REMARKS AND EXAMPLES CONCERNING UNORDERED
BAIRE-LIKE AND ULTRABARRELLED SPACES

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1. Following Saxon [12], p. 153, we call a locally convex space Baire-like (unordered Baire-like) if it cannot be covered by an increasing (arbitrary) sequence of nowhere dense absolutely convex subsets. Thus for a locally convex space we have the following implications:

\[
\text{Baire} \Rightarrow \text{unordered Baire-like} \Rightarrow \text{Baire-like} \Rightarrow \text{barrelled}.
\]

Recently, Saxon [12] (p. 158, Example 2.2, and p. 157, Example 1.4) showed that unordered Baire-like normed spaces need not be Baire and that Baire-like normed spaces need not be unordered Baire-like, whereas, on account of [1], p. 274, a metrizable locally convex space is barrelled iff it is Baire-like.

It is our first purpose here to give some more simple examples. In fact, Theorem 1 enables us to give examples of normed unordered Baire-like spaces which are not ultrabarrelled (and hence not Baire). Recall that, according to [10], p. 249, a linear topological space \((X, \beta)\) is called ultrabarrelled if \(\alpha \subset \beta\) for every linear topology \(\alpha\) on \(X\) which is \(\beta\)-polar, i.e., has a base of neighbourhoods of zero consisting of \(\beta\)-closed sets. Further characterizations of ultrabarrelled spaces may be found in [5], p. 295 ff., and [14], p. 10 ff. Clearly, if \(X\) is Baire, then it is ultrabarrelled. On the other hand, ultrabarrelled linear topological spaces need not be Baire, as the strongest linear topology on an infinite-dimensional linear space shows. Furthermore, any strict inductive limit of an increasing sequence of Fréchet spaces is ultrabarrelled (see [5], p. 297, Corollary 2), hence barrelled, but clearly not Baire-like.

Our Theorem 2 provides examples of metrizable (and even normed) ultrabarrelled locally convex spaces which are not unordered Baire-like, and hence not Baire.

Summarizing, there seems to be no evident relation between "ultrabarrelled" and "unordered Baire-like", which is not surprising at all.
In view of this, and the above-mentioned result of [1], we note that the following characterization results from Corollary 3 of [4], p. 558:

A metrizable linear topological space $X$ is ultrabarrelled iff it satisfies the following condition:

Let $(A_i^{(m)}; i, m \in \mathbb{N})$ be a double sequence of closed balanced subsets of $X$ such that

(a) $A_i^{(m)} \subseteq A_{i+1}^{(m+1)}$ and $A_i^{(m)} + A_{i+1}^{(m+1)} \subseteq A_i^{(m)}$ ($i, m \in \mathbb{N}$),
(b) $\bigcup \{A_i^{(m)}; m \in \mathbb{N}\}$ is absorbent ($i \in \mathbb{N}$).

Then for every $i \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $A_i^{(m)}$ is a neighbourhood of zero in $X$.

In Section 3, we consider the space $m_0(\mathcal{A})$ of $\mathcal{A}$-simple scalar-valued functions defined on a set $I$, where $\mathcal{A}$ is a $\sigma$-algebra of subsets of $I$, equipped with the usual supremum-norm. In addition to some known curious properties of $m_0(\mathcal{A})$ like those of being barrelled, non-Baire, etc., we prove that it is not ultrabarrelled and contains no infinite-dimensional separable barrelled subspace. For a special case $I = \mathbb{N}$ and $\mathcal{A} = \mathcal{P}(\mathbb{N})$, the $\sigma$-algebra of all subsets of $\mathbb{N}$, these results are due to N. J. Kalton and A. Pełczyński, respectively (unpublished). The extension of Pełczyński’s result to $m_0(\mathcal{A})$ requires no new techniques while our proof of the result of Kalton is entirely different from the original one (cf. Remark 2). In fact, we prove a stronger result: No infinite-dimensional subspace $X$ of $m_0(\mathcal{A})$ admits a linear topology $\xi$ stronger than the norm-topology such that $(X, \xi)$ is either Baire or metrizable and ultrabarrelled. We also give an alternative proof of the main result of Batt et al. (see [2], Theorem 1) concerning summable sequences in $m_0(\mathcal{A})$.

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2. Let us start by observing that a locally convex space $(X, \xi)$ is unordered Baire-like iff it is barrelled and has the following property:

\((*)\) Given a sequence $(A_n; n \in \mathbb{N})$ of absolutely convex closed sets covering $X$, some $A_n$ is absorbent (i.e., is a barrel in $(X, \xi)$).

Now, if $\xi \supseteq \eta$ are linear topologies on a linear space $X$ such that $(X, \xi)$ satisfies $(*)$, then so does $(X, \eta)$. In particular, this is certainly the case where $(X, \xi)$ is Baire.

It follows that any non-Baire barrelled space $X$ which admits a stronger linear Baire topology will provide an example of a non-Baire unordered Baire-like locally convex space. Such spaces really do exist, as was already shown by Robertson [10], and from the results in Section 7 of [10], p. 255, we immediately have the following

**Theorem 1.** Let $(X, \xi)$ be a non-locally convex $F$-space (i.e., a metrizable complete linear topological space) such that the strongest locally convex
topology \( c(\xi) \) weaker than \( \xi \) is Hausdorff. Then \((X, c(\xi))\) is a metrizable (hence Mackey) unordered Baire-like locally convex space which is not ultrabarrelled. Furthermore, if \((X, \xi)\) is locally bounded, then \((X, c(\xi))\) is normed.

The most simple spaces to which Theorem 1 applies are (as in [10], p. 256) the classical sequence spaces \( l_p \), where \( 0 < p < 1 \), equipped with the topology \( \xi \) defined by the \( F \)-norm

\[
|\langle t_n; n \in N \rangle|_p := \sum_{n \in N} |t_n|^p.
\]

Then \((l_p, |\cdot|_p)\) is a locally bounded non-locally convex \( F \)-space which is continuously embedded as a dense subspace in \((l_1, |\cdot|_1)\), and its dual is identified in a standard way with \( l_\infty \), the dual of \((l_1, |\cdot|_1)\) (cf. [3], p. 822).

Hence \( c(\xi) \) is simply the topology induced on \( l_p \) by the norm \( |\cdot|_1 \), and thus \((l_p, |\cdot|_1)\) is a normed unordered Baire-like space which is not ultrabarrelled (and hence not Baire).

**Theorem 2.** Let \((X, \xi)\) be an infinite-dimensional \( F \)-space which admits a biorthogonal sequence \( \{(u_n, f_n); n \in N\} \) such that

\[
Z := Y + \text{lin}\{u_n; n \in N\}, \quad \text{where} \quad Y := \{x \in X; f_n(x) = 0 \text{ for all } n \in N\},
\]

is dense in \( X \). Then \((X, \xi)\) contains a dense subspace which is ultrabarrelled but not unordered Baire-like.

**Proof.** Let \( \mathcal{F} \) be an ultra-filter on \( N \) such that \( \{n\} \notin \mathcal{F} \) for all \( n \in N \). For \( A \in \mathcal{F} \) let

\[
E_A := \overline{Y + \text{lin}\{u_n; n \notin A\}^\xi} \quad \text{and} \quad E := \bigcup \{E_A; A \in \mathcal{F}\}.
\]

Clearly, \( E \) is a linear subspace of \( X \) containing \( Z \), hence \( E \) is dense in \((X, \xi)\). Furthermore, \( E \) is the union of the sequence of closed hyperplanes \( f_n^{-1}(0) \cap E \neq \emptyset \), whence \( E \) in its relative topology \( \xi \cap E \) is not unordered Baire-like. Let \( \eta \) be the strongest linear topology on \( E \) such that \( \eta \cap E_A = \xi \cap E_A \) for all \( A \in \mathcal{F} \), i.e., \((E, \eta)\) is the linear inductive \( (\eta^*) \)-inductive as in [5], p. 286) limit of the \( F \)-spaces \((E_A, \xi \cap E_A) (A \in \mathcal{F}) \). By Corollary 1 to Theorem 3.2 of [5], p. 297, \((E, \eta)\) is ultrabarrelled. We are going to show that \( \eta = \xi \cap E \). Clearly, it suffices to prove \( \eta \subset \xi \cap E \).

Let \( U \) be an \( \eta \)-closed neighbourhood of zero in \((E, \eta)\). We prove first the existence of an open neighbourhood \( V \) of zero in \((E, \xi \cap E)\) such that \( V \cap Z \subset U \). Let \( \{V_n; n \in N\} \) be a base of the neighbourhoods of zero in \((X, \xi)\) satisfying \( V_{n+1} \subset V_n \) \( (n \in N) \), and assume that \( V_n \cap Z \neq U \) for all \( n \in N \). Since for every \( r \in N \) the space \( Y + \text{lin}\{u_n; n > r\} \) is a closed subspace of finite codimension in \((Z, \eta \cap Z)\), we find inductively a partition \( \{I(k); k \in N\} \) of \( N \) into disjoint consecutive finite sets \( I(k) \) \( (k \in N) \), a sequence \( \{y_k; k \in N\} \) in \( Y \), and a sequence \( \{\lambda_i; i \in N\} \) of scalars such that

\[
y_k + \sum_{i \in I(k)} \lambda_i u_i \in V_k \setminus U \quad \text{for all } k \in N.
\]
Now let
\[ M := \bigcup \{ I(2k-1); \, k \in \mathbb{N} \} \quad \text{and} \quad N := \bigcup \{ I(2k); \, k \in \mathbb{N} \}. \]

\( \mathcal{F} \) being an ultra-filter, either \( M \in \mathcal{F} \) or \( N \in \mathcal{F} \). Consequently, there is \( A \in \mathcal{F} \) such that \( V_n \cap E_A \neq U \) for all \( n \in N \) which contradicts \( \eta \cap E_A = \xi \cap E_A \). Thus there is an open neighbourhood \( V \) of zero in \((E, \xi \cap E)\) such that \( V \cap Z \subset U \). Since
\[ \overline{V \cap (Z \cap E_A)}^n \supset V \cap E_A \quad \text{for all} \quad A \in \mathcal{F}, \]
we have \( V \subset \overline{U}^n = U \), which proves that \( U \) is a neighbourhood of zero in \((E, \xi \cap E)\).

Theorem 2 applies especially when \((X, \xi)\) is a separable Fréchet space, since then in \( X \) there exists a biorthogonal sequence \( \{(u_n, f_n); \, n \in \mathbb{N}\} \) such that \( \text{lin}\{u_n; \, n \in \mathbb{N}\} \) is dense in \((X, \xi)\) (a result of Klee, cf. [9], p. 118).

Thus, in particular, we may construct dense ultrabarrelled not unordered Baire-like subspaces in every Banach space with basis. We do not know whether or not the assertion of Theorem 2 holds for all infinite-dimensional Fréchet spaces. (P 1031)

3. If \( \mathcal{A} \) is an algebra of subsets of a set \( I \), we denote by \( m_0(\mathcal{A}) \) the linear space of all \( \mathcal{A} \)-simple scalar-valued functions defined on \( I \); \( \tau \) denotes the topology induced on \( m_0(\mathcal{A}) \) by the usual supremum-norm \( \| \cdot \|_\infty \). If \( \mathcal{A} \) is infinite, then \( (m_0(\mathcal{A}), \tau) \) is easily seen to be non-Baire, and thus not complete. If \( \mathcal{A} \) is a \( \sigma \)-algebra, then, as in [12], p. 157, Example 1.4, one proves easily that \( (m_0(\mathcal{A}), \tau) \) is barrelled.

For the proof of Theorem 3 we use the following two lemmas, whereby we write \( \text{card}(A) \) for the cardinality of the set \( A \).

**Lemma 1.** If \( X \) is a linear subspace contained in
\[ X_n := \{ x \in m_0(\mathcal{A}); \, \text{card}(x(I)) \leq n \}, \]
then \( \dim X \leq n \).

**Proof.** Suppose that this is not true and take any \( y \in X \) with
\[ \text{card}(y(I)) = m := \max \{ \text{card}(z(I)); \, z \in X \}. \]

Since \( \dim X > n \geq m \), there exists \( z \in X \) such that, for some \( t \in y(I) \),
\[ \text{card}(z(y^{-1}(t))) \geq 2. \]

Then taking \( \varepsilon > 0 \) sufficiently small and setting \( x := y + \varepsilon z \), we have \( \text{card}(x(I)) = m \) and \( x \in X \), which is a contradiction.

**Lemma 2.** If \( \mathcal{B} \) is a subalgebra of \( \mathcal{A} \), then \( m_0(\mathcal{B}) \) is a closed subspace of \((m_0(\mathcal{A}), \tau)\).
Proof. Take any non-zero \( x \in m_0(\mathcal{A}) \) and write it in the form
\[
x = \sum_{i=0}^{m} t_i \chi_{A_i}
\]
with pairwise disjoint \( A_i \in \mathcal{A}, t_i \neq t_j \) for \( i \neq j \), and \( t_0 = 0 \). Let
\[
a := \min\{|t_i - t_j|; i \neq j\},
\]
and then choose any \( y \in m_0(\mathcal{A}) \) such that \( \|x - y\|_\infty < a/2 \); say
\[
y = \sum_{k=0}^{n} s_k \chi_{B_k},
\]
where \( B_k \) are pairwise disjoint members of \( \mathcal{A} \), \( s_k \neq s_l \) for \( k \neq l \), and \( s_0 = 0 \).

If \( B_k \cap A_i \neq \emptyset \) for some \( i \) and \( k \), then \( B_k \subset A_i \). Otherwise, for some \( j \neq i \) we would have \( B_k \cap A_j \neq \emptyset \), and hence
\[
|s_k - t_i| < a/2 \quad \text{and} \quad |s_k - t_j| < a/2,
\]
so that \( |t_i - t_j| < a \), which is impossible. It follows that each \( A_i \) is the union of a subsequence of \( B_0, B_1, \ldots, B_n \) so that \( A_i \in \mathcal{A} \), and thus \( x \) is \( \mathcal{A} \)-simple.

**Theorem 3.** Let \( \mathcal{A} \) be an infinite algebra of subsets of a set \( I \). Then:

(a) No infinite-dimensional subspace \( X \) of \( m_0(\mathcal{A}) \) admits a linear Baire topology \( \xi \) which is stronger than \( \tau \cap X \). Thus, in particular, no infinite-dimensional subspace \( X \) of \( m_0(\mathcal{A}) \) admits an F-space topology \( \xi \) stronger than \( \tau \cap X \).

(b) No infinite-dimensional subspace \( X \) of \( m_0(\mathcal{A}) \) admits a metrizable ultrabarrelled linear topology \( \xi \) which is stronger than \( \tau \cap X \). Thus, in particular, no infinite-dimensional subspace of \( m_0(\mathcal{A}) \) is ultrabarrelled.

(c) Every separable subspace of \( (m_0(\mathcal{A}), \tau) \) is of at most countable dimension. Thus no infinite-dimensional separable subspace of \( (m_0(\mathcal{A}), \tau) \) is barrelled.

Proof. Our proof of (a) and (b) uses some ideas found in [13], p. 981, and in [8], Section 4. For each \( n \in N \) let
\[
X_n := \{ x \in m_0(\mathcal{A}); \text{ card}\{x(I)\} \leq n \}.
\]
Suppose that \( X \) is an infinite-dimensional subspace of \( m_0(\mathcal{A}) \) and \( \xi \) is a linear topology on \( X \) satisfying \( \xi = \tau \cap X \). Since each \( X_n \) is \( \tau \)-closed, \( X_n \cap X \) is \( \xi \)-closed and, clearly, balanced for all \( n \in N \). Furthermore, we have
\[
X_n \cap X + X_n \cap X \subset X_n \cap X \quad (n \in N)
\]
and
\[
X = \bigcup \{X_n \cap X; n \in N\}.
\]
If $\xi$ is metrizable and ultrabarrelled, then by Corollary 2 of [6], p. 683, there exists $k \in N$ such that $X_k \cap X$ is a $\xi$-neighbourhood. This implies

$$X = \bigcup \{m \cdot (X_k \cap X); m \in N\} \subseteq X_k,$$

hence $\dim X \leqslant k$ by Lemma 1.

If $\xi$ is a Baire topology, we also infer that some $X_k \cap X$ is a $\xi$-neighbourhood, which leads to the same contradiction. Thus we have proved (a) and (b).

To show (c) it is enough to prove the first assertion, since it is well known that a metrizable locally convex space of countably infinite dimension cannot be barrelled.

Let $X$ be a separable subspace of $m_0(\mathcal{A})$, and let $D$ be a countable dense subset of $(X, \tau \cap X)$. Then there exists a countable subalgebra $\mathcal{A}$ of $\mathcal{A}$ such that $D \subseteq m_0(\mathcal{A})$. Since $m_0(\mathcal{A})$ is closed by Lemma 2, we have $X \subseteq m_0(\mathcal{A})$, and hence

$$\dim X \leqslant \dim m_0(\mathcal{A}) \leqslant \aleph_0.$$

Remarks. (a) Using similar methods, H. Pfister has independently proved that $\{m_0(\mathcal{P}(I)), \tau\}$ is not ultrabarrelled and that every Banach disk in $m_0(\mathcal{P}(I))$, provided with the relative product topology from $K'$, must be finite dimensional (unpublished).

(b) Kalton's original proof of the fact that $\{m_0(\mathcal{P}(N)), \tau\}$ is not ultrabarrelled is very ingenious so that we would like to present it here. We shorten $m_0(\mathcal{P}(N))$ to $m_0$. Suppose that $(m_0, \tau)$ is ultrabarrelled and let $(F, |\cdot|)$ be the $F$-space, constructed by Rolewicz and Ryll-Nardzewski (see [11], p. 329 ff.), containing a sequence $(x_n; n \in N)$ which is subseries summable but not bounded multiplier summable. For each $n \in N$ define $T_n: l_\infty \rightarrow F$ by

$$T_n(t) := \sum_{i=1}^{n} t_i x_i \quad (t = (t_i; i \in N) \in l_\infty).$$

Then each $T_n$ is a continuous linear map of $(l_\infty, \|\cdot\|_\infty)$ into $(F, |\cdot|)$, and $(T_n; n \in N)$ converges pointwise on $m_0$. Hence, by the Banach-Steinhaus theorem for ultrabarrelled spaces (cf. [10], p. 250), $(T_n|_{m_0}; n \in N)$ is equicontinuous. Since $m_0$ is dense in $(l_\infty, \|\cdot\|_\infty)$, we infer that $(T_n; n \in N)$ is also equicontinuous and converges pointwise on $l_\infty$. This implies that $(x_n; n \in N)$ is bounded multiplier summable, which is a contradiction.

(c) Replacing the phrase "no infinite-dimensional subspace" in Theorem 3 (a) and (b) by "no subspace $X$ satisfying $\sup \{\text{card} \{x(I); x \in X\} = \infty\}$", we may also prove the assertions of Theorem 3 (a) and (b) for the space $m_0(\mathcal{A}, E)$ of all $\mathcal{A}$-simple functions with values in a normed space $(E, \|\cdot\|)$, provided with the topology induced by the supremum-norm.
(d) It would be interesting to know the coarsest ultrabarrelled topology on \(m_0(\mathcal{A})\), stronger than \(\tau\). (P 1032)

Finally, we present an alternative and somewhat shorter proof for the main result of Batt et al. [2], Theorem 1, which — up to some variants — was also obtained independently by H. Pfister (unpublished).

**Theorem 4.** Let \(\mathcal{A}\) be a \(\sigma\)-algebra of subsets of a set \(I\), and let \(\tau\) and \(\pi\) be the topologies on \(m_0(\mathcal{A})\) of uniform and pointwise convergence on \(I\), respectively. Let \((x_n; n \in N)\) be a sequence in \(m_0(\mathcal{A})\). Then:

(a) If \((x_n; n \in N)\) is bounded multiplier \((BM)\) summable in \((m_0(\mathcal{A}), \pi)\), then \(\dim\lim\{x_n; n \in N\} < \infty\).

(b) If \((x_n; n \in N)\) is subfamily \((SF)\) summable in \((m_0(\mathcal{A}), \tau)\), then \(\dim\lim\{x_n; n \in N\} < \infty\).

**Proof.** We shorten \(m_0(\mathcal{A})\) to \(m_0\). We first show that (b) \(\Rightarrow\) (a). Suppose that \((x_n; n \in N)\) is BM-summable for \(\pi\) and define a linear map \(T: l_\infty \to m_0\) by

\[
T(t) := \sum_{i=1}^{\infty} t_i x_i \quad (t = (t_i; i \in N) \in l_\infty),
\]

where the sum is taken with respect to \(\pi\). By the Banach-Steinhaus theorem, \(T: (l_\infty, \|\cdot\|_\infty) \to (m_0, \pi)\) is continuous. Since \(\tau\) is \(\pi\)-polar, \(T: (l_\infty, \|\cdot\|_\infty) \to (m_0, \tau)\) is continuous as well. It follows that \((x_n; n \in N)\) is \(\tau\)-bounded and thus, again by the \(\pi\)-polarity of \(\tau\), \((s_n x_n; n \in N)\) is SF-summable with respect to \(\tau\) for each \((s_n; n \in N) \in l_1\). Choosing \((s_n; n \in N) \in l_1\) such that \(s_n \neq 0\) for all \(n \in N\), from (b) we obtain

\[
\dim\lim\{s_n x_n; n \in N\} = \dim\lim\{x_n; n \in N\} < \infty.
\]

Now suppose that (b) is false; then without loss of generality we may assume the sequence \((x_n; n \in N)\) to be linearly independent. Since \((x_n; n \in N)\) is SF-summable, the formula

\[
m(A) := \sum_{n \in A} x_n \quad (A \subset N)
\]

defines a countably additive vector measure \(m: \mathcal{P}(N) \to (m_0, \tau)\). Then, with \(X_n\) as in the proof of Theorem 3, a result of [7], p. 46, Lemma 2, implies \(m(A) \in X_r\) for some \(r \in N\) and all finite subsets \(A\) of \(N\), and we may suppose that \(r\) is the smallest integer for which this holds. Choose a finite subset \(A\) of \(N\) such that

\[
x := \sum_{n \in A} x_n
\]

assumes precisely \(r\) distinct values. Then, as \(\|x_m\|_\infty \to 0\) \((m \to \infty)\) and \((x_n; n \in N)\) is linearly independent, there exists \(m \in N \setminus A\) such that \(x_m\) assumes at least two different values on the set \(x^{-1}(t)\) for some \(t \in x(I)\).
and, at the same time, $\|x_m\|_\infty$ is small enough to assure that

$$x + x_m = \sum_{n \in A \cup \{m\}} x_n$$

assumes at least $r + 1$ distinct values. This, however, contradicts the choice of $r$.

We refer to [2] for various consequences of Theorem 4.

**REFERENCES**


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