $[z_2(a_0)+1, s]$. (4.2) and continuity of F yield

$$x_{t_{n(k)}}^{a_{n(k)}} = F(t_{n(k)}, x_0^{a_{n(k)}}, a_{n(k)}) \to F(t', x_0, a_0) = x_{t'}, \quad \text{as } k \to +\infty.$$

Therefore $x_{t'} \leqslant 0$. $0 < x$ on $(b_1(a_0), z_2(a_0)+1]$ and $t' \geqslant z_2(a_0)+1$ imply $t'-1 \geqslant z_2(a_0)+1$, and $x_{t'} \leqslant 0$ with $t' \geqslant z_2(a_0)+2 > b_1(a_0)+1$ is now a contradiction to (c).

(e) *Existence of* b_2: Recall $0 < \dot{x}$ on $[b_1, z_2+1)$ – see (i) and Proposition 4.2 (iv) – which gives $0 < x(z_2+1)$. $2\xi < x < 0$ on (z_2, b_1) yields $\dot{x} < 0$ on (z_2+1, b_1+1). Suppose $0 \leqslant x(b_1+1)$. With (d), $0 < x < -\xi$ on (b_1, b_1+1). Therefore $0 < \dot{x}$ on (b_1+1, b_1+2), and $0 < x$ on $(b_1+1, b_1+2]$. By continuity there exists $t > b_1+2$ close to b_1+2 with $0 < x$ on $[t-1, t]$. Again by (d), $x < -\xi$. Now the analogue of Proposition 3.1 (iv) for f implies a contradiction to (d).

It follows that $x(b_1+1) < 0$, and $\dot{x} < 0$ in (z_2+1, b_1+1), and there exists a unique zero b_2 of x in (z_2+1, b_1+1). Note

$$(***) \qquad 0 < x < -\xi \quad \text{on } (b_1, b_2).$$

(f) *Existence of* b_3: We have $x(b_2) = 0$, $\xi < x < 0$ on $(b_2, b_1+1]$, by the results from (e) and (b). (***) gives $0 < \dot{x}$ on (b_1+1, b_2+1). With (c), applied to $t = b_2$: $x(b_2+1) > 0$. It follows that there exists a unique zero b_3 of x in (b_2, b_2+1), and $0 < \dot{x}$ on (b_1+1, b_2+1).

(g) Now it is obvious how to prove assertion (iii) by induction.

(h) *Proof of* (iv). Let $n \in N$. Assertion (iii) implies

$$\max_{[b_{n+2}, b_{n+3}]} |x| = |x(b_{n+1}+1)| = |x(b_{n+1}+1) - x(b_{n+2})|$$

$$\leqslant \varkappa \int_{b_{n+2}-1}^{b_{n+1}} |x(t)|\,dt \quad \text{(with } |x| < -\xi \text{ on } [b_1, +\infty),\ b_1+1 < b_{n+2}, \text{ and with (3.5) and } a_0 \leqslant a_+)$$

$$\leqslant \varkappa \int_{b_n}^{b_{n+1}} |x(t)|\,dt \quad \text{(since } b_n+1 < b_{n+2})$$

$$\leqslant \varkappa \max_{[b_n, b_{n+1}]} |x| \quad \text{(since } b_{n+1} - b_n < 1).$$

Use $\varkappa \in (0, 1)$.

5. Verification of the hypotheses for bifurcation

A. The linearization $(t, \varphi) \to D_2 F(t, 0, a_0)\varphi$ of the nonlinear semiflow $F(\cdot, \cdot, a_0)$ at the stationary point $0 \in C$ coincides with the strongly continuous semigroup $T(\cdot, \cdot, a_0)$ from section 2 so that (3.1) and $0 < a_0 \leqslant a_+$ yield the hypothesis on the spectrum of the generator of $(t, \varphi) \to D_2 F(t, 0, a_0)\varphi$.

B. Recall that for all a in the open neighborhood $A \Subset (0, 3\pi/2)$ of a_0, $\eta\Phi_a \in B_2$ and $\eta\Phi_a + u_a(\eta\Phi_a) = X_0^*(\ldots) \in B_2 \subset B_1 \subset \{\varphi \in C\colon |\varphi| < -\xi\}$.

C. Choose $\delta_0 > 0$ according to Lemma 2.1. $\lim_{P_{a_0} \ni \varphi \neq 0 \to 0} |u_{a_0}(\varphi)|/|\varphi| = 0$ implies that there exists $\delta \in (0, \delta_0)$ with $0 < \delta\Phi_{a_0} + u_{a_0}(\delta\Phi_{a_0}) < -\xi$ (compare part d) of the proof of Proposition 3.2). Take $r > 0$ so small that

$$r < \delta\Phi_{a_0} + u_{a_0}(\delta\Phi_{a_0}) < \delta\Phi_{a_0} + u_{a_0}(\delta\Phi_{a_0}) + r < -\xi. \tag{5.1}$$

Consider an open ball $B_3 \subset B_2$ and an open neighborhood $\tilde{A} \subset A$ of a_0, as quaranteed by Lemma 2.1, and furthermore so small that

$$r < \delta\Phi_a + u_a(\delta\Phi_a) < \delta\Phi_a + u_a(\delta\Phi_a) + r < -\xi \text{ for all } a \in \tilde{A}, \tag{5.2}$$

$$F(t, \varphi, a) \in B_1 \text{ for all } (t, \varphi, a) \in R_0^+ \times (B_3 \cap S_a) \times \{a\},\ a \in \tilde{A}. \tag{5.3}$$

Inequality (5.2) can be achieved by means of continuity (see II, Section 2, and (5.1)); (5.3) is obtainable from the exponential estimate in II (iii), Section 2.

$$|\varphi - [\delta\Phi_a + u_a(\delta\Phi_a)]| < r \text{ and } a \in \tilde{A} \text{ imply } 0 < \varphi < -\xi, \tag{5.4}$$

since for all $t \in [-1, 0]$, $0 < [\delta\Phi_a + u_a(\delta\Phi_a)](t) - r < \varphi(t) < [\delta\Phi_a + u_a(\delta\Phi_a)](t) + r < -\xi$, by (5.2).

D. Choose $t_+ > z_2(a_0)+1$ with $x_t^{a_0} \in B_3$ for all $t \geqslant t_+$ (Proposition 4.4).

E. By Corollary 4.1, there exists $t_- < t_+$ with

$$p_R\, p_{a_0} x_t^{a_0} < p_R\, s_{a_0}(q_{a_0} x_t^{a_0}) \quad \text{for all } t \leqslant t_-.$$

F. Recall from Proposition 4.3 that $A_2 \subset A$ is open, and that the map $A_2 \ni a \to z_2(a) \in R$ is continuous. It follows that $A_2 \ni a \to x^a(z_2(a)+1) \in R$ is continuous as well. We have $a_0 \in A_2$ and $0 < x^{a_0}(z_2(a_0)+1)$. Therefore there is an open neighborhood $A' \subset \tilde{A} \cap A_2$ of a_0 with $a^* < \inf A'$, and $0 < x^a(z_2(a)+1)$ for all $a \in A'$. In particular, $b_1(a) < z_2(a)+1 < +\infty$ for all these a, and $0 < \dot{x}^a(b_1(a))$. Following the hints in the proof of Proposition 4.3, we infer that the map $A' \ni a \to b_1(a) \in R$ is continuous.

G. Choice of ε. Proposition 4.4 allows to find β in $(z_2(a_0), b_1(a_0))$ with $\xi < x^{a_0} < -\xi$ on $[\beta, +\infty)$. Continuity of F and the map $A' \ni a \to b_1(a) \in R$ and (4.2) altogether imply that there exists $\varepsilon > 0$ with $(a_0 - \varepsilon, a_0 + \varepsilon)$ contained in A' such that for every $a \in (a_0 - \varepsilon, a_0 + \varepsilon)$,

$$x_{t_+}^a \in B_3, \tag{5.5}$$

$$\beta < b_1(a) < z_2(a_0)+1, \tag{5.6}$$

$$\xi < x^a < -\xi \text{ on } [\beta, t_+]. \tag{5.7}$$

From (5.6) and (5.7),

$$\xi < x^a < -\xi \text{ on } [b_1(a), t_+] \text{ for all } a \in (a_0 - \varepsilon, a_0 + \varepsilon). \tag{5.8}$$

H. For every $a \in (a_0 - \varepsilon, a_0)$ there exists $s > b_1(a)$ with $x^a(s) = \xi$.

Proof. For such a, $a_* < \inf A' \leqslant a_0 - \varepsilon < a < a_0$, so that there exists $t > z_2(a)+1$ with $x_t^a < 0$ (see the construction of a_0 in Section 4; $[a_*, a_0) \subset A_0$). $0 < \dot{x}^a$ on $[z_2(a), z_2(a)+1)$, $b_1(a) < z_2(a)+1$ and $x^a(b_1(a)) = 0$ imply $t > b_1(a)+1$. With (5.8), $\xi < x_t^a < 0$. Now $f < 0$ on $(2\xi, 0)$ yields $x^a(s) = \xi$ for some $s > t > b_1(a)+1$.

I. For all $a \in (a_0 - \varepsilon, a_0)$, $x_{t_+}^a \notin S_a$.

Proof. Let a in $(a_0 - \varepsilon, a_0)$ be given. We have $x_{t_+}^a \in B_3$. Assume $x_{t_+}^a$ is in $B_3 \cap S_a$. By (5.3), $x_t^a \in B_1$ on $[t_+, +\infty)$. With (5.8), $\xi < x^a < -\xi$ on $[b_1(a), +\infty)$, a contradiction to H.

J. For all $a \in (a_0 - \varepsilon, a_0)$, $p_R\, p_a x_{t_+}^a < p_R\, s_a(q_a x_{t_+}^a)$.

Proof. (5.5) and Lemma 2.1 show that $s_a(q_a x_{t_+}^a)$ is defined. Part I above shows $p_a x_{t_+}^a \neq s_a(q_a x_{t_+}^a)$. Therefore, either $p_a x_{t_+}^a < s_a(q_a x_{t_+}^a)$, or $s_a(q_a x_{t_+}^a) < p_a x_{t_+}^a$. Suppose the last inequality holds true. Lemma 2.1 and (5.6) show that for some $\tau = \tau(x^a_{t_+}, a) > 0$,

(5.9) $$|x^a_{t_+ + \tau} - [\delta\Phi_a + u_a(\delta\Phi_a)]| < r$$

and

(5.10) $$x^a_t \in B_1 \text{ on } [t_+, t_+ + \tau].$$

Using (5.9) and (5.4), we infer $0 < x^a_{t_+ + \tau} < -\xi$, and the analogue of Proposition 3.1 (iv) for f instead of g yields

(5.11) $$0 < x^a \text{ on } [t_+ + \tau, +\infty).$$

From (5.10) and (5.8), $\xi < x^a$ on $[b_1(a), t_+ + \tau]$. Together with (5.11), $\xi < x^a$ on $[b_1(a), +\infty)$, a contradiction to H.

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