COLLOQUIUM MATHEMATICUM

VOL. XX 1969 FASC. 1

SOME PRESERVATION THEOREMS (II)

BY

B. WĘGLORZ (WROCŁAW)

This paper [1] is a continuation of [3]. Here we consider a composition of operations defined as follows:

$\mathfrak{B} \in (\mathcal{O}_1 \circ \mathcal{O}_2)(\mathfrak{A})$ if and only if there exists a $\mathfrak{C}$ such that $\mathfrak{B} \in \mathcal{O}_1(\mathfrak{C})$ and $\mathfrak{C} \in \mathcal{O}_2(\mathfrak{A})$.

Such a composition has been considered by H. J. Keisler ([1], Theorems 1 and 2) and A. I. Malcev ([2], Corollary 4). We obtain all those theorems as corollaries (Corollary 1, 2 and 3) from our general Theorem.

The aim of this paper is to show a general preservation theorem for such a composition of operations.

The terminology, notation and general assumptions are the same as in [3], but all operations considered here have an additional property that $\mathfrak{A} \in \mathcal{O}(\mathfrak{A})$ for each structure $\mathfrak{A}$.

THEOREM. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two operations. Then $\Delta(\mathcal{O}_1 \circ \mathcal{O}_2) = \Delta(\mathcal{O}_2 \circ \mathcal{O}_1) = \Delta(\mathcal{O}_1) \cap \Delta(\mathcal{O}_2)$. If, moreover, both $\mathcal{O}_1$ and $\mathcal{O}_2$ are perfect and summable, and we have $(\mathcal{O}_2 \circ \mathcal{O}_1)(\mathfrak{X}) \subseteq (\mathcal{O}_1 \circ \mathcal{O}_2)(\mathfrak{X})$ for each structure $\mathfrak{X}$, then $\mathcal{O}_1 \circ \mathcal{O}_2$ is also perfect and summable.*

For the proof of this theorem we need some lemmas.

LEMMA 1. *For any perfect and summable operations $\mathcal{O}_1$ and $\mathcal{O}_2$, and for any two structures $\mathfrak{A}$ and $\mathfrak{B}$, the following conditions are equivalent:*

(i) *The set $\big(Th(\mathfrak{A}) \cap \Delta(\mathcal{O}_2)\big) \cup \big(Th(\mathfrak{B}) \cap \Delta(\mathcal{O}_1^*)\big)$ is consistent.*

(ii) *There are elementary extensions $\mathfrak{A}'$ and $\mathfrak{B}'$ of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, such that $\mathfrak{B}' \in (\mathcal{O}_1 \circ \mathcal{O}_2)(\mathfrak{A}')$.*

Proof. (i) $\Rightarrow$ (ii). Let $\mathfrak{C}$ be an arbitrary model of the set described in (i). Then $Th(\mathfrak{A}) \cap \Delta(\mathcal{O}_2) \subseteq Th(\mathfrak{C})$ and $Th(\mathfrak{B}) \cap \Delta(\mathcal{O}_1^*) \subseteq Th(\mathfrak{C})$. Using perfectness and summability of $\mathcal{O}_1$ and $\mathcal{O}_2$, we can construct structures $\mathfrak{A}' \succ \mathfrak{A}$, $\mathfrak{B}' \succ \mathfrak{B}$ and $\mathfrak{C}' \succ \mathfrak{C}$ such that $\mathfrak{C}' \in \mathcal{O}_2(\mathfrak{A}')$ and $\mathfrak{C}' \in \mathcal{O}_1^*(\mathfrak{B}')$. The

---

[1] All results presented here were obtained when the author worked in the Institute of Mathematics of the Sibirian Branch of the Soviet Academy of Sciences at Novosibirsk.

last condition is, by definition of $\mathcal{O}_1$, equivalent to $\mathfrak{B}' \in \mathcal{O}_1(\mathfrak{C}')$, which shows that $\mathfrak{B}' \in (\mathcal{O}_1 \circ \mathcal{O}_2)(\mathfrak{A}')$.

(ii) $\Rightarrow$ (i). Without loss of generality we can assume $\mathfrak{B} \in (\mathcal{O}_1 \circ \mathcal{O}_2)(\mathfrak{A})$. Thus there is a $\mathfrak{C}$ such that $\mathfrak{B} \in \mathcal{O}_1(\mathfrak{C})$ and $\mathfrak{C} \in \mathcal{O}_2(\mathfrak{A})$. So $\mathfrak{C} \in \mathcal{O}_1^*(\mathfrak{B}) \cap \mathcal{O}_2(\mathfrak{A})$, and, consequently, this implies that $\mathfrak{C}$ is a model of the set $\big(Th(\mathfrak{A}) \cap \Delta(\mathcal{O}_2)\big) \cup \big(Th(\mathfrak{B}) \cap \Delta(\mathcal{O}_1^*)\big)$.

LEMMA 2. *For any two operations $\mathcal{O}_1$ and $\mathcal{O}_2$, and any two structures $\mathfrak{A}$ and $\mathfrak{B}$, the following conditions are equivalent*:

(i) *The set $\big(Th(\mathfrak{A}) \cap \Delta(\mathcal{O}_2)\big) \cup \big(Th(\mathfrak{B}) \cap \Delta(\mathcal{O}_1^*)\big)$ is consistent.*

(ii) $\Gamma_{\mathcal{O}_1 \circ \mathcal{O}_2}(\mathfrak{A}) = \{\varphi \in \Delta(\mathcal{O}_1):$ *for some* $\psi \in Th(\mathfrak{A}) \cap \Delta(\mathcal{O}_2),\ \vdash \psi \rightarrow \varphi\} \subseteq Th(\mathfrak{B})$.

Proof. $\neg$ (ii) $\Rightarrow$ $\neg$ (i). Suppose that for a $\varphi \in \Delta(\mathcal{O}_1)$ there is a $\varphi \in \Delta(\mathcal{O}_2)$ such that $\vdash \psi \rightarrow \varphi$ and $\mathfrak{A} \models \psi$, and we have $\varphi \notin Th(\mathfrak{B})$. Hence $\neg \varphi \in Th(\mathfrak{B})$, and, by Proposition 1.7 of [3], $\neg \varphi \in \Delta(\mathcal{O}_1^*)$, whence $\neg \varphi \in Th(\mathfrak{B}) \cap \Delta(\mathcal{O}_1^*)$. Since $\varphi \in Th(\mathfrak{A}) \cap \Delta(\mathcal{O}_2)$ and $\vdash \psi \rightarrow \varphi$, we conclude that $\big(Th(\mathfrak{A}) \cap \Delta(\mathcal{O}_2)\big) \cup \big(Th(\mathfrak{B}) \cap \Delta(\mathcal{O}_1^*)\big)$ has as logical consequences φ and $\neg \varphi$, and thus it is inconsistent.

$\neg$ (i) $\Rightarrow$ $\neg$ (ii). The sets $Th(\mathfrak{A}) \cap \Delta(\mathcal{O}_2)$ and $Th(\mathfrak{B}) \cap \Delta(\mathcal{O}_1^*)$ are closed under conjunctions, whence there are a $\neg \varphi \in Th(\mathfrak{B}) \cap \Delta(\mathcal{O}_1^*)$ and a $\psi \in Th(\mathfrak{A}) \cap \Delta(\mathcal{O}_2)$ such that $\vdash \neg(\neg \varphi \wedge \psi)$, which is equivalent to $\vdash \psi \rightarrow \varphi$. Hence $\varphi \in \Gamma_{\mathcal{O}_1 \circ \mathcal{O}_2}(\mathfrak{A})$ and $\varphi \notin Th(\mathfrak{B})$.

REMARK. From $\Gamma_{\mathcal{O}_1 \circ \mathcal{O}_2}(\mathfrak{A}) \subseteq Th(\mathfrak{B})$ it follows that

$$Th(\mathfrak{A}) \cap \Delta(\mathcal{O}_1) \cap \Delta(\mathcal{O}_2) \subseteq Th(\mathfrak{B}),$$

but not conversely. An easy example can be constructed using our Theorem.

LEMMA 3. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two arbitrary operations. Then $\Delta(\mathcal{O}_1 \circ \mathcal{O}_2) = \Delta(\mathcal{O}_1) \cap \Delta(\mathcal{O}_2)$.*

Proof. The inclusion $\Delta(\mathcal{O}_1 \circ \mathcal{O}_2) \supseteq \Delta(\mathcal{O}_1) \cap \Delta(\mathcal{O}_2)$ is obvious. The converse one follows from the fact that for each structure $\mathfrak{X}$ and operations $\mathcal{O}_1$ and $\mathcal{O}_2$ we have $(\mathcal{O}_1 \circ \mathcal{O}_2)(\mathfrak{X}) \supseteq \mathcal{O}_1(\mathfrak{X})$ and $(\mathcal{O}_1 \circ \mathcal{O}_2)(\mathfrak{X}) \supseteq \mathcal{O}_2(\mathfrak{X})$.

In the next lemmas $\mathcal{O}_1$ and $\mathcal{O}_2$ will always be two perfect summable operations, $\mathfrak{A}$ will be an arbitrary structure and A an elementary class defined by $Th(\mathfrak{A})$, i.e. $Th(A) = Th(\mathfrak{A})$. Finally, for a given operation $\mathcal{O}$, define the operation $\hat{\mathcal{O}}$ as follows:

$\mathfrak{B} \in \hat{\mathcal{O}}(\mathfrak{A})$ *if and only if there are structures* $\mathfrak{B}' \equiv \mathfrak{B}$ *and* $\mathfrak{A}' \equiv \mathfrak{A}$ *such that* $\mathfrak{B}' \in \mathcal{O}(\mathfrak{A}')$.

LEMMA. *The classes $(\hat{\mathcal{O}}_2 \circ \hat{\mathcal{O}}_1)(A)$ and $(\hat{\mathcal{O}}_1 \circ \hat{\mathcal{O}}_2)(A)$ are elementary (i.e. both belong to EC_Δ).*

Proof. This is trivial since $\Gamma_{\mathcal{O}_2\circ\mathcal{O}_1}(\mathfrak{A})$ is the set of axioms for $(\hat{\mathcal{O}}_2\circ\hat{\mathcal{O}}_1)(A)$ and $\Gamma_{\mathcal{O}_1\circ\mathcal{O}_2}(\mathfrak{A})$ for $(\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2)(A)$.

LEMMA 5. *If for each structure $\mathfrak{X}$ we have*

$$(\mathcal{O}_2\circ\mathcal{O}_1)(\mathfrak{X}) \subseteq (\mathcal{O}_1\circ\mathcal{O}_2)(\mathfrak{X}),$$

then the class $L = (\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2)(A)$ is closed under operations $\hat{\mathcal{O}}_1$, $\hat{\mathcal{O}}_2$, $\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2$ and $\hat{\mathcal{O}}_2\circ\hat{\mathcal{O}}_1$.

Proof. Let us observe that we have the following inclusions:

(a) $\hat{\mathcal{O}}_1\circ(\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2) \subseteq \hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2$;

(b) $\hat{\mathcal{O}}_2\circ(\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2) \subseteq (\hat{\mathcal{O}}_2\circ\hat{\mathcal{O}}_1)\circ\hat{\mathcal{O}}_2 \subseteq (\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2)\circ\hat{\mathcal{O}}_2 \subseteq \hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2$;

(c) $(\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2)\circ(\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2) \subseteq \hat{\mathcal{O}}_1\circ(\hat{\mathcal{O}}_2\circ\hat{\mathcal{O}}_1)\circ\hat{\mathcal{O}}_2 \subseteq \hat{\mathcal{O}}_1\circ(\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2)\circ\hat{\mathcal{O}}_2 \subseteq \hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2$;

(d) $(\hat{\mathcal{O}}_2\circ\hat{\mathcal{O}}_1)\circ(\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2) \subseteq (\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2)\circ(\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2) \subseteq \hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2$.

From these inclusions it is easy to see that L has the required property.

LEMMA 6. *Let $L = (\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2)(A)$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ have the property assumed in Lemma 5. Then there is a set of axioms of the class L which is a subset of $\Delta(\mathcal{O}_1)\cap\Delta(\mathcal{O}_2)$.*

Proof. By Lemma 5 (c) it is visible that L can be characterized by a set of sentences from $\Delta(\mathcal{O}_1\circ\mathcal{O}_2)$ but this set is, by Lemma 3, a subset of $\Delta(\mathcal{O}_1)\cap\Delta(\mathcal{O}_2)$.

Proof of the Theorem. First part follows from Lemma 3. It is also visible that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are summable, then so is $\mathcal{O}_1\circ\mathcal{O}_2$. Perfectness of $\mathcal{O}_1\circ\mathcal{O}_2$ follows from the fact that for each structure $\mathfrak{A}$ the class $(\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2)(A)$ is axiomatizable by a set $\Sigma\subseteq\Delta(\mathcal{O}_1)\cap\Delta(\mathcal{O}_2)$. Since $\mathfrak{A}\in(\mathcal{O}_1\circ\mathcal{O}_2)(A)$, we have $\Sigma\subseteq Th(\mathfrak{A})$, whence the condition $Th(\mathfrak{A})\cap\Delta(\mathcal{O}_1)\cap\Delta(\mathcal{O}_2)\subseteq Th(\mathfrak{B})$ implies $\mathfrak{B}\in(\hat{\mathcal{O}}_1\circ\hat{\mathcal{O}}_2)(A)$, and thus $\mathcal{O}_1\circ\mathcal{O}_2$ is perfect by Lemma 1, q.e.d.

Now we apply our theorem to obtain results of Keisler and Malcev mentioned in the introduction. For this purpose let us denote by

$\mathscr{H}$ the operation of taking homomorphic images;

$\mathscr{S}$ the operation of taking isomorphs of substructures;

$\mathscr{R}$ the operation of taking isomorphs of reduced powers;

$\mathscr{L}$ the operation of taking isomorphs of limit reduced powers ([2]).

It is well known that all these operations are perfect and all, except $\mathscr{R}$, are summable.

COROLLARY 1 (H. J. Keisler). *$\mathscr{H}\circ\mathscr{S}$ is perfect and $\Delta(\mathscr{H}\circ\mathscr{S}) = \Delta(\mathscr{H})\cap\Delta(\mathscr{S})$.*

Proof. For each structure $\mathfrak{X}$ we have $(\mathscr{S}\circ\mathscr{H})(\mathfrak{X})\subseteq(\mathscr{H}\circ\mathscr{S})(\mathfrak{X})$, thus our Theorem can be applied.

([2]) An elementary theory of the limit reduced power was considered in [4].

COROLLARY 2 (H. J. Keisler). *$\mathscr{S}^* \circ \mathscr{H}$ is perfect and $\Delta(\mathscr{S}^* \circ \mathscr{H}) = \Delta(\mathscr{S}^*) \cap \Delta(\mathscr{H})$.*

Proof. For each structure $\mathfrak{X}$ we have $(\mathscr{H} \circ \mathscr{S}^*)(\mathfrak{X}) \subseteq (\mathscr{S}^* \circ \mathscr{H})(\mathfrak{X})$. Thus our Theorem can be applied.

COROLLARY 3 (A. I. Malcev). *$\mathscr{S} \circ \mathscr{R}$ is perfect and $\Delta(\mathscr{S} \circ \mathscr{R}) = \Delta(\mathscr{S}) \cap \Delta(\mathscr{R})$.*

Proof. At first let us remark that for each structure $\mathfrak{X}$ we have $(\mathscr{S} \circ \mathscr{R})(\mathfrak{X}) = (\mathscr{S} \circ \mathscr{L})(\mathfrak{X})$ and $(\mathscr{R} \circ \mathscr{S})(\mathfrak{X}) \subseteq (\mathscr{L} \circ \mathscr{S})(\mathfrak{X}) \subseteq (\mathscr{S} \circ \mathscr{R})(\mathfrak{X})$. Thus applying our Theorem, we see that $\mathscr{S} \circ \mathscr{L}$ is perfect and $\Delta(\mathscr{S} \circ \mathscr{L}) = \Delta(\mathscr{S} \circ \mathscr{R}) = \Delta(\mathscr{S}) \cap \Delta(\mathscr{L}) = \Delta(\mathscr{S}) \cap \Delta(\mathscr{R})$, which finishes the proof [3].

---

[3] For the proof of the equality $\Delta(\mathscr{L}) = \Delta(\mathscr{R})$ see also [4].

REFERENCES

[1] H. J. Keisler, *Theory of models with generalized atomic formulas*, The Journal of Symbolic Logic 25 (1960), p. 1-26.

[2] А. И. Мальцев, *Несколько замечаний о квазимногообразиях алгебраических систем*, Алгебра и Логика 5 ((1966), p. 3-9.

[3] B. Węglorz, *Some preservation theorems*, Colloquium Mathematicum 17 (1967), p. 269-276.

[4] A. Wojciechowska, *Limit reduced powers*, ibidem (to appear).

Reçu par la Rédaction le 29. 12. 1967