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LIMIT DISTRIBUTION OF THE TIME OF COINCIDENCE IN RENEWAL STREAMS

Abstract. In the paper we give some conditions, sufficient for the time of coincidence in two renewal streams to have the exponential asymptoticity.

1. Introduction. Let $\{X_k^{(i)}, k \geqslant 1\}$, $i = 1, 2$, be two sequences of independent, positive, integer-valued random variables. Let

$$S_k^{(i)} = \sum_{j=1}^{k} X_j^{(i)}, \quad k \geqslant 1,$$

denote the renewal stream, and let

$$N^{(i)}(j) = \max\{k \geqslant 0\colon S_k^{(i)} < j\} \quad \text{and} \quad \gamma^{(i)}(j) = S^{(i)}_{N^{(i)}(j)+1} - j$$

denote the renewal process and the residual time process for the stream i, respectively. Moreover, let $\{U_j^{(i)}, j \geqslant 0\}$ denote the sequence of random variables defined as $U_j^{(i)} = 1$ when $\gamma^{(i)}(j) = 0$ and $U_j^{(i)} = 0$ otherwise. Then let $\{u_j^{(i)}, j \geqslant 0\}$ denote the renewal sequence for the stream $\{S_k^{(i)}, k \geqslant 1\}$, namely $u_j^{(i)} = \mathrm{E}U_j^{(i)}$. The random variable

$$T = \min\{j \geqslant 1\colon U_j^{(1)} = U_j^{(2)} = 1\}$$

is called the *moment of first coincidence of renewals* in the considered renewal streams.

The distribution of the moment of first coincidence is discussed in [2]. In this paper the question about the exponential asymptoticity of the distribution of the moment of coincidence was stated. Our paper gives some answers to this question.

2. The limit theorems. We consider two arrays of random variables $\{X_{kn}^{(i)}, k \geqslant 1, n \geqslant 1\}$, $i = 1, 2$, such that for fixed n the random variables are independent and for fixed n and i they are identically distributed with the discrete distribution $\{p_{kn}^{(i)}, k \geqslant 1\}$ on positive integers. The renewal stream generated by the sequence $\{X_{kn}^{(i)}, k \geqslant 1\}$ is denoted by $\{S_{kn}^{(i)}, k \geqslant 1\}$, and we denote the renewal process by $N_n^{(i)}(j)$ and the residual time process by $\gamma_n^{(i)}(j)$. Let $T^{(n)}$ denote the moment of first coincidence of renewals in streams

$\{S_{kn}^{(1)}, k \geqslant 1\}$ and $\{S_{kn}^{(2)}, k \geqslant 1\}$. It is known (see [2]) that

$$\mathrm{E}T^{(n)} = \theta_n = 1/\lambda_n^{(1)}\lambda_n^{(2)}, \quad \text{where } \lambda_n^{(i)} = 1/\mathrm{E}X_{1n}^{(i)}.$$

THEOREM 1. *Suppose that the following assumptions are fulfilled:*

$$\lim_{n\to\infty} \lambda_n^{(2)} = 0, \tag{1}$$

$$\lim_{n\to\infty} \sum_{i=0}^{\infty} \Big|\sum_{j=1}^{\infty} \big(u_{i+jn}^{(2)} - \lambda_n^{(2)} - \sum_{k=0}^{i-1} (u_{i+kn}^{(2)} - \lambda_n^{(2)})\, p_{i-kn}^{(1)}\big)\, p_{jn}^{(1)}\Big| = 0, \tag{2}$$

$$\lim_{n\to\infty} \lambda_n^{(2)} \int_x^{\infty} P\{X_{1n}^{(1)} > \theta_n t\}\, dt = 0 \quad \textit{for } x > 0. \tag{3}$$

Then

$$\lim_{n\to\infty} P\{T^{(n)}/\theta_n > x\} = \exp\{-x\} \quad \textit{for } x \geqslant 0. \tag{4}$$

An intermediate step in the proof of Theorem 1 is the examination of the case where the sequence $\{X_{kn}^{(2)}, k \geqslant 1\}$ generates a stationary renewal stream for every n. In other words, we assume that $X_{1n}^{(2)} = \gamma_n^{(2)}$ is a residual random variable,

$$P\{\gamma_n^{(2)} = k\} = \lambda_n^{(2)} \sum_{i=k+1}^{\infty} p_{in}^{(2)}, \quad k \geqslant 0.$$

In this "semistationary" model the moment of first coincidence of renewals $T^{(n)}$ is defined by

$$T_{\gamma_n^{(2)}}^{(n)} = \min\{j \geqslant 0\colon\ U_{jn}^{(1)} = U_{jn}^{(2)} = 1\}.$$

THEOREM 2. *Suppose that assumptions* (1)–(3) *of Theorem* 1 *are fulfilled. Then*

$$\lim_{n\to\infty} P\{T_{\gamma_n^{(2)}}^{(n)}/\theta_n > x\} = \exp\{-x\} \quad \textit{for } x \geqslant 0.$$

3. Proofs. For simplicity of notation in the proofs, we omit the index n in the notation of random variables and their distributions.

Let us define for any integer-valued, nonnegative random variables the metric

$$V(\xi, \eta) = \sum_{i=0}^{\infty} |P\{\xi = i\} - P\{\eta = i\}|.$$

In the proofs of both theorems we use the following

LEMMA 1. *If $\{p_k, k \geqslant 1\}$ is a probability distribution on positive integers and $\{u_k, k \geqslant 0\}$, $\{\gamma(j), j \geqslant 0\}$ and γ are a renewal sequence, a residual time process and a residual random variable, respectively, generated by this distribu-*

tion, then

$$V(\gamma(j), \gamma) = \sum_{i=0}^{\infty} \Big|u_{j+i} - \lambda - \sum_{k=0}^{i-1} (u_{k+j} - \lambda) p_{i-k}\Big|,$$

where

$$1/\lambda = \sum_{k=1}^{\infty} k p_k.$$

Proof. Since

$$\begin{aligned} P\{\gamma(j) = i\} &= p_{j+i} + \sum_{k=1}^{j-1} u_k p_{j+i-k} \\ &= p_{j+i} + \sum_{k=1}^{j+i-1} u_k p_{j+i-k} - \sum_{k=j}^{j+i-1} u_k p_{j+i-k} \\ &= u_{j+i} - \sum_{k=0}^{i-1} u_{k+j} p_{i-k}, \end{aligned}$$

we have

$$P\{\gamma(j) = i\} - P\{\gamma = i\} = u_{j+i} - \lambda - \sum_{k=0}^{i-1} (u_{k+j} - \lambda) p_{i-k},$$

which implies Lemma 1.

Let T_i be a random variable defined as

$$T_i = \min\{j \geqslant i \colon U_j^{(1)} = U_j^{(2)} = 1 \mid U_i^{(2)} = 1\} - i.$$

Proof of Theorem 2. First we notice that $T_{\gamma(2)}$ can be represented as

$$T_{\gamma(2)} = (1 - U_0^{(2)})(X + T_{\gamma(2)(X)}),$$

where we wrote X instead of $X_1^{(1)}$ for simplicity of the notation. Hence, putting

$$a(s) = \mathrm{E}\exp\{-sT_{\gamma(2)}/\theta\},$$

we have

$$\frac{1}{\lambda^{(2)}}[\mathrm{E}\exp\{-s(1 - U_0^{(2)})(X + T_{\gamma(2)(X)})/\theta\} - a(s)] = 0. \tag{5}$$

Since the stream $\{S_k^{(2)}, k \geqslant 1\}$ is stationary, the left-hand side of (5) equals

$$\frac{1}{\lambda^{(2)}}[\lambda^{(2)} + (1 - \lambda^{(2)})\,\mathrm{E}(\exp\{-s(X + T_{\gamma(2)(X)})/\theta\} \mid U_0^{(2)} = 0) - a(s)]$$

$$= \frac{1}{\lambda^{(2)}}[\lambda^{(2)} + (1 - \lambda^{(2)})(\mathrm{E}(\exp\{-s(X + T_{\gamma(2)(X)})/\theta\} \mid U_0^{(2)} = 0) - a(s)(1 - s\lambda^{(2)}))$$

$$+(1-\lambda^{(2)})a(s)(1-s\lambda^{(2)})-a(s)\big]$$
$$=1-a(s)(1+s-s\lambda^{(2)})$$
$$+\frac{1}{\lambda^{(2)}}(1-\lambda^{(2)})\big(\mathrm{E}\big(\exp\{-s(X+T_{\gamma^{(2)}(X)})/\theta\}\mid U_0^{(2)}=0\big)-a(s)(1-s\lambda^{(2)})\big).$$

Thus (5) takes the following form:

$$a(s)(1+s-s\lambda^{(2)})-1$$
$$=(1-\lambda^{(2)})\frac{1}{\lambda^{(2)}}\mathrm{E}\left[\left(\exp\{-s(X+T_{\gamma^{(2)}(X)})/\theta\}\mid U_0^{(2)}=0\right)-a(s)\left(1-\frac{sX}{\theta}\right)\right].$$

We show now that the right-hand side of this identity converges to zero. Since in virtue of the assumption (1) the expression $1-\lambda^{(2)}$ tends to one, it is sufficient to show the convergence to zero of the following expression:

$$\frac{1}{\lambda^{(2)}}\left|\mathrm{E}\left[\left(\exp\{-s(X+T_{\gamma^{(2)}(X)})/\theta\}-\exp\{-sT_{\gamma^{(2)}(X)}/\theta\}(1-sX/\theta)\right)\mid U_0^{(2)}=0\right]\right|$$
$$+\frac{1}{\lambda^{(2)}}\left|\mathrm{E}\left[\left(\exp\{-sT_{\gamma^{(2)}(X)}/\theta\}(1-sX/\theta)\mid U_0^{(2)}=0\right)-a(s)(1-sX/\theta)\right]\right|.$$

It can be estimated by the sum of two expressions

$$\frac{1}{\lambda^{(2)}}\mathrm{E}\left|\exp\{-sX/\theta\}-1+sX/\theta\right|$$
$$+\frac{1}{\lambda^{(2)}}\left|\mathrm{E}\left[\left(\exp\{-sT_{\gamma^{(2)}(X)}/\theta\}\mid U_0^{(2)}=0\right)-\exp\{-sT_{\gamma^{(2)}}/\theta\}(1-sX/\theta)\right]\right|.$$

By (3) it is easy to show that the first of them tends to zero (see [3a], p. 51, or [3b], p. 57).

We prove now that the second expression also tends to zero. It can be written in the form

$$\frac{1}{\lambda^{(2)}}\left|\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}p_j^{(1)}\mathrm{E}\exp\{-sT_i/\theta\}\left(P\{\gamma^{(2)}(j)=i\mid U_0^{(2)}=0\}-P\{\gamma^{(2)}=i\}\right)(1-sj/\theta)\right|$$
$$\leqslant\frac{1}{\lambda^{(2)}}\left|\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}p_j^{(1)}\mathrm{E}\exp\{-sT_i/\theta\}\left(P\{\gamma^{(2)}(j)=i\mid U_0^{(2)}=0\}-P\{\gamma^{(2)}=i\}\right)\right|$$
$$+\frac{1}{\lambda^{(2)}}\left|\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}p_j^{(1)}\mathrm{E}\exp\{-sT_i/\theta\}\left(P\{\gamma^{(2)}(j)=i\mid U_0^{(2)}=0\}-P\{\gamma^{(2)}=i\}\right)sj/\theta\right|.$$

Notice that

$$P\{\gamma^{(2)}(j)=i\mid U_0^{(2)}=0\}-P\{\gamma^{(2)}=i\}$$

$$= \frac{1}{P\{U_0^{(2)}=0\}}[P\{\gamma^{(2)}(j)=i\} - P\{\gamma^{(2)}(j)=i,\ U_0^{(2)}=1\}$$
$$- P\{\gamma^{(2)}=i\}(1-P\{U_0^{(2)}=1\})],$$

and from the fact that the stream with index two is stationary it follows that

$$\frac{1}{\lambda^{(2)}}[P\{\gamma^{(2)}(j)=i \mid U_0^{(2)}=0\} - P\{\gamma^{(2)}=i\}]$$
$$= \frac{1}{1-\lambda^{(2)}}[P\{\gamma^{(2)}(j)=i \mid U_0^{(2)}=1\} - P\{\gamma^{(2)}=i\}].$$

Therefore

$$\frac{1}{\lambda^{(2)}}\Big|\sum_{i=0}^{\infty}\sum_{j=1}^{\infty} p_j^{(1)} \mathrm{E}\exp\{-sT_i/\theta\}(P\{\gamma^{(2)}(j)=i \mid U_0^{(2)}=0\} - P\{\gamma^{(2)}=i\})(1-sj/\theta)\Big|$$
$$\leqslant \frac{1}{1-\lambda^{(2)}}\Big[V(\gamma^{(2)}(X),\ \gamma^{(2)})$$
$$+\sum_{j=1}^{\infty} p_j^{(1)} \sum_{i=0}^{\infty} |P\{\gamma^{(2)}(j)=i \mid U_0^{(2)}=1\} - P\{\gamma^{(2)}=i\}|\, sj/\theta\Big]$$
$$\leqslant \frac{1}{1-\lambda^{(2)}}[V(\gamma^{(2)}(X),\ \gamma^{(2)})+2\lambda^{(2)}].$$

By Lemma 1 we have

$$V(\gamma^{(2)}(X),\ \gamma^{(2)}) = \sum_{j=1}^{\infty} p_j^{(1)}\, V(\gamma^{(2)}(j),\ \gamma^{(2)})$$
$$= \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} p_j^{(1)} \Big|u_{j+i}^{(2)} - \lambda^{(2)} - \sum_{k=0}^{i-1}(u_{k+j}^{(2)} - \lambda^{(2)})\Big|.$$

By (2) this expression tends to zero. Now we prove the main result.

Proof of Theorem 1. Since the assumptions of Theorem 2 are fulfilled, $\mathrm{E}\exp\{-sT_{\gamma(2)}/\theta\}$ is convergent to $1/(1+s)$ for $s \geqslant 0$, and for the proof it is sufficient to show that

$$\mathrm{E}\exp\{-sT/\theta\} - \mathrm{E}\exp\{-sT_{\gamma(2)}/\theta\} \tag{6}$$

is convergent to zero for $s \geqslant 0$.

Notice that $T = X + T_{\gamma(2)(X)}$, where we wrote X instead of $X_1^{(1)}$ for simplicity of the notation. Using this we can express (6) in the form

$$|\mathrm{E}\exp\{-s(X+T_{\gamma(2)(X)})/\theta\} - \mathrm{E}\exp\{-sT_{\gamma(2)(X)}/\theta\}$$
$$+\mathrm{E}\exp\{-sT_{\gamma(2)(X)}/\theta\} - \mathrm{E}\exp\{-sT_{\gamma(2)}/\theta\}|.$$

This expression can be estimated by

$$\mathrm{E}\,|\exp\{-sX/\theta\}-1|+\sum_{i=0}^{\infty}\mathrm{E}\,|\exp\{-sT_i/\theta\}|\,|P\{\gamma^{(2)}(X)=i\}-P\{\gamma^{(2)}=i\}|$$
$$\leqslant \mathrm{E}\,\{sX/\theta\}+V(\gamma^{(2)}(X),\gamma^{(2)}).$$

Assumptions (1) and (2), and Lemma 1 prove that (6) converges to zero.

4. Remarks. The exponential distribution of the moment of coincidence in the discussed limit problem is suggested by the famous Rényi's Theorem dealing with the convergence of rarified renewal streams to the Poisson stream. In the notation of the present paper, Rényi's assumptions are of the following form: the first renewal stream is fixed with a finite expectation of lifetime and the second one is a Bernoulli trial stream with probability of success tending to zero. Note that Brown in [1] examines limit properties of the N_n sequence of point processes, thinned by p_n processes. However, like Rényi, Brown assumes that the sequence of thinning events conditional on N_n and p_n is an independent sequence.

Our assumptions (1) and (2) are similar to Rényi's ones. Note that the condition

$$\lim_{n\to\infty}(\sup_{j\geqslant 1}p_{jn}^{(1)})\sum_{i=1}^{\infty}i\,|u_{in}^{(2)}-\lambda_n^{(2)}|=0 \tag{7}$$

implies easily the assumption (2). The sum which appears in condition (7) is the measure of the distance between the renewal stream and the Bernoulli stream with the same expectation of lifetime. This sum is finite when the third moment of lifetime is finite (see [4]). The assumption (3) means the convergence of the sequence of random variables $\{X_{1n}^{(1)}/\theta_n,\ n\geqslant 1\}$ to zero in the Khinchin sense. This assumption appears in Soloviev's limit theorems about the asymptoticity of rare events in regenerative processes (see [3a], p. 48, or [3b], p. 53).

The example mentioned below shows that the condition

$$\lim_{n\to\infty}\sup_{j\geqslant 1}p_{jn}^{(i)}=0 \quad \text{for } i=1,2 \tag{8}$$

is not sufficient for the exponential asymptoticity of the moment of coincidence.

Example. Consider two renewal processes which are probability copies of the same process. Let

$$p_{jn}^{(1)}=p_{jn}^{(2)}=rpq^{j-1}+(1-r)\,\delta_{sj}, \quad j=1,2,\ldots,$$

where $p=1/n$, $q=1-p$, $r=q$, $s=n^2$, and δ is the Kronecker delta. Then

$$u_i^{(1)}=u_i^{(2)}=rp(q+rp)^{i-1}, \quad u_i=r^2p^2(q+rp)^{2(i-1)},$$

$$P\{T^{(n)} = i\} = r^2 p^2 \left((q+rp)^2 - r^2 p^2\right)^{i-1}, \quad i = 1, 2, \ldots, n^2-1,$$

and now

$$\lim P\{T^{(n)} \geqslant n^2\} = 1-(1-\exp\{-3\})/3.$$

Conditions (1) and (8) are satisfied. We have $\theta_n = (2n-1)^2$ but hypothesis (4) implies that the limit equals $\exp\{-1/4\}$. We should notice that in this case the condition (3) is not satisfied.

References

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